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## The Ideal in Mathematics: A Spinozist-Marxian Elaboration and Revision of the *Theory of Knowledge Objectification*

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### Abstract

*The theory of knowledge objectification, initially presented and developed by Luis Radford, has gained some traction in the field of mathematics education. As with any developing theory, its presentation contains statements that may contradict its stated intents; and these problems are exacerbated in its uptake into the work of other scholars. The purpose of this study is to articulate a Spinozist-Marxian approach based on the works of the Russian philosopher E. V. Ilyenkov. In this approach, the objectification exists not in things—semiotic means that mediate interactions—but as a real relation between people. An important aspect of the theory of knowledge objectification is the role mediation, a process where some third entity intervenes (intercedes) or stands between two entities of concern. However, one consequence of the approach developed in this study is that the (problematic) concept of mediation is unnecessary and can be abandoned. A concrete classroom example from Radford's own studies is used to exemplify and develop pertinent issues. I show how words (signs) are not mediators standing between interlocutors but are events common to them. In particular, the societal nature of the ideal—a synecdoche of relations between objects that reflect relations between people—should be added and the notion of (sign) mediation no longer is required.*

**Keywords:** Idealism; materialism; consciousness; praxis; mediation.

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## Introduction

### Purpose and Contribution of this Study

The purpose of this study is to propose changes to the *theory of knowledge objectification*, which is used in mathematics education to theorize interactional and learning phenomena. It has been developed as part of cultural-historical activity theory (Radford, 2008). A main trait of the theory of knowledge objectification is that “semiotic means of objectification—e.g. objects, artifacts, linguistic devices and signs ... are intentionally used by individuals in social processes of meaning production,” where *objectification* denotes the fact that something is brought “in front of someone’s attention or view” (Radford, 2002, p. 14). In other words, something ideal from the mind of an individual is given an equivalent or representative material form that objectifies the ideal form in a thing, and, as such, is situated and thus mediating between people (i.e., in social processes). In this article, I show that in this view form Radford presents it, the theory is inconsistent with (a) Ilyenkov’s (1977) Spinozist-Marxian conception of the relation between the ideal and the material and (b) the theory of the later Vygotsky, who was in the process of developing a Spinozist-Marxian theory of cognition that abandons the idea of mediation. Following Ilyenkov and Vygotsky, I articulate a radical alternative—which, because both scholars are grounded in their Marxian reading of Spinoza, I also refer to as Spinozist-Marxian.<sup>1</sup> In this alternative, the ideal, general, and universal mathematical forms and indeed every higher psychological function exist not in the thing (semiotic means) but *as real, physical relation between people*. That is, in the revised theory, knowledge is objectified, appearing *as* the real, physical relations of people. In this take of the late Vygotsky, objects, artifacts, linguistic devices, and signs have

<sup>1</sup> This is not a place to argue that both Vygotsky and Ilyenkov pursue Spinozist-Marxian approaches. Let it merely be said that Vygotsky noted in the months before his death that he wanted to “Bring Spinozism to life in Marx[ist] psychology” (in Zavershneva, 2010, p. 66), and there are 96 and 177 explicit references, respectively, to Spinoza and Marx in Ilyenkov (1977). In numerous published works, I thus have used the term Spinozist-Marxian to refer to the theoretical underpinnings of these scholars (e.g., Roth & Jornet, 2017).

“once and for always *lost the role of mediator* between subject and object, response and stimulus” (Mikhailov, 2001, p. 18, emphasis added).

Following the brief survey of the uses of the theory of knowledge objectification (section 1.2), I articulate in section 2, and closely following Ilyenkov (1977), a Spinozist-Marxian account of ideal (mathematical) forms that radically differs from the way it appears in the theory of knowledge objectification. A classroom fragment is presented in section 3 together with two analyses, the first exemplifying the original theory of knowledge objectification and the second representing the revised, Spinozist-Marxian account (Roth, 2017, 2018). Section 4 discusses issues that require reframing in the theory of objectification: (a) the idea of externalization and (b) the phenomenon of (semiotic) mediation.

### **The Theory of Knowledge Objectification: State of the Field**

The theory of knowledge objectification is an important recent framework for understanding knowing, knowledge, and learning of mathematics from a cultural-historical activity theoretic perspective (e.g., Radford, 2002, 2008, 2015). It has enjoyed a considerable uptake in the mathematics education literature. Researchers draw on it to conceptualize, for example, the role of multi-modal articulations of thought in “semiotic nodes,” ensembles of words, gestures, objects, or tools, that externalize what otherwise are subjective thoughts (e.g., Arzarello, Robutti, & Thomas, 2015; Chahine, 2013; Kaput, 2009). This externalization leads to and constitutes a form of “desubjectified (numerical) communication,” which has the effect of a “higher level of objectification” on the part of the student (Moutsios-Rentzos, Spyrou, Peteinara, 2014, p. 48). The theory is used as a means of describing a “subject’s knowledge, understanding and skills at some point in the learning process” (Font, Godino, & Gallardo, 2013, p. 113). Because such externalizations tend to be—but, as some suggest, not always are (e.g., Zurina & Williams, 2011)—for interlocutors, the theory inherently articulates learning as a social process (e.g., Carlsen, 2010; Radford, 2009). The theory also

leads to a different, ethics-based definition of the modes and relations of mathematical productions in the classroom (Radford, 2012).

