Scandinavian Political Studies, Vol. 2 – New series – No. 2 1979
ISSN 0080–6757
Nordic Political Science Association

Research Note

The Maximum Distortion and the Problem of the First Divisor of Different P.R. Systems

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The mathematical analysis of the relationship between parties' vote shares and seat shares in a given electoral district originally presented by Stein Rokkan has given rise to many applications (Rokkan 1968). The mathematical theory of 'boundary conditions' for different electoral systems provides a means of understanding the nature of representative democracy. Different threshold formulae provide useful analytical tools in comparative empirical studies of different electoral systems as well as important information about alternative electoral systems for political decision-makers pondering constitutional or electoral reform. These formulae also enable political parties engaged in an electoral campaign or in negotiations concerning common electoral coalitions to increase their information about alternative strategies, thus making it possible for the parties to base their choice on results obtained by threshold formulae (Laakso 1978).

The purpose of this paper is to analyze maximum distortion formulae of different electoral systems. Particular emphasis is placed on the Sainte Laguë and d'Hondt methods which are widely used in Scandinavia (Denmark, Norway, and Sweden apply the modified Sainte Laguë rule, and Finland and Iceland the d'Hondt rule). The second main question is the first divisor of electoral formulae. This is a very central point when applying the modified Sainte Laguë rule. Is there any theoretical reason to defend the 1.4 divisor accepted in most of the Scandinavian countries? What is the consequence of different arrangements of the first divisor for the proportionality of elections?

In order to answer these questions one must first define the threshold formulae for different electoral methods, as they are the constituent elements of the maximum distortion formulae. Some simple empirical examples are also presented to illustrate theoretical calculations.

1. Thresholds for Proportional Representation

Throughout this paper the following notations are used:

- n = the number of parties in a given electoral district
- m = the number of seats in a given electoral district (district magnitude)
- k = the first divisor of a number series
- v_r = the threshold of representation
- v_w = the threshold for winning all the seats

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According to Raschauer (1971), the general formula for number series methods of P.R. can be presented as follows:

$a \cdot m - b$

where a and b are constants. With different values for a and b the electoral formulae most widely used are solved. Because this kind of analysis has been carried out in another paper only the final results are presented here (for details, see Laakso 1978). The general formulae for v_w and v_r are as follows:

$$v_w = \frac{am-b}{am-b+k(n-1)} \tag{1}$$

$$v_r = \frac{k}{am-b+k(n-1)} \tag{2}$$

The threshold of representation (v_r) means the minimum share of votes needed to win the first seat (see Rae, Hanby & Loosemore 1971). By definition the v_w threshold is analogously the minimum share of votes required to win all the seats. For different systems of P.R. the v_w and v_r indices are presented in Table 1:

Table 1. The Threshold of Winning All the Seats (v_w) and the Threshold of Representation (v_r) for Different Systems of P.R.

Electoral method	The threshold of winning all the seats	The threshold of representation
d'Hondt	$v_w = \frac{m}{m+n-1}$	$v_r = \frac{1}{m+n-1}$
Sainte Laguë	$v_w = \frac{2m-1}{2m+n-2}$	$v_r = \frac{1}{2m+n-2}$
The modified (1.4) Sainte Laguë	$v_w = \frac{2m-1}{2m+1.4n-2.4}$	$v_r = \frac{1}{2m+1.4n-2.4}$
The Danish method	$v_w = \frac{3m-2}{3m+n-3}$	$v_r = \frac{1}{3m+n-3}$
Imperial	$v_w = \frac{m+1}{m+2n-1}$	$v_r = \frac{2}{m+2n-1}$

To make it easier to understand the behavior of different indices, an empirical example is constructed and calculations are plotted in Figure 1, where the v_r index is presented as a function of n and m. The index values for v_r are calculated in cases where the number of parties is 6 (the approximate case in Norway and Sweden) or 10 (the approximate case in Denmark and Finland) and m acquires values from 1 to 20.

