CONVERGENCE AND STABILITY IN THE NEOCLASSICAL GROWTH MODEL

By HANS BREMS*

1. *Introduction*

In the neoclassical growth model entrepreneurs are assumed to produce a single good from labor and an immortal capital stock of that good. Thus investment is simply the act of setting aside part of output for installation as capital stock. Capital stock is the result of accumulated savings under an autonomously given propensity to save. Available labor force is growing autonomously, and there is always full employment.

Traditionally the one-good neoclassical growth model ignores the price of that good. But if one wants to examine the stability of neoclassical growth equilibrium, he will find price a convenient stabilizing variable. Let us, therefore, try to solve the neoclassical growth model for price and the proportionate rate of growth of price. In doing so we shall assume the money wage rate and its growth rate to be autonomously given.

2. *Notation*

Variables

C = consumption
g_C = proportionate rate of growth of consumption C
g_I = proportionate rate of growth of investment I
g_L = proportionate rate of growth of employment L
g_P = proportionate rate of growth of price P
g_S = proportionate rate of growth of capital stock S
g_X = proportionate rate of growth of output X
g_Y = proportionate rate of growth of national money income Y
I = investment
$\varkappa$ = physical marginal productivity of capital stock

* Professor, University of Illinois at Champaign-Urbana. The present article grew out of my discussion in *Nationaløkonomisk Tidsskrift* 1969, 107. bind, 1.-2. hefte, pp. 43-48 of Ølgaard's contribution to *Udviklingslinjer i makroøkonomisk teori*, eds. Niels Thygesen and P. Nørregaard Rasmussen, Copenhagen 1969.

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L = labor employed
P = price of good
R = revenue of entrepreneurs
S = capital stock of goods
W = wage bill
X = output
Y = national money income
Z = profits

Parameters

α, β = parameters of production function
c = propensity to consume
F = available labor force
g_F = proportionate rate of growth of available labor force F
g_M = proportionate rate of growth of multiplicative factor M
g_w = proportionate rate of growth of money wage rate w
M = multiplicative factor in production function
w = money wage rate

For time coordinate we shall use t. The flow variables C, I, and X are measured in physical units consumed, invested, or produced, respectively, per annum of the good produced. All flow variables refer to the instantaneous rate of that variable measured on a per annum basis.

3. *The Equations of the Model*

To the seven variable growth rates listed in Sec. 2 apply the definition of a proportionate rate of growth of a variable v as

(1) through (7) $$g_v = \frac{dv}{dt}\frac{1}{v}$$

Define investment as the derivative of capital stock with respect to time:

(8) $$I = \frac{dS}{dt}.$$

Let entrepreneurs apply the Cobb-Douglas production function

(9) $$X = ML^{\alpha}S^{\beta}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1$, and $M > 0$.

Let profit maximization under pure competition equalize real wage rate and physical marginal productivity of labor:

$$(10) \qquad \frac{w}{P} = \frac{\partial X}{\partial L} = \alpha \frac{X}{L}.$$

Define the physical marginal productivity of capital as

$$(11) \qquad \varkappa = \frac{\partial X}{\partial S} = \beta \frac{X}{S}.$$

Define revenue as

$$(12) \qquad R = PX.$$

It follows from (12) applied to (11) that entrepreneurs will be earning the profits

$$(13) \qquad Z = \beta R.$$

Under full employment available labor force must equal labor employed:

$$(14) \qquad F = L$$

Define the wage bill as money wage rate *times* employment

$$(15) \qquad W = wL.$$

Define national money income as the sum of wage bill and profits bill:

$$(16) \qquad Y = W + Z.$$

Let consumption be a fixed proportion of output:

$$(17) \qquad C = cX$$

where $0 < c < 1$.

Output equilibrium requires output to equal the sum of consumption and investment demand for it, or inventory would either accumulate or be depleted. Thus

$$(18) \qquad X = C + I.$$

Eqs. (1) through (18) contain the 18 variables listed in Sec. 2. Can we solve such a nonlinear system?

4. *The Convergence of Neoclassical Growth Equilibrium*

By the convergence of equilibrium we mean its tendency to settle down at stationary proportionate rates of growth[1].

