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Existence of Equilibrium in CAPM: Further Results

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1. Introduction

In the two-period mean-variance capital asset pricing model (CAPM), a general equilibrium may fail to exist because of non-monotonicity of preferences and satiation as well as because of the possibility of short-selling. This paper provides some new existence results.

Nielsen (1989) exhibited sufficient conditions for existence of a general equilibrium in a model which allowed for satiation but was somewhat more general than the CAPM. These conditions were applied to the CAPM without a riskless asset in Nielsen (1990b). The conclusion was, essentially, that a general equilibrium exists if the investors agree on the expected returns to all assets, and if their risk aversion at a particular point satisfies a certain inequality. The inequality ensures that satiation occurs only outside the relevant range of portfolios or combinations of standard deviation and mean. Nielsen (1992) derived a number of conditions that ensure positivity of equilibrium prices.

Nielsen (1990a) derived complete necessary and sufficient conditions for existence of a general equilibrium in two special situations: one where utility is linear in mean and variance, and the other where the market portfolio minimizes the ratio of mean to standard deviation of return. Homogeneous beliefs were assumed.

For the case where there is a riskless asset, Nielsen (1990b) showed that there exists a general equilibrium if the investors either agree on all expected returns or have sufficiently large limiting risk aversion.

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Nielsen (1990a) derived complete necessary and sufficient conditions for existence of a general equilibrium in two special situations: one where utility is linear in mean and variance, and the other where the market portfolio minimizes the ratio of mean to standard deviation of return. Homogeneous beliefs were assumed.

For the case where there is a riskless asset, Nielsen (1990b) showed that there exists a general equilibrium if the investors either agree on all expected returns or have sufficiently large limiting risk aversion.

Using a quite different technique, Allingham (1991) has derived an alternative existence result for a general equilibrium with positive prices in the CAPM without a riskless asset and with homogeneous beliefs. It assumes two vector inequalities for each investor, involving the parameters of the model and the measure of risk aversion. Instead of the standard assumption that utility is concave or quasi-concave in mean and standard deviation, it relies on the more restrictive assumption that utility is concave in mean and variance.

The present paper derives an existence theorem which generalizes Allingham's result. It assumes a single numerical inequality for each investor, and it depenses with the assumption that utility is concave in mean and variance. The theorem does not in general guarantee positivity of prices.

However in the special case that corresponds to Allingham's result, positive prices are indeed guaranteed.

We show that the inequalities involved do not need to hold globally, but only in a relevant range of portfolios, specifically at all individually rational allocations. That is significantly weaker than the global requirement, because the set of individually rational allocations is compact (under the assumption of homogeneous beliefs, which is maintained in this part of the analysis).

Mainly as a technical device we introduce the concept of equality-constrained general equilibria. They are almost the same thing as general equilibria, except that the investors maximize utility subject to an equality budget constraint instead of an inequality. One nice thing about equality-constrained general equlibria is that they virtually always exist. The method of proof in our main general equilibrium existence result is to pick an equality-constrained general equilibrium and show that it is, in fact, a general equilibrium. There is, however, something to be said for equality-constrained general equilibrium concept in its own right. If the mean-variance model is interpreted as a model of a market for contracts that may turn out to be liabilities, then one may argue that the individual investor cannot freely dispose of a part of his endowment. He must choose to hold a portfolio which is equal in value to his endowment, and so he optimizes subject to an equality budget constraint.

We show the existence of equality-constrained general equilibrium in a model which is somewhat more general than the mean-variance model. In fact, we show something slightly stronger. There exists what we call a modified general equilibrium, where all satiable investors maximize subject to an equality constraint, while the insatiable investors maximize subject to an inequality constraint.

The paper is based in part on Nielsen (1985).

The plan of the paper is this. Section 3 proves the existence of modified general equilibria and equality-constrained general equilibria in a generalized model. Section 2 applies the results of Section 3 to the CAPM and then uses them to prove the main existence result for general equilibria.

2. Equilibrium in CAPM

This section explores the conditions for existence of equilibrium in the two-period mean-variance CAPM. First, we derive the conditions for existence of an equality-constrained general equilibrium. The results of that analysis are then used to prove our main existence result for general equilibrium.

There are n assets. A *portfolio* is represented by an n-vector x, where the j'th entry indicates the number of shares of the j'th asset included in the portfolio. Short-selling is allowed, so that the number x_j of shares of asset j held in a portfolio may be negative.