An advance over its alternatives is that the “theory of knowledge objectification ... brings together both the sensuous and the cultural” (Radford, Edwards, & Arzarello, 2009, p. 91); it therefore is held as not separating knowledge from its instantiation in concrete situations. Radford (e.g., 2013, 2015) grounds the theory in the work of G.W.F. Hegel (e.g., Radford, 2013, 2015), who showed that the thinking self (the internal) knows itself only when it turns (part of) itself into something outside of itself, allowing the latter to be the object of the former’s activity. But the Hegelian “outside” is not in the material world but merely (past) thought outside of (current) thinking—“Hegel’s objective idealism” (Marx & Engels, 1978, p. v). Purely ideal forms, such as those that are characteristic of mathematics, remain invisible and unknown unless they manifest (reveal) themselves in some objective form entirely outside of mind (Ilyenkov, 2012). It therefore makes sense that the theory of knowledge objectification moves beyond Hegel in focusing on the materiality of the means of objectification. Ideal mathematical forms only become real through their *objectification*—a fact that constitutes the heart of the theory of knowledge objectification as Radford develops it. This articulation allows the theory of knowledge objectification to make a connection between the individual and objective mathematical (abstract) ideas, conceived as historically developed cultural forms, and school mathematics, that is, “students’ encounters with historically constituted mathematical meanings” (Radford, 2009, p. 123). Here, the “meanings” are equivalent to what classical philosophers name the (purely) *ideal* forms (e.g. of mathematics): “meaning is related to a hypothetical or idealized state of affairs” (Radford, 2006, p. 45).

The theory of knowledge objectification therefore concerns the appearance of the ideal (i.e., thought, mind) in objective, external, and material form, for example, in how

knowledge is made available in speech, written words, diagrams, gestures, and other body movements (Arzarello et al., 2015; LaCroix, 2014; Moutsios-Rentzos et al., 2014). It is here that the problems arise for the theory, because knowledge, the ideal, is separated from its material manifestations. An indication of this separation is the need to introduce signs, tools, bodies (i.e., actions, gestures), or activity that *mediate* between (a) individual students and their (cultural) object of knowledge (Bartolini-Bussi & Mariotti, 2008; Radford, 2002), (b) bodily senses and mind (Radford, 2003), (c) cognition and praxis (Radford, 2006), (d) observer and observed (Radford, 2006), (e) (subjective) knowledge or cognition and its objective expression (Radford, Bardini, & Sabena, 2006; Radford, 2012, 2015), (f) mental plane and social plane (Radford, 2008), (g) student and student or student and teacher (Bartolini-Bussi & Mariotti, 2008; Radford & Roth, 2011), (h) knowledge and experience (Radford, 2009), or (i) knowledge as pure possibility and reality (Radford, 2013). This role of (sign) mediation is problematic, however, as Mikhailov (2001), another Spinozist philosopher, points out in a text on the relation between the self and other: “the external corporeal existence of other people, their real-objective behavior, their activity with things, their voices and gestures and, consequently, the object-related nature of all the conditions of their lives (all that is *other*), is not mediated for individuals” (p. 20, original emphasis, underline added). Thus, there is no place for the concept of mediation in a revised approach to mathematical knowing and learning (Roth, 2018).

### **The Sensible–Supersensible Nature of Cultural Objects**

The theory of knowledge objectification presents a real materialist solution to the relation between thought and the world, mind, and matter. This creates a problem, for, appearing as things (e.g., signs and artifacts), these externalizations stand as a third material thing *between* persons or between a person and the world; as such, they are said to mediate the relations (cf. Vygotsky, 1989). A Spinozist-Marxian approach, on the other hand, focuses

on irreducible social relations rather than sign mediation. In subsection 2.1, I articulate the position that ideal forms exist *as* societal relations between people (Marx & Engels, 1978); in subsection 2.2, I extend the general discussion of ideal forms to mathematical objects.

### **The Emergence of Ideal Forms Exemplified by Marx's Analysis of *Value***

Most theoretical approaches locate ideal forms in the mind. In contrast, the theory of knowledge objectification, what matters to classroom learning is the knowledge that is given objective presence in the form of material things (e.g., objects, artifacts, linguistic devices, and signs). The theoretical alternative presented here goes even further: in it, all higher psychological functions (thus knowledge and ideal mathematical forms) exist *as* social relation (not *in* material things) (Vygotsky, 1989). This approach is grounded in the way Marx conceived of the ideal and everything else (e.g. personality) that distinguishes humans from other animals. The ideal, abstract, supersensible exchange- *value* form emerges from the material, concrete, sensible use-value form of a commodity (Marx & Engels, 1962).

