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ON TRANSFINITE DIAMETERS IN $\mathbb{C}^d$ FOR GENERALIZED NOTIONS OF DEGREE

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Abstract

We give a general formula for the C-transfinite diameter $\delta_C(K)$ of a compact set $K \subset \mathbb{C}^2$ which is a product of univariate compacta where $C \subset (\mathbb{R}^+)^2$ is a convex body. Along the way we prove a Rumely type formula relating $\delta_C(K)$ and the C-Robin function $\rho_{V_{C,K}}$ of the C-extremal plurisubharmonic function $V_{C,K}$ for $C \subset (\mathbb{R}^+)^2$ a triangle $T_{a,b}$ with vertices $(0,0)$, $(b,0)$, $(0,a)$. Finally, we show how the definition of $\delta_C(K)$ can be extended to include many *nonconvex* bodies $C \subset \mathbb{R}^d$ for $d-$circled sets $K \subset \mathbb{C}^d$, and we prove an integral formula for $\delta_C(K)$ which we use to compute a formula for $\delta_C(\mathbb{B})$ where $\mathbb{B}$ is the Euclidean unit ball in $\mathbb{C}^2$.

1. Introduction

The multivariate transfinite diameter, introduced by Leja in 1957, is a well-known concept in pluripotential theory and approximation theory. It is defined as follows. Let $e_1(z), \dots, e_j(z), \dots$ be a listing of the monomials $e_j(z) = z^{\alpha(j)} = z_1^{\alpha_1(j)} \dots z_d^{\alpha_d(j)}$ in $\mathbb{C}^d$ where the multi-indices $\alpha(j)$ are arranged in the usual graded lexicographic (grlex) order, i.e., $\alpha \prec \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and the first left non-zero entry of $\alpha - \beta$ is negative. Let d_n be the dimension of the space $\mathscr{P}_n(\mathbb{C}^d)$ of polynomials of degree at most n. For $\zeta_1, \dots, \zeta_{d_n} \in \mathbb{C}^d$, let

$$\begin{aligned} \mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n}) &:= \det[e_i(\zeta_j)]_{i,j=1,\dots,d_n} \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \end{aligned} \tag{1.1}$$

and for a compact subset $K \subset \mathbb{C}^d$ let

$$V_n = V_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n} \in K} |\mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n})|.$$

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Then
$$\delta_d(K) := \limsup_{n\to\infty} V_n^{1/\ell_n} \tag{1.2}$$
is the transfinite diameter of K where $\ell_n := \sum_{j=1}^{d_n} \deg(e_j)$ (note that we could use any order $\prec$ on multiindices which satisfies $\alpha \prec \beta$ if $|\alpha| < |\beta|$).

The existence of the limit is not obvious. Zaharjuta [15] verified the existence of the limit by introducing directional Chebyshev constants $\tau(K, \theta)$ and proving
$$\delta_d(K) = \exp\left(\frac{1}{|\sigma|}\int_{\sigma^0} \log \tau(K,\theta)\, d|\sigma|(\theta)\right), \tag{1.3}$$
where
$$\sigma := \left\{(x_1, \dots, x_d) \in \mathbb{R}^d : 0 \le x_i \le 1, \sum_{j=1}^{d} x_i = 1\right\};$$
$$\sigma^0 = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d : 0 < x_i < 1, \sum_{j=1}^{d} x_i = 1\right\};$$
and $|\sigma|$ is the $(d-1)$-dimensional measure of σ. We mention that inequalities between the transfinite diameter and different notions of capacity in pluripotential theory are known and, in particular, a compact set has zero transfinite diameter if and only if it is pluripolar, see [9].

In the recently developed pluripotential theory associated to a convex body $C \subset (\mathbb{R}^+)^d$ (cf. [1]), notions of C-extremal plurisubharmonic (psh) function $V_{C,K}$ and C-transfinite diameter $\delta_C(K)$ of a compact set $K \subset \mathbb{C}^d$ generalize the corresponding notions in the standard setting. Their definitions are recalled in the next section, and we include a brief discussion of Ma'u's recent work [11] on C-transfinite diameter. We also recall the notion of C-Robin function $\rho_{V_{C,K}}$ associated to $V_{C,K}$ as defined in [8] for $C \subset (\mathbb{R}^+)^2$ a triangle $T_{a,b}$ with vertices $(0, 0)$, $(b, 0)$, $(0, a)$. The C-Robin function describes the precise asymptotic behavior of $V_{C,K}$; i.e., the behavior of $V_{C,K}(z)$ for $|z|$ large.