Take the derivative of (9) and (14) with respect to time, divide the derivatives by (9) and (14), respectively, and find

$$g_X = g_M + \alpha g_F + \beta g_S. \tag{19}$$

From the consumption function (17), the equilibrium requirement (18), and the definitions (1) through (8) find

$$g_S = (1 - c)X/S. \tag{20}$$

Using (1) through (7), differentiate (20) with respect to time and find

$$\frac{dg_S}{dt} = g_S(g_X - g_S).$$

Insert (19) into this and express it

$$\frac{dg_S}{dt}\ \frac{1}{g_S} = \alpha(g_M/\alpha + g_F - g_S). \tag{21}$$

Now there are three possibilities: If $g_S > g_M/\alpha + g_F$ then

$$\frac{dg_S}{dt}\ \frac{1}{g_S} < 0. \tag{22}$$

If

$$g_S = g_M/\alpha + g_F \tag{23}$$

then

$$\frac{dg_S}{dt}\ \frac{1}{g_S} = 0. \tag{24}$$

And if $g_S < g_M/\alpha + g_F$ then

$$\frac{dg_S}{dt}\ \frac{1}{g_S} > 0. \tag{25}$$

We conclude that as long as g_S is greater than the value (23), (22) shows it to be falling. And as long as g_S is equal to that value, (24) shows it to re-

1. The convergence of equilibrium should not be confused with the stability of equilibrium: Convergence involves no violation of any equilibrium condition, indeed all equations (1) through (18) remain satisfied throughout Sec. 4. A stability test necessarily involves such violation, as we shall see in Secs. 6-11.

main so. And as long as g_S is less than that value, (25) shows it to be rising.

Now g_S cannot alternate around the value (24), for differential equations trace continuous time paths, so to get from a value on one side of (23) to a value on the other side, g_S would have to pass through the value (23). But then (24) would keep it there.

Finally g_S cannot converge toward anything else than the value (23), for if it did, then by letting enough time elapse we could make the left-hand side of (21) less than any arbitrarily assignable positive constant ε, however small, without the same being possible for the right-hand side.

We have trapped g_S, then: Either it equals $g_M/\alpha + g_F$ from the outset, or if it does not, it will converge toward that value.

Insert (23) into (19) and find

$$g_X = g_M/\alpha + g_F. \tag{26}$$

Once we have found g_S and g_X we may guess the rest:

(27) $g_C = g_X$ (28) $g_I = g_X$

(29) $g_P = g_w - g_M/\alpha$ (30) $g_Y = g_F + g_w$

To convince himself that those are indeed solutions, the reader should take derivatives with respect to time of *Eqs.* (8) through (10), (16) with (12) through (15) inserted into it, (17) and (18), use the definitions (1) through (7), insert solutions (23) and (26) through (30), and see that each equation is satisfied.

5. *Equilibrium Price*

At a particular time, the reader may wish to know, what is the equilibrium level of price P? Use (10) to express P, use (9) to express L/X and to find that

$$L/S = M^{-1/\alpha} (X/S)^{1/\alpha}$$

Use (1) through (8), (17) and (18) to see that $X/S = g_S/(1 - c)$, then take all this together and find the solution for equilibrium price to be

$$P = \alpha^{-1} M^{-1/\alpha} [g_S/(1 - c)]^{\beta/\alpha} w \tag{31}$$

where g_S stands for (23).

6. *Stability of Price Equilibrium*

By stability of equilibrium we mean its ability to restore itself after a disturbance. A disturbance violates the output equilibrium condition (18), so let us replace that condition by a definition of excess demand for output.

$$D = C + I - X. \tag{32}$$

Initially $D = 0$, but now let price P change by dP. The effect of the change upon excess demand is

$$\frac{dD}{dP} = \frac{\partial D}{\partial P}. \tag{33}$$

If we can show that (33) is negative, we have demonstrated that a price lower than equilibrium ($dP < 0$) will create positive excess demand, i.e. inventory depletion, and that a price higher than equilibrium ($dP > 0$) will create negative excess demand, i.e. inventory accumulation. We have, in other words, shown equilibrium to be stable.

7. *Two Asymmetries in Applying the Stability Test*

Showing that (33) is negative will have to be done twice, once for $dP < 0$ and once for $dP > 0$. The reasons for this inconvenience are two fundamental asymmetries built into the neoclassical growth model.

The first asymmetry is that capital stock can rise but not fall. Once installed, capital stock is immortal, hence no cost is involved in using it. Furthermore, according to (9) and (11) the physical marginal productivity of capital stock is always positive. Consequently, no already installed unit of capital stock should be left idle. The second asymmetry is that at full employment, employment can fall but not rise.