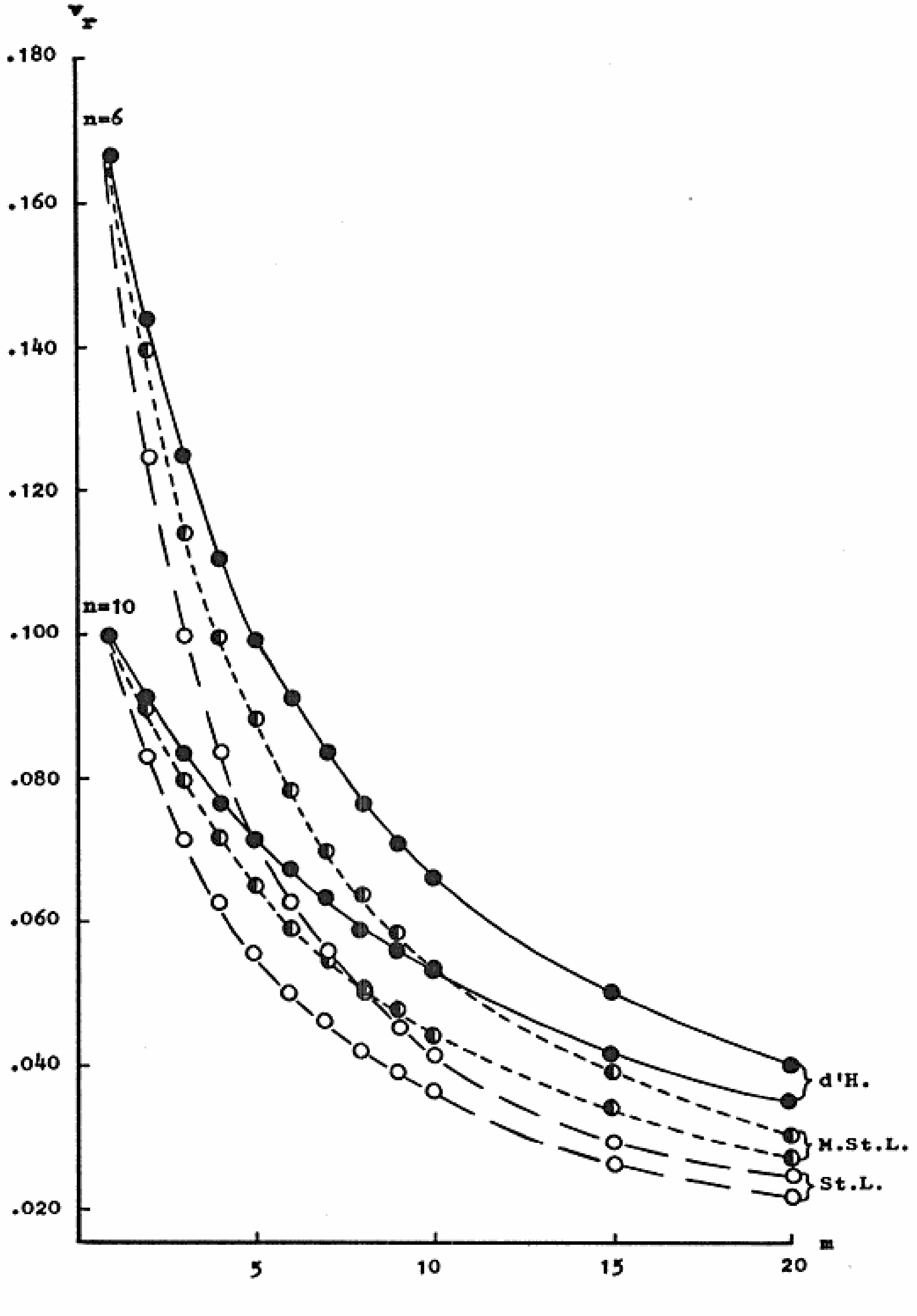

Figure 1. The Threshold of Representation (v_r) for Different Electoral Rules as a Function of the Number of Mandates (m) and the Number of Parties (n).

From Figure 1 it is easy to observe that the more representatives elected or the more parties there are struggling for mandates, the lower the threshold of representation. The Sainte Laguë methods have a somewhat lower threshold of representation compared with that of d'Hondt. Later we shall see that the order remains the same when comparing the maximum distortion formulae.

2. The General Formula for the Theoretical Limits of Maximum Distortion

Loosemore & Hanby (1971) measured the theoretical limits of maximum distortion with the formula

$$D = \frac{1}{2} \sum_{i=1}^{n} \left| v_i - s_i \right| \tag{3}$$

where v_i and s_i are the vote and seat shares of party i respectively. The formula is based on the assumption that the maximum attainable distortion occurs when one party wins all the seats with the minimum possible vote share while the (n–1) other parties win no seats, but divide their votes equally (for details, see Loosemore & Hanby 1971, 470).