In classical pluripotential theory, which corresponds to the special case where C is the standard unit simplex $\Sigma \subset (\mathbb{R}^+)^d$, it is very difficult to find explicit formulas for extremal psh functions V_K (and hence their Robin functions) or to find precise values of transfinite diameters $\delta_d(K)$ for $K \subset \mathbb{C}^d$. In 1962, Schiffer and Siciak [14] proved that if $K = E_1 \times \cdots \times E_d \subset \mathbb{C}^d$ is a product of planar compact sets E_j, then $\delta_d(K) = \prod_{j=1}^{d} D(E_j)$ where $D(E_j)$ is the univariate transfinite diameter of E_j. Their proof used an intertwining of univariate Leja sequences for the sets E_j. Then in 1999, Bloom and Calvi [4] proved a more general result: if $K = E \times F$ where $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$, then
$$\delta_{n+m}(K) = \left(\delta_m(E)^m \cdot \delta_n(F)^n\right)^{1/(m+n)}. \tag{1.4}$$

Their proof used orthogonal polynomials associated to certain measures, called Bernstein-Markov measures, on K. In 2005, Calvi and Phung Van Manh [6] recovered the Bloom-Calvi result (1.4) by generalizing the Schiffer-Siciak method in introducing "block" Leja sequences for the component sets.

In [12], Rumely gave a remarkable formula relating transfinite diameter and Robin function in this classical setting. Using this formula, Blocki, Edigarian and Siciak [3] gave a very short proof of the general product formula (1.4). In §3, based on results in [1] and [8], we prove a Rumely type formula relating $\delta_C(K)$ and $\rho_{V_{C,K}}$ for $C = T_{a,b} \subset (\mathbb{R}^+)^2$ and we use this in section 4 to prove a formula for $\delta_C(K)$ when $K = E \times F$ is a product of univariate compacta. We modify the Bloom-Calvi proof using orthogonal polynomials in section 5 to give a product formula for the C-transfinite diameter when C is a general convex body in $(\mathbb{R}^+)^2$. In particular, for such C which are symmetric with respect to the line $y = x$, we obtain the striking result that the C-transfinite diameter of $K = E \times F$ is the same for these C. Finally, in section 6, we show how the C-transfinite diameter $\delta_C(K)$ can be extended to include many *nonconvex* bodies $C \subset \mathbb{R}^d$ for d-circled sets $K \subset \mathbb{C}^d$, and we exhibit an integral formula for $\delta_C(K)$. We use this to directly compute a formula for $\delta_{C_p}(\mathbb{B})$ for the Euclidean unit ball $\mathbb{B} \subset \mathbb{C}^2$ for a natural one-parameter family of symmetric $C = C_p$ (section 6) which explicitly yields different values for different p.

2. *C*-transfinite diameter and *C*-Robin function

Let C be a convex body in $(\mathbb{R}^+)^d$. We assume throughout that

$$\epsilon\Sigma \subset C \subset \delta\Sigma \quad \text{for some } \delta > \epsilon > 0, \tag{2.1}$$

where

$$\Sigma := \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \le x_i \le 1, \sum_{i=1}^{d} x_i \le 1\right\}$$

is the standard unit simplex. We set

$$\text{Poly}(nC) = \left\{p(z) = \sum_{J \in nC \cap \mathbb{N}^d} c_J z^J = \sum_{J \in nC \cap \mathbb{N}^d} c_J z_1^{j_1} \cdots z_d^{j_d},\ c_J \in \mathbb{C}\right\},$$

$n = 1, 2, \ldots$, and for a nonconstant polynomial p we define

$$\deg_C(p) = \min\{n \in \mathbb{N}\colon p \in \text{Poly}(nC)\}.$$

Next, we define the logarithmic indicator function

$$H_C(z) := \sup_{J \in C} \log |z^J| := \sup_{(j_1,\ldots,j_d) \in C} \log\bigl(|z_1|^{j_1} \cdots |z_d|^{j_d}\bigr)$$

in order to define

$$L_C = L_C(\mathbb{C}^d) := \{u \in PSH(\mathbb{C}^d) : u(z) \le H_C(z) + c_u\},$$

and

$$L_C^+ = L_C^+(\mathbb{C}^d) = \{u \in L_C(\mathbb{C}^d) : u(z) \ge H_C(z) + C_u\}$$

where c_u, C_u are constants depending on u and $PSH(\mathbb{C}^d)$ denotes the class of plurisubharmonic functions on $\mathbb{C}^d$. In particular,

$$\text{if } \ p \in \mathrm{Poly}(nC) \quad \text{then } \ u(z) := \frac{1}{\deg_C(p)} \log|p(z)| \in L_C.$$

These classes are generalizations of the classical Lelong classes $L := L_\Sigma$, $L^+ := L_\Sigma^+$ when $C = \Sigma$. The C-extremal function of a compact set $K \subset \mathbb{C}^d$ is defined as the uppersemicontinuous (usc) regularization

$$V_{C,K}^*(z) := \limsup_{\zeta \to z} V_{C,K}(\zeta) \text{ of } V_{C,K}(z) := \sup\{u(z) \colon u \in L_C, u \le 0 \text{ on } K\}.$$

If $C = \Sigma$, we simply write $V_K := V_{\Sigma,K}$. As in this classical setting, $V_{C,K}^* \equiv +\infty$ if and only if K is pluripolar; and when this is not the case, the complex Monge-Ampère measure $(dd^c V_{C,K}^*)^d$ is supported in K. We call K *regular* if $V_K = V_K^*$; i.e., V_K is continuous. This is equivalent to $V_{C,K}$ being continuous for any C. Our definition of dd^c is such that $(dd^c \log^+ \max[|z_1|, \dots, |z_d|])^d$ is a probability measure.