There are m investors $i = 1,..., m$. All investors have *choice set* $\mathbb{R}^n$, which means that there are no short sales constraints. Each summarizes his beliefs about the total (gross) returns per share of the assets in a mean vector $\bar{R}^i$ and a covariance matrix Ω_i. These may, in general, differ across investors. The mean return to a portfolio x according to i's beliefs is $x'\bar{R}^i$, and the standard deviation is $\sigma_i(x) = (x'\Omega_i x)^{1/2}$. At times, it will be assumed that the investors agree on the expected returns or the variances and covariances, in which case we write $\bar{R}^i = \bar{R}$, $\Omega_i = \Omega$, and $\sigma_i(x) = \sigma(x)$.

A portfolio x is *riskless* (as judged by investor i) if $\sigma_i(x) = 0$. It will be assumed that either there is no riskless asset or else the first asset is riskless while the remaining assets are risky. More specifically, in the first case (no riskless asset), it is assumed that the full covariance matrices Ω_i are positive definite. In the second case (where the first asset is riskless), the total return to the first (riskless) asset is assumed to be positive, and the covariance matrices of returns to the remaining assets are assumed to be positive definite. These assumptions imply that there are *no redundant assets or portfolios:* All portfolios $e \neq 0$ have $(\sigma_i(e), e'\bar{R}^i) \neq (0,0)$ for all i.

Investor i has utility function $W_i(v,\mu)$ which is a function of the variance and mean of total portfolio return. It is defined for $v \geq 0$ and for values of μ. The corresponding utility function for standard deviation and mean is $U_i(\sigma, \mu) = W_i(\sigma^2, \mu)$.

ASSUMPTION 1. *W_i is continuously differentiable (also at $v = 0$) with $W'_{iv} < 0$ and $W'_{i\mu} > 0$, and U_i is quasi-concave.*

The investor's utility function for portfolios is

$$V_i(x) = W_i(x'\Omega_i x, x'\bar{R}) = U_i(\sigma_i(x), x'\bar{R}^i).$$

It is continuous and strongly quasi-concave. By definition, the latter means that $V_i(tx + (1-t)y) > V_i(y)$ whenever x and y are portfolios with $V_i(x) > V_i(y)$.

As discussed in Nielsen (1987), V_i may not be monotone, and it may exhibit satiation if there is no riskless asset.

An *allocation* is an m-tuple $(x^i) = (x^1,..., x^m)$ consisting of a portfolio x_i for each i. The investors are endowed with an *initial portfolio allocation* (ω^i). The *market portfolio* $\omega = \sum_i \omega^i$ indicates the total number of shares available of each asset. An *attainable allocation* is an allocation (x^i) such that $\sum_i x^i = \omega$.

A *general equilibrium* is a pair $(p, (x^i))$, where $p \neq 0$ is a price system (an n-vector) and (x^i) is an attainable allocation, such that for each i, $p'x^i \leq p'\omega^i$, and if y^i is a portfolio with $p'y^i \leq$, then $V^i(y^i) \leq V_i(x^i)$. The initial allocation (ω^i) is exogenously given, while the asset price vector p and the equilibrium portfolio allocation (x^i) are endogenous.

The possibility of satiation and the unboundedness of the investors' choice sets may lead to non-existence of a general equilibrium in this model, cf. Nielsen (1990a). One possible way to deal with this problem is to consider equality-constrained general equilibria instead.

An *equality-constrained general equilibrium* is a price vector $p \neq 0$ and an attainable allocation (x^i) such that for each investor i, x^i is optimal for i if he has wealth $w^i = p'\omega^i$ and his budget constraint is an equality. In other words, $p'x^i = p'\omega^i$, and if y^i is a portfolio in X^i with $V_i(y^i) > V_i(x^i)$, then $p'y^i \neq p'x^i$.

The wealth of investor i is $w^i = p'\omega^i$, and now his budget constraint says $p'x^i = w^i$ rather than $p'x^i \leq w^i$. In ordinary consumption theory, one idea behind the inequality budget constraint $p'x^i \leq w^i$ is that there is a numeraire good which does not affect preferences. The consumer first exchanges his endowment for units of the numeraire and then uses some or all of the numeraire to purchase ordinary goods (or assets). If the numeraire has a positive value, then the budget constraint is an inequality when the value of the amount of the numeraire retained is not included in the value of the consumption bundle. A difficulty with this view is that there seems to be no reason why a numeraire good would have positive value in (and out of) equilibrium.