The simple value form of a commodity comes as “value expression” or “value relation” (Fig. 1 – Form I). In general form, this may be expressed as  $z \text{ Com. A} = u \text{ Com. B}$ , or  $= v \text{ Com. C}$  etc. Here, in the relation between 20 yards of linen and 1 coat, the actual proportion may be accidental (contingent). Such exchange relation is the contingent self-expression of a particular *social* relation and thus is not universal. The total or expanded form of value appears as in relative relations as in (Fig. 1 – Form II). In this second form, the value of linen remains constant whether it is traded for 1 coat, 10 lbs. of tea, 40 lbs. of coffee, etc. Here, “The bodily form of each of these commodities figures now as a particular equivalent form” (Marx & Engels, 1962, p. 78). Form II thus expresses the situation where weavers trade their linen for everything else they need. Any *general* expression of value is excluded, because the value of any commodity is expressed in terms of the value of many other commodities. The general form of value, the condition for the universal money form, makes

one commodity the measure of all the others. This is expressed in a new kind of system of relation (Fig. 1 – Form III)

**Figure 1**

*Forms of value according to Capital*

|                                    |                   |                |
|------------------------------------|-------------------|----------------|
| <u>Form I</u>                      | <u>Form III</u>   |                |
| 5 beds = 1 house; or               | 1 coat =          | }              |
| 20 yards of linen = 1 coat         | 10 lbs. tea =     |                |
|                                    | 40 lbs. coffee =  |                |
| <u>Form II</u>                     | 1 quarter wheat = |                |
| 20 yards of linen = 1 coat         | 2 ounces gold =   |                |
| 20 yards of linen = 10 lbs. tea    | etc. commodity =  |                |
| 20 yards of linen = 40 lbs. coffee |                   | 20 yards linen |
| ...                                |                   |                |

Three forms of values according to Capital: I – Elementary or accidental form; II – Total or expanded form; III – General form.

Form III no longer is merely contingent on one *social* relation, for, as it is valid in society as a whole, it is *universal*. This is so because the value of all commodities, but that of linen, now is expressed in terms of linen. As a result, commodities express their value simply (in elementary form) and unitarily, because with respect to the same commodity. Therefore, “its value is elementary and the same for all, therefore general” (Marx & Engels, 1962, p. 79). That is, the value form is general (universal) because in every single trade relation of a society, the same referent serves to establish the values of all other commodities. Money becomes the symbol, which requires general recognition, and, therefore, has to be a *societal* thing that can be replaced by other symbols (Marx & Engels, 1983). In each commodity, therefore, the ideal form (the general, exchange-value) *coincides* with the particular (use-value).

The ideal value-form, thus, “is understood in *Capital* precisely as the reified ... form of societal-human life-activity” (Ilyenkov, 2012, p. 180). “The production of life ... now appears as a twofold relation—on the one hand, as a natural, on the other as a societal

relation” (Marx & Engels, 1978, p. 29). Contrasting the way it appears in the theory of knowledge objectification and constructivist theory, the ideal (also supersensible, abstract, general) exists in this take neither in the object (commodity, mathematical expression) nor in the human (mind). Instead, it exists in the inherently soci(et)al relations of real people, relations that reflect the relation between persons and things. The ideal and mind, therefore, are societal products existing precisely *as* societal relations (Mamardašvili, 1986). The ensemble of societal relations is reflected in the relations among commodities generally and in generalized value specifically. Importantly, however, the “proper societal relationship of people of common labor (value) is presented by consciousness as occurring outside the societal relations of things, as consciousness of the suprasensible properties of the latter” (p. 109). That is, the notion of the ideal as residing in things is a fiction of consciousness.

### **From Commodities to Mathematical Objects**

The preceding analysis has been extended to mathematical objects (Ilyenkov, 2012). Thus, the ideal, “is the pattern of the real, object-oriented activity of man, consistent with the form of the thing outside the head, outside the brain” (p. 162). Related to mathematics, the ideal includes its truths, logical (dialectical) categories, topological structures, imaginary numbers, regularities in natural numbers, and, everything else that mathematicians investigate. Importantly, mathematics is no less material than economy, for it “investigates the real material world, even though it examines it from its own special perspective, from its own specifically mathematical point of view” (Ilyenkov, 2012, p. 183). As a result, if we were to “declare the topological structure to be exclusively a psychological phenomenon, as subjective idealism tends to do” it would mean to “deny mathematical science, and in the end the whole of mathematical natural science, of the objective and necessary meaning of its constructions” (p. 183). The ideality of mathematical forms therefore does not come from their mental characteristics but from the fact that the material forms of things are



manifestations of something very different with which the material form has nothing in common. Ideality is “but the form of societal-human activity represented in the thing, reflecting objective reality; or conversely, the form of human activity, which reflects objective reality, represented as a thing, as an object” (p. 176). If the ideal were to be approached as something existing in the human head, as “some purely psychological or psycho-physiological, mental phenomenon” then we would be already “helpless before a subjective-idealist understanding of the object of contemporary mathematical knowledge” (p. 183).

### **Primacy of the Societal in Mathematical Learning:**

#### **Sketch of an Exemplary Case**

In the preceding sections, I distinguish the theory of knowledge objectification, which focuses on the materialization of knowledge in things (objects, signs, artifacts, linguistic devices) from a Spinozist-Marxian conception of the ideal existing *as* soci(et)al relations, a conception that can be found in the works of Ilyenkov, the notes of the late Vygotsky, and the writings of Mikhailov. In this section, I contrast the two takes in terms of the different accounts they provide for the same fragment from a mathematics lesson. The presentation of the theory of knowledge objectification frequently uses examples from curricular units in which students learn fundamentals of algebra in tasks where the generalization takes the form  $a \cdot (n + b) + c$  (e.g., Radford, 2006, 2013; Radford & Roth, 2011). The following lesson fragment is drawn from the most extensively described case I found in the literature, which is available with a lot of context and a lengthy transcription (Roth & Radford, 2011). In this fragment, the fourth grader Mario and his peers work on a task that requires modeling the efforts of a girl, who, after receiving a piggybank containing \$6, decides to save \$3 each week. The intended abstraction here is of the form  $3n + 6$ , where  $n$  stands for the number of weeks.

