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Savage's Independence Axiom and the Von Neumann-Morgenstern Substitution Axiom

Anders Borglin
Institute of Economics, University of Copenhagen

SUMMARY: *We consider the relation between a Savage-agent, an agent in the sense of Savage (1954), and a v. N-M-agent, an agent in the sense of von Neumann and Morgenstern (1944). It is shown that, if the Savage-agent is consistent with the v.N-M-agent, in the sense that his preferences among random variables agree with the v.N-M-agent's preferences among the corresponding distributions, then the v.N-M.agent satisfies the Substitution axiom on a subset if and only if the Savage-agent satisfies the Independence axiom on a corresponding subset and certain induced agents, called relatives of the Savage-agent, are also consistent with the same v.N-M.agent.*

1. Introduction

In the theory of choice under uncertainty there are two main approaches, one initiated by Savage (1954), and the other one initiated by von Neumann-Morgenstern (1944). We will discuss the relationship between an important axiom in Savage's theory, the Independence axiom, and an equally inportant axiom in von Neumann-Morgensterns' theory, the Substitution axiom.[1] We will show that a Savage-agent, an agent in the sense of Savage, satisfies the Independence axiom "to the same extent" that the corresponding v.N-M.-agent, an agent in the sense of von Neumann-Morgenstern, satisfies the Substitution axiom. It turns out that, in order to do so, we have to assume, not only that the Savage-agent himself, but also that certain derived agents, called relatives of the given Savage-agent, are consistent with the corresponding v.N-M-agent.

Savage's approach to the theory of choice under uncertainty, is closely related to the theory of consumer choice, under uncertainty in the theory of general equilibrium. In that theory, there are no external probabilities and an agent's preferences depict his attitudes towards the outcomes (prizes, consequences) as well as his, subjective, probability judgements.

1. This axiom is also referred to as "the Independence Axiom" or "the Cancellation Axiom". We prefer to use "the Substitution Axiom" to avoid misunderstandings.

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1. Introduction

In the theory of choice under uncertainty there are two main approaches, one initiated by Savage (1954), and the other one initiated by von Neumann-Morgenstern (1944). We will discuss the relationship between an important axiom in Savage's theory, the Independence axiom, and an equally inportant axiom in von Neumann-Morgensterns' theory, the Substitution axiom.[1] We will show that a Savage-agent, an agent in the sense of Savage, satisfies the Independence axiom "to the same extent" that the corresponding v.N-M.-agent, an agent in the sense of von Neumann-Morgenstern, satisfies the Substitution axiom. It turns out that, in order to do so, we have to assume, not only that the Savage-agent himself, but also that certain derived agents, called relatives of the given Savage-agent, are consistent with the corresponding v.N-M-agent.

Savage's approach to the theory of choice under uncertainty, is closely related to the theory of consumer choice, under uncertainty in the theory of general equilibrium. In that theory, there are no external probabilities and an agent's preferences depict his attitudes towards the outcomes (prizes, consequences) as well as his, subjective, probability judgements.

1. This axiom is also referred to as "the Independence Axiom" or "the Cancellation Axiom". We prefer to use "the Substitution Axiom" to avoid misunderstandings.

In the von Neumann-Morgenstern theory, which was developed along with their work in game theory, probability is "objective" and could well be thought of as originating from random devices, like roulette wheels, dealings of cards etc.

Uncertainty arises in Savage's theory because Nature chooses among "states of the World". Once the agent has chosen an act (random variable) Nature's choice determines the prize (consequence, outcome) the agent will actually receive. Thus, if S is the set of "states of the world", X is the set of prizes, the objects of choice are the functions from S to X, that is, random variables.[2] These will typically be denoted by ξ, η and the set of random variables is $\Xi(S, X)$. The agent is assumed to have a preference relation. $\gtrsim_S$ on the set of random variables, which is a total preorder.