The maximum distortion formula is thus determined as follows:

$$D = \frac{1}{2} [(1-v_w) + (n-1)(v_r-0)]$$

D is solved by means of the formulae for v_w and v_r presented above (formulae (1) and (2)):

$$D = \frac{k(n-1)}{am-b+k(n-1)} \tag{4}$$

The D index receives values ranging from 0 to 1. The maximum possible distortion occurs when one party wins all the seats with no votes (D = 1). In turn, the minimum possible distortion presupposes that $v_r = 0$ and $v_w = 1$. In reality these boundary conditions are hardly possible, but they are useful in that they present a notional limiting case.

For different electoral methods of P.R. the maximum distortion is presented in Table 2.

The D index enables us to calculate which of the P.R. methods is the most proportional. A simple empirical application is presented in Table 3.

From Table 3 it is simple to rank the P.R. systems in order of proportionality: the Danish method (the most proportional), the Sainte Laguë, the modified (1.4) Sainte Laguë, the d'Hondt, and the Imperial methods.

Table 2. The Maximum Distortion (D) for Different Systems of P.R.

Electoral method	The maximum distortion
d'Hondt	$D = \frac{n-1}{m+n-1}$
Sainte Laguë	$D = \frac{n-1}{2m+n-2}$
The modified (1.4) Sainte Laguë	$D = \frac{1.4(n-1)}{2m+1.4n-2.4}$
The Danish method	$D = \frac{n-1}{3m+n-3}$
Imperial	$D = \frac{n-1}{\frac{m}{2}+n-\frac{1}{2}}$

Table 3. The Maximum Distortion (D) for Different Systems of P.R. when n = 10 and m gets Values 5, 10, 15 and 20 Respectively

	the number of mandates (m)			
	5	10	15	20
d'Hondt	0.643	0.474	0.375	0.310
Sainte Laguë	0.500	0.321	0.237	0.188
The mod. (1.4) Sainte Laguë	0.583	0.399	0.303	0.244
The Danish method	0.409	0.243	0.173	0.134
Imperial	0.750	0.621	0.529	0.462

3. The Role of the First Divisor of a Number Series

The only number series method linked to a particular stipulation barrier against party system fragmentation is the Sainte Laguë rule applied in Denmark, Norway and Sweden. The 'normal' first divisor of 1.0 is replaced by a threshold barrier of 1.4. Is this theoretically justified? Why precisely a 1.4 divisor? This problem is analyzed in part by studying the dependence of maximum distortion (D) on the numerical values of the first divisor (k).

In Table 4 the D index is presented as a function of k (the first divisor) and m (the number of seats in a given electoral constituency). Figure 2 illustrates graphically the ensuing calculations.