## The Lesson Fragment

In the featured episode, the children have placed differently colored chips that distinguish between the initial amount and those amounts subsequently saved. The next step of the task consists in filling a table of values the first row of which contains the “number of the week.” In the second row, students are to enter the “amount saved” in the form  $3 + 6$ ,  $3 + 3 + 6$ ,  $3 + 3 + 3 + 6$ , and so on. The third row contains as header “or,” and the existing numbers and multiplication signs suggest (as someone already in the know can easily see), a contraction of the second-row cells to  $[1 \times] 3 + 6$ ,  $2 \times 3 + 6$ ,  $3 \times 3 + 6$  and so on, respectively. The existing account suggests that Mario indicates having trouble with the task and, after some attempts on his own and after asking his peers, calls the teacher (Roth & Radford, 2011). Mario tells her that he does not understand, and the teacher (Jeanne) then engages in the initially failing attempt to help him fill the cells in the forms provided above.

- 196 J: how many thrEE dollars are you going to have?  
 197 (1.47)  
 198 J: three dOLLars, three dOLL 

|       |
|-------|
| ars   |
| three |

, three dOLLars (0.23)  
 ((points to the 2 '3's in week  
 2 and simultaneously points  
 with left hand to the first,  
 second, and third goblet))  
 199 M:  
 200 J: what are you going to write here?  
 201 M: three?  
 202 (2.59) ((Jeanne moves finger to the cell on his left))  
 203 M: <<p>plus three? plus three? >  
 204 J: yES:: ((he writes))

After succeeding to produce in the same manner—i.e., in joint social work—the product formulation  $3 \times 3 + 6$ , which Mario fills in the corresponding cell of the third row (Week 3), the two move on to produce, in the same give-and-take manner, the contents of the cells corresponding to Week 4. The teacher Jeanne leaves Mario to his own, saying that she thinks he understands now. Mario continues for a while, completing the table, and then states, “Me, I understand now.”

## A Theory of Knowledge Objectification Account

The published analysis presents this as an instant of knowledge objectification, whereas “knowledge objectification is not realized” (Roth & Radford, 2011, p. 49) while Mario was working on his own, and in the initial, failing attempts with the teacher. This is not surprising, “because mathematical meanings are general, [and therefore] they cannot be noticed through observation” (Radford, 2009, p. 119). Yet studies using the theory of objectification suggest that students should see the ideal in physical arrangements, such as when a student’s failure is explained as failing to realize “that the spatiality of the terms provides us with clues that are interesting from an algebraic viewpoint” (Radford, 2013, p. 25). An earlier presentation of the theory frames what eventually happens in the form of a “factual generalization as a scheme abstracted from actions” (Radford, 2003, p. 46). Nevertheless, it is possible to read existing accounts from an individualistic frame, because objectification is said to exist in becoming conscious of the fact that the contents of the goblets, 9, 12, 15 ... dollars are in fact the result of the steps  $(3 + 6)$ ,  $3 + (3 + 6)$ ,  $(3 + (3 + (3 + 6)))$  ... that may be translated into the shorthand of  $n \times \$3 + \$6$  (or  $3n + 6$ ). The text then states: “Realizing or becoming aware of these new forms of seeing the saving process is what objectification is about” (p. 57).

In the theory of knowledge objectification, teaching and learning are not two different forms of action (activity) but are conceived as two manifestations of the “shared work of the students and the teacher in a space of joint action, in the course of which students become familiar with historically and culturally constituted forms of thinking” (Radford & Roth, 2011, p. 237). In the present case, these forms exist in a specific, algebraic form of thinking about the regularities that occur when someone saves the same amount of money in equal time intervals. When students arrive in some form of stating what the structure “ $3n + 6$ ” abstractly expresses, then they can be said to have achieved “the objectification of the

formula” (p. 237). Objectification occurs at the moment that Mario becomes aware of the structure, because, “to remark the remarkable, that is, the mathematical structure, and to grasp it, to make it an object of consciousness is ... what objectification is about” (p. 240).

Objectification thus includes two levels of societal activity: its material, object-oriented practice and its reflection in the conscious mind.

This account shows that the teacher did not state the generalization so that this description affords understanding objectification as an individualistic process of externalization, of something previously unseen or indeed something in principle invisible (i.e., knowledge). When Mario translates the 21 ( $15 + 6$ ) chips in the goblet numbered “5” to  $3 + 3 + 3 + 3 + 3 + 6$  and then into  $5 \times 3 + 6$ , this could be characterized as the objectification of knowledge pre-existing its objectification (externalization) in graphic form. This is not to say that Roth and Radford see or conceive the situation in this way, but their description affords (even if they did not intend it) an individualistic take. However, individuals could never make the abstraction  $3n + 6$ . Instead, Marx shows that the formation of mind emerges “from the relations arising among the objects in the system, and from their place and differentiation among these relations” (Mamardašvili, 1986, p. 107). The ideal form objectified in the different amounts of chips in the six goblets “is a form of societal-human life-activity, which exists not in that life-activity,” not in Mario’s doing school mathematics, but “as a form of the external thing, which represents, reflects, another thing” (Ilyenkov, 2012, p. 184). At the same time, the relation (formula) is a form that lies outside the thing, outside the individual goblet or ensemble of goblets, thus existing “as a form of human life-activity ... ‘inside man’” (p. 184), that is, inside Mario and Jeanne. The original theory of knowledge objectification does not make thematic the fact that the ideal, for Mario, first comes to exist *as* the relation with Jeanne.