The Independence axiom is concerned with two pairs. ξ, η and ξ', η', of such random variables, with the property that there is a partition, $\{A, S \setminus A\}$, of S such that

the restriction to A of ξ and ξ' are equal
the restriction to A of η and η' are equal
the restriction to S \ A of ξ and η are equal
the restriction to S \ A of ξ' and η' are equal

Then according to the Independence axiom: $\xi \gtrsim_S \eta$ if and only if $\xi' \gtrsim_S \eta'$. The rationale for the axiom is as follows. If Nature chooses s in $S \setminus A$ then ξ and η give the agent the same price. Hence the agent must base his evalution of ξ and η on the prizes he will receive if nature chooses s in A. But ξ and η are equal to ξ' an η' respectively, on A, and ξ' is equal to η' on $S \setminus A$. Thus the evaluation of ξ' and η' should be based on the prized received, if Nature chooses s in A. But then ξ' and η' should be ordered in the same way as ξ and η.

The von Neumann-Morgenstern theory takes the probability measures on X, to be denoted by $\Pi(X)$, as the objects of choice. A v.N-M agent is thus implicitly assumed to be disinterested in the choice of Nature: only the induced probabilities for different prizes matter. The preferences among the objects of choice is given by a total preorder, $\gtrsim$, on a all of $\Pi(X)$ or, possibly, some subset. Since the probability distributions form a convex set it makes sense to take convex combinations and the Substitution axiom is concerned with the interplay between preferences and convex combinations. Thus: let $\pi', \pi'', \pi \in \Pi(X)$, $0 \leq \alpha \leq 1$. Then[3]

$$\pi' \gtrsim \pi'' \text{ if and only if } \alpha\pi' + (1-\alpha)\,\pi \gtrsim \alpha\pi'' + (1-\alpha)\,\pi.$$

2. In this section no mention will be made of the necessary measurability conditions. A function defined on a measurable space with values in a measurable space, is a random variable also if there is no probability measure defined on the domain.
3. Often the Substitution Axiom is formulated with an "only if". Cf. Kreps (1988).

This axiom is often motivated by considering two compound lotteries. (Cf. Duffie (1988) or Kreps (1988). The first one gives the distribution π' as a prize, with probability α, and the distribution π, with probability $(1-\alpha)$. The second one gives π'', with probability α, and π with probability $(1-\alpha)$. Since, in both lotteries, π is received with probability $(1-\alpha)$, the preferences concerning the other prizes ought, according to the axiom, to determine the preferences among the two compound lotteries.

In Savage's theory the Independence axiom, together with the other axioms, determine a probability measure on S, reflecting the subjective probability judgements of the agent. Here we will assume that such a measure, P, is given. For a random variable, $\xi: S \to X$, the distribution of ξ is then given by $P(\xi^{-1}(\cdot))$, assigning to each event (measurable subset of X) the probability of the event.

With the probability measure P given, it makes sense to inquire if a Savage-agent, $[\Xi(S, X), \gtrsim_S]$ is consistent with a v.N-M-agent, $[\Pi, \gtrsim]$.

By this we mean: $\xi \gtrsim_S \eta$ if and only if $P(\xi^{-1}(\cdot)) \gtrsim P(\eta^{-1}(\cdot))$.

Assume that $[\Xi(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$. The idea that there should be a relation between Savage's Independent axiom and the v.N-M-Substitution axiom stems from the following reasoning.

Let $\pi', \pi'', \pi \in \Pi$ and let $0 < \alpha < 1$. Let $\hat{\pi} = \alpha\pi' + (1-\alpha)\pi$ and $\bar{\pi} = \alpha\pi'' + (1-\alpha)\pi$. Assume that there is a subset, A, of S, with $P(A) = \alpha$, and hence $P(S \setminus A) = 1 - \alpha$, such that we can find:

$$\xi_A' : A \to X \text{ such that } \frac{1}{P(A)} P(\xi_A'^{-1}(\cdot)) = \pi'$$

$$\xi_A'' : A \to X \text{ such that } \frac{1}{P(A)} P(\xi_A''^{-1}(\cdot)) = \pi''$$

$$\xi_{S\setminus A} : A \to X \text{ such that } \frac{1}{P(S \setminus A)} P(\xi_{S\setminus A}^{-1}(\cdot)) = \pi$$