An alternative way to think of the organization of the market is to imagine that although there is no numeraire good with positive value, the consumer has the option not to spend all his wealth. He could conceivably throw away some of his endowment or some of his consumption bundle. Although he is not forced to spend all his wealth, he will in fact do so unless he reaches a satiation point.

It may be argued that the situation is different in a market for contracts without limited liability. The investor holds his wealth in the form of an initial portfolio of contracts. He cannot just throw some of the contracts away since they oblige him to make a payment in some situations. The only way he can get rid of some contracts is to exchange them in the market for other contacts with the same value. Consequently, his chosen portfolio has to have the same value as his initial portfolio, implying that the budget constraint is an equality.

A portfolio e is a *direction of improvement* for investor i at ω^i if $V_i(\omega^i + te) \geq V_i(\omega^i)$ for all $t \geq 0$.

CONDITION 1. *Positive semi-independence of directions of improvement: If for each i, e^i is a direction of improvement for investor i at ω^i, and if $\sum_i e^i = 0$, then $e^i = 0$ for all i.*

The following proposition is proved in the next section.

PROPOSITION 1. *Condition 1 implies the existence of an equality-constrained general equilibrium.*

In order to interpret Condition 1, let $\bar{\alpha}_i$ denote the limiting slope at large values of σ or μ of investor i's indifference curve in (σ, μ)-space through the standard deviation and mean $(\sigma_i(\omega^i), \omega^i \bar{R}^i)$ of his initial endowment portfolio. Also, for any portfolio $e \neq 0$, let $\alpha_i(e)$ denote the amount of mean per unit of standard deviation of return to e according to investor i's beliefs. A portfolio $e \neq 0$ is a direction of improvement for i at ω^i if and only if $\alpha_i(e) \geq \bar{\alpha}_i$. See Nielsen (1987, 1990b) for details.

If W_i is concave, as assumed by Allingham (1991), then $\bar{\alpha}_i = +\infty$, all i's indifference curves have this limiting slope, and there is necessarily satiation. No such assumption is imposed here.

The following proposition is a direct consequence of Proposition 1 above and Proposition 1 of Nielsen (1990b).

PROPOSITION 2. *Interpretation of Condition 1 and existence of an equality-constrained general equilibrium.*

1. Condition 1 is equivalent to the following: If for each i, e^i is a portfolio with $\alpha_i(e^i) \geq \bar{\alpha}_i$ (or $e^i = 0$) and if $\sum_i e^i = 0$, then $e^i \bar{R}^i = 0$ for all i.

2. If the investors agree on all expected returns, then Condition 1 holds and there exists an equality constrained general equilibrium.

3. If $\bar{\alpha}_i = \infty$ for all i, then Condition 1 holds and there exists an equality constrained general equilibrium.

One might attempt to construct an ordinary general equilibrium from an equality-constrained general equilibrium in the following way. Consider an augmented model with a numeraire good that does not affect preferences. An equality-constrained general equilibrium in the augmented model corresponds to an ordinary general equilibrium in the original model if either the price of the numeraire is positive or it is negative but negative holdings of the numeraire are allowed. However, in the latter case, Condition 1 is violated; and in general, there is no reason why the equality-constrained general equilibrium price of the numeraire would be different from zero. The main existence theorem below imposes a condition which ensures that an equality-constrained general equilibrium is a general equilibrium without introducing a numeraire good.

From now on, we shall maintain the assumption of homogeneous beliefs:

ASSUMPTION 2. *Homogeneous beliefs: $\bar{R}^i = \bar{R}$ and $\Omega_i = \Omega$ for all i.*

In order to compute and exploit an expression for the gradient of V_i, it is useful to introduce some notation. If x is a portfolio, let

$$\gamma_i(x) = -2W'_{iv}(x'\Omega x, x'\bar{R})/W_{i\mu}(x'\Omega x, x'\bar{R}).$$

Then $\gamma_i(x) > 0$. The gradient of V_i is

$$V'_i(x) = 2W'_{iv}\Omega x + W'_{i\mu}\bar{R}$$
$$= W'_{i\mu}[\bar{R} - \gamma_i(x)\Omega x].$$

If (x^i) is an allocation, set

$$\gamma = \left[\sum_i(\gamma_i(x^i))^{-1}\right]^{-1}.$$

Then $\gamma > 0$. Note that γ is a function of the allocation (x^i), even though it is suppressed in the notation.