8. *The Desired Capital Intensity*

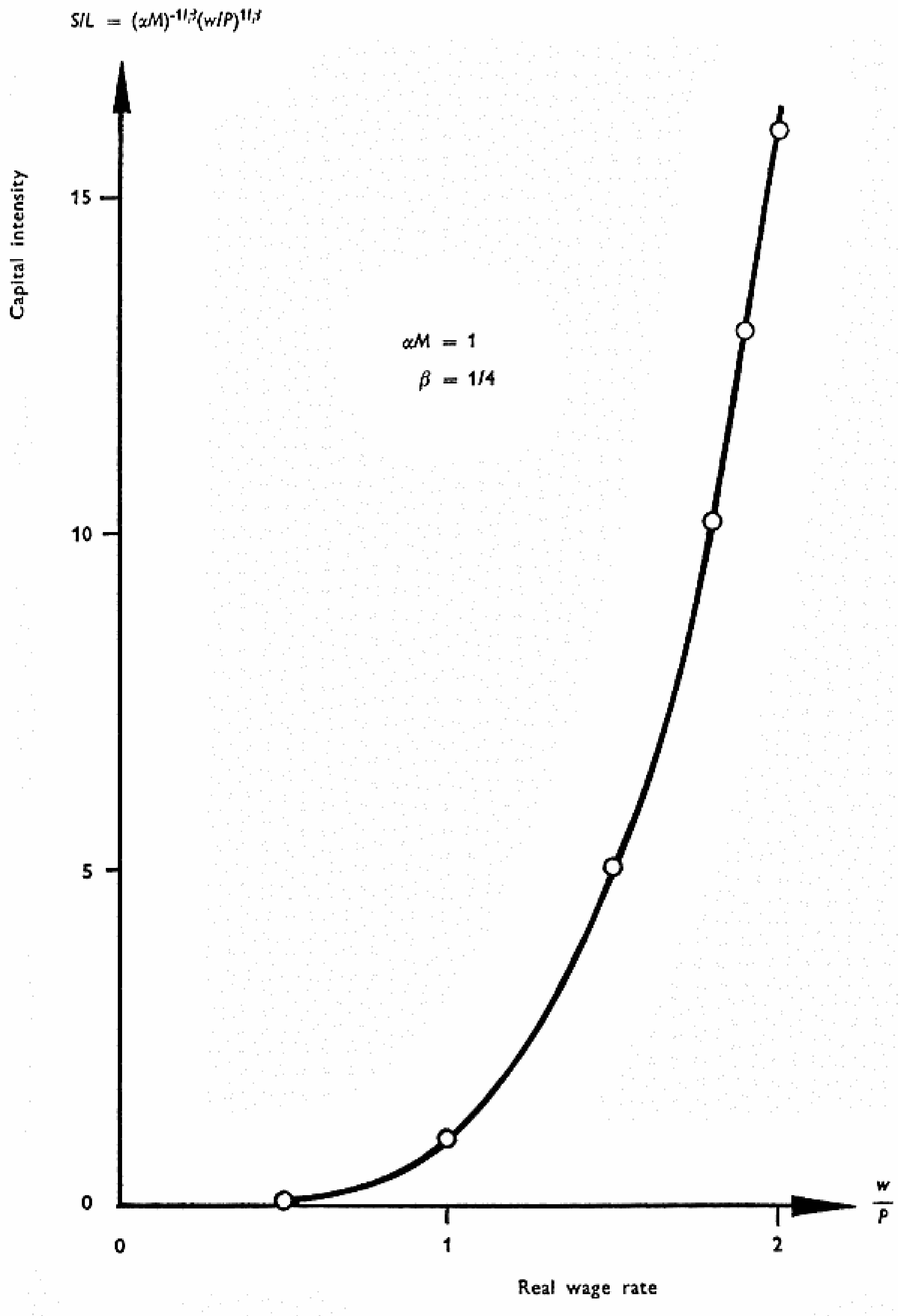
The clue to the stability of the neoclassical growth model is its flexible capital intensity. Insert the production function (9) into the profit-maximization consequence (10) and find how a profit-maximizing entrepreneur's desired capital intensity responds to the real wage rate facing him:

$$S/L = (\alpha M)^{-1/\beta}(w/P)^{1/\beta}. \tag{34}$$

The elasticity of desired capital intensity with respect to the real wage rate is, then, $1/\beta$. The high elasticity for realistic values of β is apparent from Figure 1, drawn for $\alpha M = 1$ and $\beta = 1/4$.

9. *Real Wage Rate Too High*

First let us examine the disequilibrium case $dP < 0$. A lower price means a higher real wage rate, which raises the desired capital intensity, i.e. induces capital deepening. Since the new higher real wage rate exceeds the physical marginal productivity of labor at full employment, entrepreneurs will accomplish their capital deepening by reducing employment L at constant


$S/L = (\alpha M)^{-1/\beta}(w/P)^{1/\beta}$
Capital intensity
15
10
5
0
$\alpha M = 1$
$\beta = 1/4$
0
1
2
$\frac{w}{P}$
Real wage rate

Figur 1

capital stock S. The instant effect of the higher real wage rate, then, is a negative excess demand in the labor market.