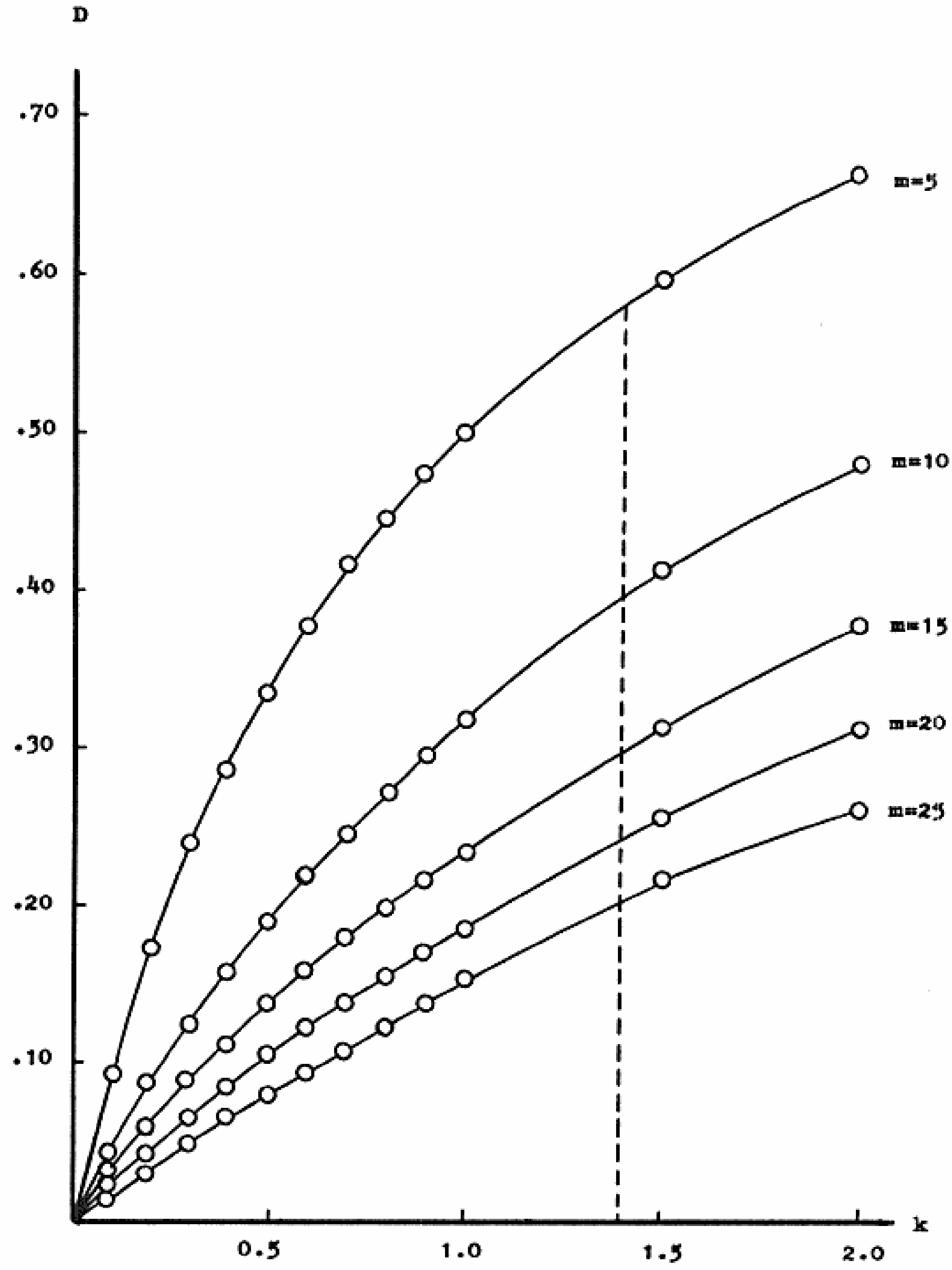

Figure 2. The Maximum Distortion (D) of the Sainte Laguë Rule as a Function of the First Divisor (k) and the Number of Mandates (m).

From the results presented it can be noticed that the lower the first divisor, the lower the maximum distortion. Minimum distortion (D = 0) is only possible if all the parties secure representatives (k = 0). In the light of these results, it is difficult to understand precisely why a divisor of 1.4 has been taken into use. There is no theoretical justification for the choice. Why not 1.3 or 1.5? In the latter case the division of the first two mandates is equal to the d'Hondt method, as to divide votes by 1.5 and 3 sequentially is the same as to divide votes by 1 and 2 as in the d'Hondt method. Arguments for the 1.4 choice can be found in political circumstances rather than in theoretical calculations (see Rokkan 1968, 14–15).

From Table 4 it can be calculated how much the maximum distortion increases as the first divisor 1.0 is replaced by 1.4, as follows:

$$\frac{D\ (k=1.4)}{D\ (k=1.0)}$$

The increase in maximum distortion depends on the number of mandates. When m = 5 the increase is 16.6% but when m = 25 the increase is already 32.3%.

In any case the introduction of the first divisor in a general form in the maximum distortion formula allows us to analyze the relative importance of this factor to the proportionality of the different methods of P.R. The proportionality of e.g. d'Hondt is increased by lowering the first divisor below 1.0. By using the maximum distortion formulae for d'Hondt and Sainte Laguë, we can determine