An allocation is *individually rational* if it is attainable and Pareto dominates the initial allocation. Let A denote the set of individually rational allocations, *i.e.*,

$$A = \{(x^i): \textstyle\sum_i x^i = \sum_i \omega^i,\ x^i \in X^i,\ V_i(x^i) \geq V_i(\omega^i) \text{ for all } i\}$$

THEOREM 1. *Existence of general equilibrium in CAPM without a riskless asset. Assume that all assets are risky and that*

$$(\bar{R} - \gamma\Omega\omega)'(\Omega^{-1}\bar{R} - \gamma_i(x^i)\omega^i) > 0$$

for all i and all individually rational allocations. Then there exists a general equilibrium.

PROOF: Because the investors have homogeneous beliefs, there exists an equality-constrained general equilibrium $(p,(x^i))$ by Proposition 2. We shall show that it is actually a general equilibrium (except that we may need to replace p by $-p$). The first-order condition for utility maximization implies that there exist numbers λ_i with

$$\lambda_i p = \bar{R} - \gamma_i(x^i)\Omega x^i.$$

Divide by $\gamma^i(x^i)$, sum over i, and multiply by γ to get

$$\lambda p = \bar{R} - \gamma\Omega\omega,$$

where

$$\lambda = \gamma \sum_i \frac{\lambda_i}{\gamma_i(x^i)}.$$

Since $(-p, (x^i))$ is also an equality-constrained general equilibrium, it may be assumed without loss of generality that $\lambda \geq 0$. Now,

$$\begin{aligned} \lambda\lambda_i p'\Omega^{-1}p &= \lambda p'(\Omega^{-1}\bar{R} - \gamma_i(x^i)x^i) \\ &= \lambda p'(\Omega^{-1}\bar{R} - \gamma_i(x^i)\omega^i) \\ &= (\bar{R} - \gamma\Omega\omega)(\Omega^{-1}\bar{R} - \gamma_i(x^i)\omega^i) \\ &> 0. \end{aligned}$$

Hence $\lambda > 0$, and $\lambda_i > 0$ for all i. This implies that $V_i'(x^i)$ and p point in the same direction, so that i in fact maximizes utility at x^i subject to an inequality constraint. □

A special case of the condition in Theorem 1 obtains when

$$\bar{R} - \gamma\Omega\omega \gg 0 \tag{1}$$

at all individually rational allocations and

$$\Omega^{-1}\bar{R} - \gamma_i(x^i)\omega^i \gg 0 \tag{2}$$

for all i and at all individually rational allocations. Inequality 1 will hold if

$$\bar{R} - \gamma_i(x^i)\Omega\omega^i \gg 0 \tag{3}$$

for all i and at all individually rational allocations (divide by $\gamma_i(x^i)$, sum over i, and multiply by γ). The conditions in the main theorem of Allingham (1991) amount to imposing Inequalities 2 and 3 globally (rather than only at individually rational allocations). Inequality 1 ensures that all general equilibrium prices are strictly positive. Thus, Allingham's main result is a special case of Theorem 1.

3. Equality-Constrained General Equilibria

This section explores for existence of equality-constrained general equilibria in an abstract asset market model and proves Proposition 1 in this generalized setting. In fact, we shall prove the existence of a »modified general equilibrium«, which is an equality-constrained general equilibrium where the insatiable investors actually optimize subject to an inequality budget constraint.

Assets and portfolios are represented as before. There are still m investors $i = 1,\ldots, m$. Investor i has *choice set* X^i, which is a subset of $\mathbb{R}^n$.

ASSUMPTION 3. *The choice set X is closed and convex.*

An investor's preferences among portfolios is represented by his *utility function* V^i for portfolios, which is not necessarily derived from preferences over mean and variance of return.

ASSUMPTION 4. *The utility function* V^i *is continuous and strongly quasi-concave.*

By definition, the function V^i is *strongly quasi-concave* if $V^i(tx + (1-t)y) > V^i(y)$ whenever $0 < t < 1$ and x and y are portfolios in X^i with $V^i(x) > V^i(y)$.