Of course, had the money wage rate w been a variable, such a negative excess demand would have reduced w until at the new lower price P the original real wage rate w/P had been restored. The inducement to capital deepening would then have been removed as suddenly as it arose, and the stability story would have ended. However, the money wage rate w was assumed to be a parameter, our *numéraire* if you like, so our story must go on. In (34) keep S constant, let L vary with P, and find

$$\frac{\partial L}{\partial P} = \frac{1}{\beta} \frac{L}{P}. \tag{35}$$

Use (10) and (35) to find the effect of the new lower price P upon output X, *via* employment L:

$$\frac{\partial X}{\partial P} = \frac{\partial X}{\partial L} \frac{\partial L}{\partial P} = \frac{\alpha}{\beta} \frac{X}{P}. \tag{36}$$

Capital stock S remains as in equilibrium and grows as it did in equilibrium, hence investment I must remain as in equilibrium:

$$\frac{\partial I}{\partial P} = 0. \tag{37}$$

When output falls, consumption falls according to (17):

$$\frac{\partial C}{\partial P} = \frac{\alpha}{\beta} \frac{C}{P}. \tag{38}$$

Finally insert the derivatives (36) through (38) into (33), use (17), and find

$$\frac{dD}{dP} = (c-1) \frac{\alpha}{\beta} \frac{X}{P} \tag{39}$$

which is negative, so equilibrium is stable for $dP < 0$.

10. *Real Wage Rate Too Low*

Is equilibrium also stable for $dP > 0$? In that case we collide head-on with the inability of employment to rise and the inability of capital stock to fall, as we shall now see.

A higher price means a lower real wage rate, which reduces the desired capital intensity, i.e. induces capital shallowing. Since the new lower real wage rate falls short of the physical marginal productivity of labor at full employment, entrepreneurs will try to accomplish their capital shallowing by raising employment L at constant capital stock S. The instant effect of the lower real wage rate is a positive excess demand in the labor market. Again, had the money wage rate been a variable, it would have adjusted itself and restored the original real wage rate. But again, the money wage rate is our *numéraire*, hence the inducement to capital shallowing persists.

But how can one raise an already full employment? Instead, the capital shallowing will have to be accomplished by reducing capital stock. In (34) keep L constant, let S vary with P, and find

$$\frac{\partial S}{\partial P} = -\frac{1}{\beta}\frac{S}{P}. \tag{40}$$

Once installed, however, capital stock is immortal, and there is no cost involved in using it. Furthermore, according to (9) and (11) the physical marginal productivity of capital is always positive. Consequently, no already installed unit of capital stock should be left idle. When neither employment L nor capital stock S changes, output X will not change either:

$$\frac{\partial X}{\partial P} = 0. \tag{41}$$

According to (17), consumption is unchanged if output is:

$$\frac{\partial C}{\partial P} = 0. \tag{42}$$

But what *does* change? How can the desired capital shallowing be accomplished? The answer is that while no already installed unit of capital stock should be left idle, entrepreneurs are still perfectly free to reduce future capital stock by reducing investment.

The partial derivative (40) measures the change in desired capital stock brought about by, say, one cent's worth of price change. Such a change is small compared with a whole year's investment and may therefore be accomplished by changing the latter — remember that investment, like all flow variables, was said to be an instantaneous rate measured on a per annum basis. Consequently we may also think of the partial derivative (40) as measuring the change of a year's investment brought about by one cent's worth of price change:

$$\frac{\partial I}{\partial P} = -\frac{1}{\beta}\frac{S}{P}. \tag{43}$$

Finally insert the derivatives (41) through (43) into (33) and find:

$$\frac{dD}{dP} = -\frac{1}{\beta}\,\frac{S}{P}. \tag{44}$$

(44) is negative, hence equilibrium is stable for $dP > 0$, too.

11. *Conclusion*

Our stability test relied on the sensitivity of the desired capital intensity to the real wage rate. In this reliance the test admitted two possibilities. First the possibility that in order to accomplish capital deepening, firms may employ less than the available labor force. Second the possibility that in order to accomplish capital shallowing, firms may invest less than what is being saved.

But our stability test clearly recognized two impossibilities, both relevant to firms trying to accomplish capital shallowing. First the impossibility of raising an already full employment. Second the impossibility of leaving idle any already installed unit of capital stock.

We conclude that at a parametric money wage rate w, a variable price of goods P cannot for long deviate from its equilibrium level. Pushed off its equilibrium path by a wrong price, the system will correct that price. Once the price is back at its equilibrium level it will induce a capital intensity permitting firms once again to employ the available labor force and to invest what is being saved.