Then $(\xi_A', \xi_{S\setminus A})$, defined by $(\xi_A', \xi_{S\setminus A})(s) = \xi_A'(s)$ for $s \in A$, and $(\xi_A', \xi_{S\setminus A})(s) = \xi_{S\setminus A}(s)$, for $s \in S \setminus A$, has the distribution $\hat{\pi}$ and $(\xi_A'', \xi_{S\setminus A})$, defined analogously, has the distribution $\bar{\pi}$. Since the Savage-agent is consistent with the v.N-M-agent we get:

$$\hat{\pi} \gtrsim \bar{\pi} \text{ if and only if } (\xi_A', \xi_{S\setminus A}) \gtrsim_S \xi_A'', \xi_{S\setminus A}) \qquad (*)$$

Assume that $\pi' \gtrsim \pi''$ and that the Substitution axiom is satisfied. By (*), $(\xi_A', \xi_{S\setminus A}) \gtrsim_S (\xi_A'', \xi_{S\setminus A})$ for any $\xi_{S\setminus A}$, and we get "independence on the subset A". But then we may define $\gtrsim_A$ on $\Xi\,(A, X)$ by: $\xi_A' \gtrsim_A \xi_A''$ if and only if $(\xi_A', \xi_{S\setminus A}) \gtrsim_S (\xi_A'', \xi_{S\setminus A})$ for some $\xi_{S\setminus A}$. (*) then also implies that $\xi_A' \gtrsim_A \xi_A''$. Hence the Substitution axiom implies that the preferences $\gtrsim_A$ induced by $\gtrsim_S$, on restrictions of functions in $\Xi\,(S, X)$ also agree with $\gtrsim$.

On the other hand, if the Independence axiom is satisfied then $\hat{\pi} \gtrsim \bar{\pi}$ and we would have the Substitution axiom satisfied if $\pi' \gtrsim \pi''$. This will be the case if $\xi_A' \gtrsim_A \xi_A''$ implies $\pi' \gtrsim \pi''$, that is, if the Savage-agent $[\Xi\,(A, X), \gtrsim_A]$ is also consistent with $[\Pi, \gtrsim]$.

We call a derived agent, like $[\Xi\,(A, X), \gtrsim_A]$, a relative of $[\Xi\,(S, X), \gtrsim_S]$. The considerations above suggests that if a Savage-agent is consistent with a v.N-M-agent then the v.N-M-agent satisfies the Substitution axiom if and only if the Savage-agent and his relatives are consistent with the v.N-M-agent. In the sequel we will show that a general version, using restricted versions of the Substitution axiom and the Independence axiom, of this is actually true.

Before proceeding we note the following difficulties:

(1) If Π is "large", it may be difficult to find a set of "states of the world", S, and a probability measure, P, such that the distributions of the random variables from S to X, generate all the distributions in Π[4]

(2) There seems to be nothing to ensure that $(\xi_A', \xi_{S\setminus A}) \gtrsim_S \xi_A'', \xi_{S\setminus A})$ for every $\xi_{S\setminus A}$: $S \to X$, implies $\eta' \gtrsim \pi''$. Thus even if $[\Xi\,(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$ it appears that we can not be sure that $[\Xi\,(A, X), \gtrsim_A]$ is consistent with $[\Pi, \gtrsim]$.

(3) The distributions of ξ_A' and/or ξ_A'' may not belong to the set of distributions generated by random variables $\xi \mathscr{E} \Xi\,(S, X)$. For S a finite set, the distributions generated by restrictions to subsets of S, are not related in any obvious way.

2. Notation

For easy reference we list the notation to be used in the sequel.

S: states of the world, $\mathscr{S}$: σ-algebra of subsets of $\mathscr{S}$.

$\mathscr{S} \cap A = \{B \mid B = C \cap A,\ C \in \mathscr{S}\}$. $A \in \mathscr{S}$, P: probability measure on $\mathscr{S}$.

X: set of prizes, $\mathscr{X}$: σ- algebra of subsets of X.

ξ, η : measureable functions from S to X.

ξ_A, η_A : measureable functions from A to X. $A \in \mathscr{S} \setminus \{\phi\}$.