Satiation portfolios are defined as before. Say that investor i is *satiable* if there is a satiation portfolio for i in X^i, and call him *insatiable* otherwise.

An *allocation* is an m-tuple $(x^i) = (x^1 \ldots x^m)$ consisting of a portfolio x^i in X^i for each investor. Each investor i has an *initial portfolio* ω^i.

ASSUMPTION 5. *For all i, the initial portfolio* ω^i *belongs to the interior of the choice set* X^i.

The initial portfolios constitute the *initial allocation* (ω^i). An allocation (x^i) is *attainable* if $\sum_i x^i = \sum_i \omega^i$.

The concepts of a general equilibrium and an equality-constrained general equilibrium are defined as in the previous section, except that the investors optimize only in their choice sets X^i.

A *modified general equilibrium* is a price vector $p \neq 0$ and an attainable allocation (x^i) such that for each satiable investor i, x^i is optimal for i subject to the equality budget constraint $p'x^i = p'\omega^i$, while for each insatiable investor i, x^i is optimal for i subject to the inequality budget constraint $p'x^i \leq p'\omega^i$. Formally, $p'x^i = p'\omega^i$, and if y^i is a portfolio in X^i with $V_i(y^i) > V_i(x^i)$, then $p'y^i \neq p'\omega^i$ if i is satiable, and $p'y^i > p'\omega^i$ if i is insatiable.

Obviously, a general equilibrium is a modified general equilibrium, and a modified general equilibrium is an equality-constrained general equilibrium.

If e is a direction of resession of X^i and x is a portfolio in X^i, say that e is a *direction of improvement* at x for the investor if $V_i(x + te) \geq V_i(x)$ for all $t \geq 0$.

Condition 1 and Proposition 1 make sense in the present model without adjustment. The following proposition is stronger and implies Proposition 1.

PROPOSITION 3. *Condition 1 implies the existence of a modified general equilibrium.*

The proof of proposition 3 requires the following Lemma.

LEMMA 1. *Assume that each* X^i *is bounded below. Then there exists a modified general equilibrium.*

PROOF: Modify the existence proof in Hart and Kuhn (1975) as follows. There is no production. Hart and Kuhn's Assumptions (α) (γ) and (ε) hold for all i, and Assumption (δ) holds for insatiable i. For satiable i, ,modify the definition of the demand correspondence D_i, in the truncated economy by using an equality as budget constraint instead of an inequality. In the proof of Hart and Kuhn's Theorem 5, note that if all investors are satiable, then case *(b)* implies case *(a)*. Hence, what needs to be shown is that case *(b)* is impossible if at least one investor is insatiable. In the case where $p'z < 0$, note that $p'x^i < p'\omega^i$ for some insatiable investor i, and proceed to a contradiction. In the case where $p'z = 0$, pick an insatiable investor i. The arguments of Hart and Kuhn show that x^i lies neither in the interior of T nor on the boundary of T, and hence case *(b)* is impossible. This establishes the existence of a modified general equilibrium in the truncated economy, and it is straightforward to show that it is also a modified general equilibrium in the original economy, *c.f.* Hart and Kuhn's Remark 1. □

The definition of the set A of individually rational allocations from the previous section can be directly applied here. Recall the concept of »Pareto attainable portfolios« from Nielsen (1989). A portfolio x is *Pareto attainable* for investor i if it is part of some individually rational allocation, *i.e.*, it belongs to the set

$$A^i = \{x : x = x^i \text{ for some } (x^1,\dots,x^m) \in A\},$$

where A is the set of individually rational allocations.

Proof of Proposition 3: (Modification of the existence proof in Nielsen (1989)). The set A is closed and convex. Condition 1 implies that A has no non-zero directions of recession, so A is compact, and hence each A^i is compact. For each i, let N^i be a compact, convex neighborhood of A^i, and set $Y^i = X^i \cap N^i$. Then Y^i is convex, compact and contains ω^i in its interior. Consider the truncated economy where the choise sets X^i are replaced by the sets Y^i. By Lemma 1, there exists a modified general equilibrium $(p,(x^i))$ for the truncated economy. Since (x^i) is an individually allocation in the truncated economy, it is also an individually rational allocation in the original economy, and so x^i belongs to A^i and to the interior of N^i for all i. Using strong quasi-concavity, it is easily seen that $(p,(x^i))$ is a modified general equilibrium for the original economy. □

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