## A Spinozist-Marxian Account

In this subsection, I present a Spinozist-Marxian approach, based on my readings of the late Vygotsky, Ilyenkov, and Mikhailov. In this non-dualistic account, knowledge objectively exists *as the relation between people* rather than *in* things (objects, such as goblet contents, or signs, such as numbers and multiplication/addition signs that appear in Mario's table of value) that are semiotic means and serve as mediators.

The writings of Marx, Vygotsky, Ilyenkov, Mikhailov, and others following a Spinozist line of thought suggest a reading of the classroom fragment that differs substantially. In this alternative, the very essence of the ideal lies in its societal nature, it exists as a relation between two people (Vygotsky, 1989). Accordingly, Jeanne and Mario do not only talk *about* something, engaging in joint labor, but also produce the very conditions of this labor. That is, they produce the relation. Thus, the transcription of the 15 chips (\$15) in the third goblet as  $3 + 3 + 3 + 6$ , and the subsequently jointly produced—i.e., in irreducible social actions of the {query | response | evaluation} form—multiplicative, generalizing  $3 \times 3 + 6$  here is a relation between two people that exists precisely in the jointly produced transactional turn-taking sequence (Roth, 2016). Later, when Mario fills the table of value cells for week 5 (and 6)—i.e., when he writes “ $3 + 3 + 3 + 3 + 3 + 6$ ” and, below it,  $5 \times 3 + 6$ —what *was* joint labor and a relation with Jeanne now is ascribed to individual. To constitute a parallel to Marx's account of the emergence of a generalization, Mario not only has to convert individual cases, that is, the concrete contents of the fifth goblet  $\rightarrow 3 + 3 + 3 + 3 + 3 + 6 \rightarrow 5 \times 3 + 6$  and similar steps of all other cases but also has to write or state the *general* form, for example, as “take the number of weeks times 3 and add 6” or indeed “ $3n + 6$ .” The relations in these configurations are manifestations of the societal relations in which Mario and Jeanne are parts; they have synecdochical function in that they are part of the material relation that come to be taken for the relation as a whole. But the general form

actually hides its own origin, which leads many mathematics educators to conclude, in Piagetian fashion, that the student has abstracted a pattern from the physical configuration (with the aid of the teacher). This hiding occurs because the mind presents these, as quoted above, to be “outside the societal relations of things, as consciousness of the supersensible properties of these things” (Mamardašvili, 1986, p. 109). The consciousness of a mathematical form is the result of societal relations, initially appearing *as* the relation itself. Indeed, we ought to think of consciousness not as a thing but as a form of event, an experience of prior experience (e.g., Roth, 2020). Thus, the number patterns are *not* in the material configuration or in the corresponding additions or multiplications that appear in the table of values; and they are not in Mario’s actions. Instead, in and as the living relation with Jeanne, Mario finds himself confronted with the forms of societal consciousness “as a special, internally organized ‘actuality,’ as completely ‘external’ forms of its determination” (Ilyenkov, 2012, p. 168). Any individual, from the beginning of its life, has to “reckon far more carefully with demands and restrictions expressed and institutionalised by means of tradition than with the immediately perceptible appearance of external ‘things’ and situations” (p. 168).

In this episode, the origin of that first step towards a generalization, the ideal, is a relation with another person. This is so because the very talk that concerns the composition of the \$15 in the piggybank as  $\$3 + \$3 + \$3 + \$6$  also produces the relation. This can be seen from expanded transcriptions that also include *active* reception (e.g., Roth, 2016), which in fact implements that a word is a reality for both interlocutors:

200 J: [((says)) what are you going to write here?  
 201 M: [((hears)) what are you going to write here? ((says)) three? ]

Here, the sign (word) as an event is a reality for two in the same way that commodity is a reality for two in the commodity exchange relations. Thus, the ideal exists precisely in

the form of the associated double movement of exchange-value (what J says) into use-value (what M hears) into exchange-value (what M says). That is, the ideal exists only “in the reciprocating movement of the two opposing ‘metamorphoses’—forms of activity and forms of things in their dialectically contradictory mutual transformations” (Ilyenkov, 2012, p. 192). In the same spirit, Vygotsky (1987) notes that thought only becomes in speech; but as speech, a physical thing, unfolds, the environment is changed, and, therefore, in a reciprocal movement, experience [pereživanie] and thinking (Vygotsky, 2001). In the Spinozist-Marxian approach taken here, there is no mediator (Negri, 1991; Roth & Jornet, 2019), just a double movement from abstract (thinking) to concrete (speech) and from concrete (speech) to abstract (thinking). Thus, just as the commodity does not mediate between the parties, because at the instant of the exchange it is in the hands of both, so the sign is not mediating between Jeanne and Mario because the (physical) word is the same for both; and just as the commodity does not mediate between exchange-value and use-value because these are manifestations of the social relation, so the sign does not mediate between Jeanne and Mario, as some suggest (e.g., Arieviditch & Stetsenko, 2014), because any differences are characteristics of the social relation and mind (e.g. Bateson, 1979; Mead, 1938).