$\Xi(A, X)$: the set of measureable functions defined on A with values in X, $A \in \mathscr{S} \setminus \{\phi\}$.

$\gtrsim_S$: preferences on $\Xi(S, X)$

4. Topological conditions on X, ensuring that there is an underlying set of "states of the world", S, and a (single) probability measure, P on S, generating the probability measures on X, as the distributions induced by random variables from S to X, can be found in Hildenbrand (1974) and the references given there.

$\gtrsim_A$: preferences on $\Xi(A, X), A \in \mathcal{S} \setminus \{\phi\}$, induced by $\gtrsim_S$
$\Pi(X)$: distributions on X, probability measures from $\mathcal{X}$ to $\mathbb{R}$
$\Pi(A, X)$: distributions on X induced by random variables from

$$(A, \mathcal{S} \cap A, \frac{1}{P(A)} P(\cdot)) \text{ to } (X, \mathcal{X}), A \in \mathcal{S}, P(A) > 0.$$

Π: a non-empty subset of $\Pi(X)$.
π_ξ, π_η distributions of $\xi, \eta \in \Xi(A, X)$.
π_{ξ_A}, π_{η_A} distributions of ξ_A, $\eta_A \in \Xi(A, X), A \in \mathcal{S}, P(A) > 0$ (induced by the probability measure

$$\frac{1}{P(A)} P(\cdot) : \mathcal{S} \cap A \to \mathbb{R})$$

Let $A, B \in \mathcal{S}, A, B$ disjoint and non-empty, and let $\xi_A: A \to X$, $\xi_B : B \to X$ be respectively $\mathcal{S} \cap A$-measureable and $\mathcal{S} \cap B$-measureable functions. Then (ξ_A, ξ_B) denotes the $\mathcal{S} \cap (A \cup B)$-measureable, function from $A \cup B$ to X defined to be $\xi_A(s)$, for $s \in A$, and $\xi_B(s)$, for $s \in B$. ξ_A, $A \in \mathcal{S} \setminus \{\phi\}$, will also be used to denote the restriction to A, of a function $\xi \in \Xi(S, X)$.

3. Savage Agents

DEFINITION: A *Savage-agent* is a pair. $[\Xi(A, X), \gtrsim_A]$, where $A \in \mathcal{S} \setminus \{\phi\}$, $\Xi(A, X)$ is the set of $\mathcal{S} \cap A$-measureable functions from A to X and $\gtrsim_A$ is a preference relation on $\Xi(A, X)$. For $A = S$, $\gtrsim_A$ is assumed to be a total preorder. □

The reason we want to allow for preferences, which are not total preorders is that, given the Savage-agent, $[\Xi(S, X), \gtrsim_S]$, we will define agents induced by the Savage-agent, who may have preferences, that are not total preorders. These induced agents will be the *relatives* of the Savage-agent.

Define a set $\mathcal{A}$, which is to be fixed in the sequel, by

$$\mathcal{A} = \{ (A, \xi_A', \xi_A'', \xi_{S \setminus A}) \mid A \in \mathcal{S} \setminus \{\phi, S\}, \xi_A', \xi_A'' \in \Xi(A, X), \text{ and } \xi_{S \setminus A} \in \Xi(S \setminus A)\}.$$

DEFINITION: Let $\mathcal{B}$ be a subset of $\mathcal{A}$. A Savage-agent $[\Xi(S, X), \gtrsim_S]$ *satisfies Savage-Independence on* $\mathcal{B}$ if, for $(A, \xi_A', \xi_A'', \xi_{S \setminus A}), (A, \xi_A', \xi_A'', \eta_{S \setminus A}) \in \mathcal{B}$:

$$(\xi_A', \xi_{S \setminus A}) \gtrsim_S (\xi_A'', \xi_{S \setminus A}) \text{ implies } (\xi_A', \eta_{S \setminus A}) \gtrsim_S (\xi_A'', \eta_{S \setminus A}). \ \square$$

Savage's Independence axiom corresponds to the case $\mathcal{B} = \mathcal{A}$.