Table 4. The maximum Distortion (D) of the Sainte Laguë Rule as a Function of the First Divisor (k) and the Number of Mandates (m)

k	m = 5	m = 10	m = 15	m = 20	m = 25
0.0	0.000	0.000	0.000	0.000	0.000
0.1	0.091	0.045	0.030	0.023	0.018
0.2	0.167	0.087	0.058	0.044	0.035
0.3	0.231	0.124	0.085	0.065	0.052
0.4	0.286	0.159	0.110	0.085	0.068
0.5	0.333	0.191	0.134	0.103	0.084
0.6	0.375	0.221	0.157	0.122	0.099
0.7	0.412	0.249	0.178	0.139	0.114
0.8	0.444	0.275	0.199	0.156	0.128
0.9	0.474	0.299	0.218	0.172	0.142
1.0	0.500	0.321	0.237	0.188	0.155
1.1	0.524	0.343	0.254	0.202	0.168
1.2	0.545	0.362	0.271	0.217	0.181
1.3	0.565	0.381	0.287	0.231	0.193
1.4	0.583	0.399	0.303	0.244	0.205
1.5	0.600	0.415	0.318	0.257	0.216
1.6	0.615	0.431	0.332	0.270	0.227
1.7	0.630	0.446	0.345	0.282	0.238
1.8	0.643	0.460	0.358	0.293	0.248
1.9	0.655	0.474	0.371	0.305	0.259
2.0	0.667	0.486	0.383	0.316	0.269

how much the first divisor of d'Hondt should be lowered to make this electoral method as proportional as Sainte Laguë. This can be calculated from the following:

D (d'Hondt) = D (Sainte Laguë)

By using the maximum distortion formulae presented in Table 2 we get

$$\frac{k(n-1)}{m+k(n-1)} = \frac{n-1}{2m+n-1}$$
$$\longrightarrow k = \frac{m}{2m-1}$$

In this case k is dependent only on the size of electoral constituency. The more representatives to be elected, the more the ratio m/(2m–1) approaches the value 0.5. Therefore the first divisor of the d'Hondt method would have to be about 0.5 for this method of P.R. to be as proportional as the Sainte Laguë formula.

In a manner similar to that presented above, the value of the first divisor (k) for d'Hondt as compared with the modified Sainte Laguë can be calculated as follows:

$$\frac{k(n-1)}{m+k(n-1)} = \frac{1.4(n-1)}{2m-1+1.4(n-1)}$$
$$\longrightarrow k = \frac{1.4m}{2m-1}$$

The first divisor of d'Hondt should be about 0.7 for this method of P.R. to be as proportional as the modified Sainte Laguë.

The above analysis demonstrates that the manipulation of the first divisor of a number series is a very powerful weapon in influencing the proportionality of P.R. methods. This should also be taken into account by political decision-makers considering electoral reform.

4. Conclusions

In the first part of this paper a general formula for the maximum distortion of an electoral rule was suggested. The first divisor of a number series was included in the maximum distortion formulae. In the second part the Sainte Laguë and d'Hondt methods were analyzed using empirical examples based on theoretical formulae.

The electoral rules considered can be ranked in the following order of proportionality: the Danish method, the Sainte Laguë, the mod. Sainte Laguë, the d'Hondt, and the Imperial methods. The proportionality of different electoral rules is influenced by modifying the first divisor. Proportionality is increased by lowering the first divisor below 1.0 and lowered by increasing the first divisor over 1.0 (as in the modified Sainte Laguë method). However, modification has been based on political realities rather than theoretical calculations. In other words, it is hard to find any theoretical justification for Denmark, Norway, and Sweden applying precisely a 1.4 divisor.

It is worth noting that proportionality is only one of the properties of an electoral system. In the Scandinavian countries active discussion is presently centered around the consequences of different electoral rules on party system fragmentation. The more proportional the electoral system, the more it fragmentates the party system. This tendency has given rise to dissatisfaction and there is increasing opposition to a further 'democratization' of the electoral systems.

Balinski & Young (1978) have recently presented three new 'tests' for P.R. systems. They discuss a criterion of stability and the property of different electoral systems to encourage coalitions or schisms. According to their analysis there is no 'perfect' electoral method which fully meets their criteria. The choice between different methods of P.R. should therefore be made in terms of which criteria are viewed as the most important for the situation in question (Balinski & Young 1978, 857). Their analysis demonstrates e.g. that the d'Hondt method encourages coalitions much more than the Sainte Laguë rule. One may consider this result a strong argument for d'Hondt, as the capacity for coalition formation is very important in cabinet formation and in legislative decision-making.

It must be kept in mind that the relative merits of electoral rules are based on different criteria, and constructing an absolute order of superiority is therefore inconceivable. Nevertheless, further theoretical and empirical research on the different components of the system and tests of P.R. systems will increase our knowledge about representative democracy, its justifications, and its functioning in practice.

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