The “abstraction” that translates the goblet contents of 9 red and 6 blue chips into the form of an addition of four terms and then into multiplicative form  $3 \times 3 + 6$  is achieved by, and exists *as*, the joint labor and the social relation between two people (Mario, Jeanne). The relationship that comes to exist between these forms, thus, hides the societal relation between a teacher and a student. It is in and *as* that relation that the objects come to obtain their ideal form, in other words, that the ideal reproduces itself in the labor of Mario and Jeanne and then in the actions of Mario when he does on his own what immediately before he has done together with Jeanne. When he says, “Me, I now understand,” he manifests the appearance of awareness in (verbal) consciousness of a transformation that before existed as a relation. But

the ideal now appears in the conscious mind as an attribute of the material object (goblet and chips), the no less material inscriptions in the table cells, and the relations between them. Here, consciousness [Bewußtsein] is nothing other than conscious being [bewußtes Sein] (Marx & Engels, 1978), inherently an event rather than an entitative (abstract) thing.

Jeanne leaves Mario to his own after having made a gesture “that might be glossed as ‘You got it, so what was the problem?’” (Roth & Radford, 2011, p. 45). That is, learning mathematics here then means coming to act in ways so that others (here Jeanne) can see a person (here Mario) acting mathematically. Vygotsky’s notion *sociogenesis* denotes the fact that those ways of acting, the practices, first exists appears *as* collective social behavior before it appears *as* individual, no less social behavior. The patterns (rules) in these relations, life-activity, therefore initially are encountered “as an external object, as the forms and relationships between things produced and reproduced by human labor” (Ilyenkov, 2012, p. 187). In the presented case material, Mario and Jeanne’s joint labor, their relationship, produces the relationship between the goblet contents and the mathematical forms in the corresponding cells of rows 2 and 3 of the table of value. The first step to the generalization is made when they have produced the cell contents up to Week 4. However, when Jeanne leaves, and even when Mario states that he now understands, there is no evidence that the abstraction exists in the mind. For what he produces is of the same type as Form II in Marx’s analysis (Fig. 1 – FormII):

$$\begin{aligned} 18 \text{ (12 red, 6 blue) chips} &\rightarrow 3 + 3 + 3 + 3 + 3 + 6 && \rightarrow 5 \times 3 + 6; \\ 21 \text{ (15 red, 6 blue) chips} &\rightarrow 3 + 3 + 3 + 3 + 3 + 3 + 6 && \rightarrow 6 \times 3 + 6 \end{aligned}$$

The final step—e.g., the one that relates the different cells in the same row, and, equivalent, the one that relates the number of the week to the numbers of weeks and thus to the multiplier that appears in row 3 (i.e. as  $3n + 6$ )—does not exist here as relation. Mario does not finish the worksheet, which asks him also to calculate the contents of the piggybank in weeks 10, 15, and 25. His solution might have provided an answer to the status of the



practice in his behavior. Another student in his group—the one who appeared to know what to do and who had rapidly finished the first part of the task—did not calculate the amount for week 10 using the generalization but instead acted in a way that parallels Form B. Thérèse wrote out a column of ten 3's and one 6—i.e.,  $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 6$ —which she then added by combining pairs of 3's into 6, then pairs of 6's into 12's to arrive at a final addition  $24 + 12 = 36$ . That is, she produced a full account as if she were to determine the cell corresponding to Week 10 and then moved to produce an addition. She did not make the corresponding step to the cell in row 3—i.e.,  $10 \times 3 + 6$ ; nor did she employ the generalization, which would have taken her directly from week 10 to the multiplication and the final result.

### **Key Issues to be Rethought in the Theory of Knowledge Objectification**

In contrast to the original theory of knowledge objectification, which takes knowledge to be objectified in things (objects, signs, artifacts, linguistic devices), the present study—following the Spinozist-Marxian take of the late Vygotsky, Ilyenkov, and Mikhailov—suggests that anything particular to mathematics exists *as* relation between people. Other important issues in mathematics education research, such as sign mediation, also require complete reconsideration in the light of the Spinozist-Marxian take that constitutes “the dissolution of every idea of mediation” (Negri, 1991, p. 63). There are further issues, but constrained by space limitation, the two issues are elaborated in the following.

### **Knowledge Objectification as a Process of Externalization**

The name and description of the theory of knowledge objectification orient us to the process of externalization of knowledge, of the (subjective) ideal. In this articulation there exists ghosts of the parallelism of polar opposites: internal–external, ideal–material, or individual–collective. In contrast, the revised account articulates the ideal as a transcendent unity/identity that manifests itself in the continuous, mutual transformation of the sensible

and the supersensible. Ideal (supersensible) forms are societal in their genesis and essence. In developmental terms, there is a primacy of the social relation, which constitutes the first appearance of a specifically human higher (psychological) function (Vygotsky, 1989). In teaching situations, such as that between Jeanne and Mario, we find “*a renewed division [of behavior] into two of what had been fused in one* (cf. modern labor), the experimental unfolding of a higher process ... into a *small drama*” (p. 58, original emphasis). An example of this is provided in section 3.3, where I show how what will have been Mario’s higher psychological function (behavior), the mathematical abstraction, first was the real, living relation with Jeanne.