DEFINITION: Let a Savage-agent $[\Xi(S, X), \succsim_S]$ satisfy Savage-Independence on a subset $\mathscr{B}$ of $\mathscr{A}$. For $A \in \mathscr{S} \setminus \{S, \phi\}$, define $\succsim_A$ on $\Xi(A, X)$ by:

$\xi_A' \succsim_A \xi_A''$ if and only if $(A, (\xi_A', \xi_A'', \xi_{S\setminus A}) \in \mathscr{B}$ for some $\xi_{S\setminus A} \in \Xi(A, X)$ and $(\xi_A', \xi_{S\setminus A}) \succsim_S (\xi_A'', \xi_{S\setminus A})$. □

The preference relation $\succsim_A$ depends on the set $\mathscr{B}$. If the set $\mathscr{B}$ is small the relation $\succsim_A$ may fail to be total or transitive. It may even be empty. If the Savage-agent satisfies Savage-Independence on all of $\mathscr{A}$, then $\succsim_A$ is a total preorder for $A \in \mathscr{S} / \{S, \phi\}$.

DEFINITION: Let a Savage-agent $[\Xi(S, X), \succsim_S]$ satisfy Savage-Independence on subset $\mathscr{B}$ on $\mathscr{A}$. *A $\mathscr{B}$-relative* of $[\Xi(S, X), \succsim_S]$ is the agent himself or, for $A \in \mathscr{S} \setminus \{S, \phi\}$, the Savage-agent $[\Xi(A, X), \succsim_A]$. □

The concepts above are defined without any reference to the probability measure on $(S, \mathscr{S})$.

4. v.N-M-Agents

DEFINITION: A *v.N-M-agent* is a pair. $[\Pi, \succsim]$ where Π is a subset of $\Pi(X)$ and $\succsim$ is a total preorder on Π. □

Define a set $\mathscr{C}$, which is to be fixed in the sequel, by

$$\mathscr{C} = \{ (\alpha, \pi', \pi'', \pi) / 0 \leq \alpha \leq 1, \pi', \pi'', \pi \in \Pi(X)\}$$

DEFINITION: Let $\mathscr{D}$ be a subset of $\mathscr{C}$. A v.N-M-agent, $[\Pi, \succsim]$, *satisfies the Substitution axiom on* $\mathscr{D}$, if, for $(\alpha, \pi', \pi'', \pi) \in \mathscr{D}$:

$\pi' \succsim \pi''$ if and only if $\alpha\pi' + (1-\alpha)\pi \succsim \alpha\pi'' + (1-\alpha)\pi$. □

The next definition captures the notion of a Savage-agent, or a relative, who ranks random variables taking into account only their distributions. To make sense we assume that the probability measures used are P, on $(S, \mathscr{S})$, and $\frac{1}{P(A)} P(\cdot)$, on $(A, \mathscr{S} \cap A)$, for $A \in \mathscr{S}$ such that $P(A) > 0$. A relative is *non-trivial* if $P(A) > 0$.

DEFINITION: A non-trivial $\mathscr{B}$-relative of a Savage-agent, $[\Xi(A, X), \succsim_A]$, of a Savage-agent, $[\Xi(S, X), \succsim_S]$, is *consistent* with a v.N-M-agent. $[\Pi, \succsim]$ if, for $\xi_A, \xi_A' \in \Xi(A, X)$,

$\xi_A \succsim_A \xi_A'$ implies $\pi_{\xi_A} \succsim \pi_{\xi_A'}$, and $\xi_A >_A \xi_A'$ implies $\pi_{\xi_A} > \pi_{\xi_A}'$. □

If a relative of a Savage-agent is consistent with a v.N-M-agent and $\gtrsim_A$ is a total preorder, then $\Pi(A, X) \subset \Pi$, but the inclusion might be proper. The relation $\gtrsim_A$ may be very incomplete, but if $\gtrsim_A$ is a total preorder, then the final line in the definition is equivalent to:

$$\xi_A \gtrsim_A \xi_A' \text{ if and only if } \pi_{\xi_A} \gtrsim \pi_{\xi_A'}$$

5. Result

Using the definitions introduced we can now state the following result, the proof of which is an immediate consequence of the definitions.

THEOREM. *Let the Savage-agent $[\Xi(S, X), \gtrsim_S]$ be consistent with the v.N-M-agent $[\Pi, \gtrsim]$. Let $\mathcal{B}$ be a subset of $\mathcal{A}$ and define $\mathcal{D} \subset \mathcal{C}$ by*

$$\mathcal{D} = \{ (\alpha, \pi', \pi'', \pi) \mid 0 < \alpha < 1 \textit{ and there exists } (A, \xi_A', \xi_A'' \xi_{S\setminus A}) \in \mathcal{B} \textit{ such that } \alpha = P(A), \pi' = \pi_{\xi_A'}, \pi'' = \pi_{\xi_A''}, \pi = \pi_{\xi_{S\setminus A}} \}$$

Then

The Savage-agent satisfies Savage-Independence on $\mathcal{B}$ and all the non-trivial $\mathcal{B}$-relatives are consistent with $[\Pi, \gtrsim]$

if and only if

$[\Pi, \gtrsim]$ satisfies the Substitution-axiom on $\mathcal{D}$.

PROOF: "*Only if*" If $\mathcal{B}$ is empty then $\mathcal{D}$ is empty and we are finished. Let $(\alpha, \pi', \pi'', \pi,) \in \mathcal{D}$. If $\alpha \in \{0,1\}$, then we are finished. Assume that $0 < \alpha < 1$. By the definition of $\mathcal{D}$ there is $(A, \xi_A', \xi_A'', \xi_{S\setminus A}) \in \mathcal{B}$ such that

$$P(A) = \alpha, \pi_{\xi_A'} = \pi', \pi_{\xi_A''} = \pi'' \text{ and } \pi_{\xi_{S\setminus A}} = \pi$$

Since $(A, \xi_A', \xi_A'' \xi_{S\setminus A}) \in \mathcal{B}$ and $[\Xi(S, X), \gtrsim_S]$ satisfies Savage-Independence on $\mathcal{B}$, we get $\xi_A' \gtrsim_A \xi_A''$ or $\xi_A'' >_A \xi_A'$.

We now have

$$\pi' \gtrsim \pi'' \overset{(1)}{\iff} \xi_A' \gtrsim_A \xi_A'' \overset{(2)}{\iff} (\xi_A', \xi_{S\setminus A}) \gtrsim_S (\xi_A'', \xi_{S\setminus A}) \overset{(3)}{\iff}$$

$$\iff \pi(\xi_A', \xi_{S\setminus A}) \gtrsim \pi(\xi_A'', \xi_{S\setminus A}) \overset{(4)}{\iff} \alpha\pi' + (1-\alpha)\pi \gtrsim \alpha\pi'' + (1-\alpha)\pi$$

where
(1) is true since $[\Xi(A, X), \gtrsim_A]$ is consistent with $[\Pi, \gtrsim]$
(2) is true since $[\Xi(S, X), \gtrsim_S]$ satisfies Savage-Independence on $\mathcal{B}$.
(3) is true since $[\Xi(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$
(4) by the definition of the distributions of $(\xi_A', \xi_{S\setminus A})$ and $(\xi_A'', \xi_{S\setminus A})$.

"*If*". Let $[\Pi, \gtrsim]$ satisfy the Substitution axiom on $\mathcal{D}$. Let $(A, \xi_A', \xi_A'', \xi_{S\setminus A}) \in \mathcal{B}$ and assume that $(\xi_A', \xi_{S\setminus A}) \gtrsim_S (\xi_A'', \xi_{S\setminus A})$. We want to show that $(A, \xi_A', \xi_A'', \eta_{S\setminus A}) \in \mathcal{B}$ implies $(\xi_A', \eta_{S\setminus A}) \gtrsim_S (\xi_A'', \eta_{S\setminus A})$.

If $P(A) \in \{0,1\}$ the conclusion follows from the assumption that $[\Xi(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$.