Societal relations such as that between Jeanne and Mario become higher psychological functions that are *not* biologically encoded. Thus, “man ... already posses a special plane of life-activity that is absent in the animal world—activity directed toward mastering specifically societal, purely social in origin and essence, forms of life-activity” (Ilyenkov, 2012, p. 185). It is therefore not that students have to somehow see the ideal (“meaning”) forms of mathematics in some configuration—e.g., the abstraction  $3n + 6$  in the goblet contents and their doubled representations in the cells of the table of values. Instead, the ideal (supersensible) form first will have been a real (sensible) relation with another person. It is not just any other person, but a person, like Jeanne, from whose behavior characteristic can be abstracted that fall into the category of the ideal. It is in the relational event with Mario, in their joint labor, that we observe a single behavior accomplished by two. That is, in contrast to the original theory of knowledge objectification, the Spinozist-Marxian account emphasizes the societal relation *as* the concrete locus and embodiment of the ideal (universal) of mathematics, a relation that manifests itself in the relation among mathematical objects. But having become the ideal of the object, the relation disappears thereby hiding the genetic origin of the ideal because the mind presents these as being in the things

(Mamardašvili, 1986). That is, when Mario and Jeanne are done, and when Mario fills the remaining table cells on his own, the ideal appears to exist in the goblet content and his mathematical representations. In the theory of knowledge objectification things are treated as bodies in which mathematical idealizations (“meanings”) are objectified. In other words, what appear to be the ideal aspects of the externalized objects students produce in the course of communication in mathematics classrooms are mere *projections* of (individual) consciousness onto the things.

Any theory in which knowledge is objectified into things has a problem: it is easily subsumed to an individual-based, agential account of mathematical learning—even though its formulation includes the essentially passive dimension of human life (e.g., Radford & Roth, 2011). Thus, for example, learning may be described as “students’ construction of a cognitive configurations network ... in line with the institutionally intended configurations” (Font et al., 2013, p. 113). Knowledge, in objectification (externalization), affects (brings about, causes) changes in the setting. Students are said to achieve knowledge objectification in the synchronous production of gestures, words, or actions; that is, the communicative act constitutes a “social objectification of abstract mathematical spatial-temporal relationships” (Arzarello, et al., 2015, p. 21) and constitutes *an externalization of thinking* through reasoning and explaining (Carlsen, 2010). A subjective conception, such as that of a right triangle, may be “gradually quantified to be incorporated within an axiomatic system ... thus becoming a non-arbitrary, omnitemporal object” (Moutsios-Rentzos et al., 2014, p. 31). Moreover, “level of knowledge objectification” is slated to be the same as “students’ understanding” (LaCroix, 2014), such that “higher levels of [the reactivation of] an objectification come to be used synonymously with student understanding (e.g., Moutsios-Rentzos et al., 2014). Such uses of the theory of knowledge objectification lead to metaphysical (mentalist) materialism that Marx explicitly rejected. Thus, “metaphysical

materialism, with its naïve understanding of the ‘ideal’ and the ‘material’ ... transforms into the purest subjective idealism” (Ilyenkov, 2012, p. 182). This happens when the investigator “confronts the so-called problem of ‘ideal, or abstract, objects’ in mathematical knowledge” (p. 182) while considering it only under its abstract and universal aspects. It is therefore “untenable” to describe the appearance of mind as a process where a “representation directing activity” comes to be “embodied in an object” and, thus, “gets its secondary ‘objectivized’ existence, which is accessible to sensory perception” (Leont’ev, 1978, p. 77) of others and the agent herself. In this case, mind would be a condition of activity. However, mind is “not the ‘cause’ of the manifestation of this new plane of relationships between the individual and the external world, but only the psychological form of its expression, in other words, its effect” (Ilyenkov, 2012, p. 187).

### **Sign Mediation**

The analysis in section 3.3 shows that words and phrases are events common to participants and do not stand *between* them. The resulting interaction of perspectives is taken to be “giving a binocular view in depth. This double view *is* the relationship” (Bateson, 1979, p. 133). In the revised theory offered here, every higher psychological process (e.g., mathematical thinking) first exists *as* physical relation between persons rather than being objectified in things and external to persons (e.g. signs and artifacts). But when the idea of an objectification into things goes, so does the concept of (sign) mediation.

Sign mediation is an important concept of mathematics education research (Bartolini Bussi & Mariotti, 2008). In the theory of knowledge objectification, mediation is required precisely because of the process of externalization. The “sign is placed *between* the subject and the object [another subject, brain, psychological task]” (Vygotsky, 1989, p. 61, original) and is external to both. In that approach, triangular representations are used to represent a sign  $X$  that takes an intermediary position between the stimulus  $A$  (or  $S$ ) and the response  $B$