Assume $0 < P(A) < 1$. By the definition of $\mathcal{D}$, there are $(\alpha, \pi', \pi'', \pi)$, $(\alpha, \pi', \pi'', \hat{\pi}) \in \mathcal{D}$ such that

$$P(A) = \alpha,\ \pi_{\xi_A'} = \pi',\ \pi_{\xi_A''} = \pi'',\ \pi_{\xi_{S\setminus A}} = \pi \text{ and } \pi_{\eta_{S\setminus A}} = \hat{\pi}$$

We get

$$(\xi_A', \xi_{S\setminus A}) \gtrsim_S (\xi_A'', \xi_{S\setminus A}) \overset{(1)}{\Rightarrow} \pi(\xi_A', \xi_{S\setminus A}) \gtrsim \pi(\xi_A'', \xi_{S\setminus A}) \overset{(2)}{\Rightarrow}$$
$$\Rightarrow \alpha\pi' + (1-\alpha)\pi \gtrsim \alpha\pi'' + (1-\alpha)\pi \overset{(3)}{\Rightarrow} \pi' \gtrsim \pi'' \overset{(4)}{\Rightarrow}$$
$$\Rightarrow \alpha\pi' + (1-\alpha)\hat{\pi} \gtrsim \alpha\pi'' + (1-\alpha)\hat{\pi} \overset{(5)}{\Rightarrow}$$
$$\Rightarrow (\xi_A', \eta_{S\setminus A}) \gtrsim_S (\xi_A'', \eta_{S\setminus A}).$$

where
(1) is true since $[\Xi(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$.
(2) by the definition of the distributions of $(\xi_A', \xi_{S\setminus A})$ and $(\xi_A'', \xi_{S\setminus A})$
(3) and (4) since $[\Pi, \gtrsim]$ satisfies the Substitution axiom on $\mathcal{D}$
(5) is true since $[\Xi(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$.

Let $[\Xi(A, X), \gtrsim_A]$ be a $\mathcal{B}$-relative of $[\Xi(S, X), \gtrsim_S]$. Then, since $[\Xi(S, X), \gtrsim_S]$ is consistent with $[\Pi, \gtrsim]$ and this agent satisfies the Substituton axiom on $\mathcal{D}$.

$$\xi_A' \gtrsim_A \xi_A'' \Rightarrow \text{for some } (A, \xi_A' \xi_A'', \xi_{S\setminus A}) \in \mathcal{B}:$$
$$(\xi_A', \xi_{S\setminus A}) \gtrsim_S (\xi_A'', \xi_{S\setminus A}) \Rightarrow \pi(\xi_A', \xi_{S\setminus A}) \gtrsim \pi(\xi_A', \xi_{S\setminus A}) \Rightarrow$$
$$\Rightarrow \alpha\pi_{\xi_A'} + (1-\alpha)\pi_{\xi_{S\setminus A}} \gtrsim \pi_{\xi_A''} + (1-\alpha)\pi_{\xi_{S\setminus A}} \Rightarrow \pi_{\xi_A'} \gtrsim \pi_{\xi_A''}$$

and a similar argument shows that $\xi_A' \succ_A \xi_A''$ implies $\pi_{\xi_A'} \succ \pi_{\xi_A''}$.

This shows that the $\mathscr{B}$-relative $[\Xi(A, X), \succsim_A]$ is consistent with $[\Pi, \succsim]$.

6. Conclusions

We have shown that if a Savage-agent is consistent with a v.N-M-agent then the Savage-agent satisfies Savage-Independence and his non-trivial relatives are consistent with the same v.N-M-agent if and only if the v.N-M-agent satisfies the Substitution axiom.

If one considers the Independence axiom as intuitively more appealing than the Substitution axiom, the latter could thus be derived from the former. However, in doing so one is forced to make assumptions regarding, not only the Savage-agent, but also his relatives.

The analysis points to two problems: One is to give an example of a Savage-agent who is consistent with a v.N-M-agent, but who has a non-trivial relative who is not consistent with the same v.N-M-agent. A second one is to state conditions, under which the consistency of the Savage-agent himself implies that his relatives are also consistent.

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The analysis points to two problems: One is to give an example of a Savage-agent who is consistent with a v.N-M-agent, but who has a non-trivial relative who is not consistent with the same v.N-M-agent. A second one is to state conditions, under which the consistency of the Savage-agent himself implies that his relatives are also consistent.

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