(or *R*) (Vygotsky, 1997, p. 79) or between person (child, adult) and world (Cole & Engeström, 1993, p. 25). In other places, Vygotsky had used a similar triangular representation where the *sign* [znak] takes an intermediary position between the subject *S* and his/her *brain* [mozg], in which it, qua concrete object, brings about an “instrumental stimulus” (Vygotskij, 2005, p. 1025). The recognition of the shortcomings in the theory of (the early) Vygotsky, which fails to explain the gap between internal and external processes, led to the notion of “cultural mediation” (Arievitch & Stetsenko, 2014). But the late Vygotsky already has tried using the idea of “sociocultural mediation,” which he abandoned when realizing that the internal-external (body-mind) dilemma, “one of the most painful problems for psychology [still] lurks here” (Zavershneva, 2010, p. 73). Although the concept of sign mediation “was for a long time a basic problem for Vygotsky as well,” his turn to Spinozism ultimately enabled Vygotsky “to complete an extremely important step away from [it]” (Mikhailov, 2001, p. 15). We have to step away from the concept, because it is associated with and a remnant of Cartesianism, a “divine principle” and “special passage or bridge” between the world of thought and the material world (Ilyenkov, 1977, p. 220–221). The theoretical alternative offered here, as seen in section 3.3, does not require, and allows us to do without, the notion of mediation.

Consistent with the latter take, it has been shown that we can theorize learning without this concept (Roth, 2007; Roth & Jornet, 2019). The concept of mediation is necessary only when two things are external to each other (e.g., Mikhailov, 2004), such as body (extension) and mind (thought) in the Cartesian formulation. Indeed, if Spinoza is read through a Marxian lens, then mediation no longer is necessary because the unity/identity of extension and thought exists in *one and the same* substance, which only manifests itself in radically different, mutually exclusive ways. Thus, a “general element that mediates between the idea and the thing as a middle term,” “a common element between thinking and extension

*does not exist*” (Ilyenkov, 1982, p. 23, emphasis added). In a Spinozist-Marxian account, there is no mediation between the parts (manifestations) of a phenomenon or between the parts and the whole (e.g., Negri, 1991). Near the end of this life, Vygotsky, inspired by his reading of Spinoza, denounces all his preceding work stating that “the mistake of our analysis ... there is no unity, but parallelism and correspondence” (Vygotsky, in Zavershneva, 2016, p. 117). He therefore moved away from mediation so that “in his last, ‘Spinozan’ works the idea of semiotic mediation is supplanted by the concept of the intersubjective speech field” (Mikhailov, 2006, p. 35). The preceding analysis that presents words (phrases) as events *common to* rather than standing between interlocutors is to be understood precisely in the sense of the speech field. In his personal notes, Vygotsky envisions re-writing psychological theory through a Marxist reading of Spinoza (see note 1). In that reading “the question of the mediation of thought and being as substances no longer arises” (Siebert, 2014, p. 100). Ilyenkov (1977) takes the same stance, rejecting the “complicated magic of mediation” (p. 213), noting that the contradiction between external manifestations “does not disappear just because a whole chain of mediating links has been developed between them” (p. 329).

Whereas for Marx, “‘the concrete’ is by no means a synonym for the sensually given” (Ilyenkov, 1982, p. 33), the original theory of knowledge objectification, takes the “realm of the concrete and the particular”—the source for sensible, “visible stimuli”—to be something to be “overcome” (Radford et al., 2006, p. 393). In such conceptualizations that separate the purely ideal from the sensible, mediation is required because the purely ideal, thought in terms of something abstracted and abstract, is separated from the material, the object. The function of signs—language, mathematical expressions, gestures—is to bridge between the two, as well as between the subjective experience of the subject and mathematical ideas as something objective and common the (mathematical) community and its culture. In the alternative take offered here, the external exists only as part and gestalt of a whole, and which

has to be considered within the diversity (multiplicity) of the event from which the “internal” and “external” are but abstractions. Vygotsky, although “it is not unwarranted to consider [him] the originator of basic research in this area in Russian psychology” (Mikhailov, 2001, p. 10), did step back from his earlier position, classifying it as an instance of parallelism and correspondence. The parallelism existed, as he recognized, in the separate consideration of semantics (sense, “meaning”) and the external, objective field. Signs cannot join the two poles, the inner and outer, for “what has been disconnected at the initial stage cannot be connected at the final stages. Consequently, it is necessary to find some connection in the disconnected” (Leont’ev, 1994, p. 39). With respect to language, Vygotsky found the connection in the “*undifferentiated unity of external / internal speech*” (Vygotsky, in Zavershneva, 2016, p. 123, emphasis added). In this undifferentiated unity, signs no longer serve as mediators. They are integral parts of relations (wholes), and, as such, reflect the whole. The parts are therefore common to rather than standing between persons. Each part (interlocutor) already “contains” this whole, such that no mediation is required to bring interlocutors together.

### **Coda**

The theory of knowledge objectification is based on the idea of an externalization of knowledge into material form (objects, artifacts, linguistic devices, and signs). These materials serve as mediators between subject and object (including other subjects). Because of the problems with this approach, I offer here an alternative according to which anything specifically mathematical, as any other higher psychological function (e.g., abstract ideas, universal mathematics), is objectified in the form of the physical-social relation between people. Without the externalization of knowledge into things, the theory of knowledge objectification no longer requires the problematic concept of mediation. The advance of this approach over the earlier formulation of the theory lies in the consideration of the ideal in

consciousness as an attribute of societal systems. The ideal of mathematical forms thereby becomes an objective phenomenon rather than an expression of individual human subjectivity. Concerning learning and development in relation to ideal mathematical objects, mathematics is objectified *as* social intercourse. The mathematical ideal forms that appear in consciousness are nothing but synecdoches of relations between objects reflecting real relations between people.



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