Letting d_n be the dimension of $\mathrm{Poly}(nC)$, we have

$$\mathrm{Poly}(nC) = \mathrm{span}\{e_1, \dots, e_{d_n}\}$$

where $\{e_j(z) := z^{\alpha(j)} = z_1^{\alpha_1(j)} \cdots z_d^{\alpha_d(j)}\}_{j=1,\dots,d_n}$ are the standard basis monomials in $\mathrm{Poly}(nC)$ in any order.

The definition of C-transfinite diameter $\delta_C(K)$ of a compact set $K \subset \mathbb{C}^d$ is similar to the one given in (1.1)–(1.2) except that we now use in (1.1) the basis of $\mathrm{Poly}(nC)$. Thus, for points $\zeta_1, \dots, \zeta_{d_n} \in \mathbb{C}^d$, we consider

$$\begin{aligned} \mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n}) &:= \det[e_i(\zeta_j)]_{i,j=1,\dots,d_n} \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \end{aligned}$$

and

$$V_n = V_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n} \in K} |\mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n})|. \tag{2.2}$$

Then

$$\delta_C(K) := \limsup_{n\to\infty} V_n^{1/\ell_n} \tag{2.3}$$

is the C-transfinite diameter of K where $\ell_n := \sum_{j=1}^{d_n} \deg(e_j)$.

In this generality, the existence of the limit was proved in [1]. Actually we will utilize results from [11] where a Zaharjuta-type proof of the existence of the limit in the general C-setting is given. There it is shown that

$$\delta_C(K) = \left(\exp\left(\frac{1}{\mathrm{vol}(C)}\int_{C^o} \log \tau_C(K,\theta)\, dm(\theta)\right)\right)^{1/A_C} \tag{2.4}$$

where the directional Chebyshev constants $\tau_C(K,\theta)$ and the integration in the formula are over the interior C^o of the entire d-dimensional convex body C and A_C is a positive constant depending only on C and d (defined in (2.9)). Note that in (1.3) the integration is over σ which is the extreme "face" of Σ; in fact, one can instead integrate over Σ^o (cf., §5.2 of [10]).

Apriori, in the definition of $\tau_C(K,\theta)$ the standard grlex ordering $\prec$ on $\mathbb{N}^d$ (i.e., on the monomials in $\mathbb{C}^d$) as defined in the introduction was used. This was required to obtain the submultiplicativity of the "monic" polynomial classes

$$M_k(\alpha) := \left\{p \in \mathrm{Poly}(kC) : p(z) = z^\alpha + \sum_{\beta\in kC\cap\mathbb{N}^d,\ \beta\prec\alpha} c_\beta z^\beta\right\} \tag{2.5}$$

for $\alpha \in kC \cap \mathbb{N}^d$; i.e., $M_{k_1}(\alpha_1)\cdot M_{k_2}(\alpha_2) \subset M_{k_1+k_2}(\alpha_1+\alpha_2)$. Defining Chebyshev constants

$$T_k(K,\alpha) := \inf\{\|p\|_K : p \in M_k(\alpha)\}^{1/k}, \tag{2.6}$$

for $\theta \in C^o$, this submultiplicativity allows one to verify existence of the limit

$$\tau_C(K,\theta) := \lim_{k\to\infty,\ \alpha/k\to\theta} T_k(K,\alpha) \tag{2.7}$$

as well as convexity of the function $\theta \to \ln \tau_C(K,\theta)$ on C^o.

In the proof that $\lim_{n\to\infty} V_n^{1/\ell_n}$ exists in [11], it is shown that

$$\lim_{n\to\infty} V_n^{1/nd_n} = \lim_{n\to\infty}\left(\prod_{j=1}^{d_n} T_n(K,\alpha(j))^n\right)^{1/nd_n}. \tag{2.8}$$

The asymptotic relation between nd_n and ℓ_n is that

$$\lim_{n\to\infty}\frac{\ell_n}{nd_n} = A_C := \frac{1}{\mathrm{vol}(C)}\cdot\iint_C (x_1+\cdots+x_d)\,dx_1\cdots dx_d =: M_C/\mathrm{vol}(C). \tag{2.9}$$

The following propositions will be useful in the sequel.

PROPOSITION 2.1. *For $t > 0$,*

$$\delta_{tC}(K) = \delta_C(K).$$

PROOF. We first observe that if $t \in \mathbb{N}$, since the limit in (2.3) exists,

$$\delta_C(K) = \lim_{n\to\infty} V_n^{1/\ell_n} = \lim_{n\to\infty} V_{tn}^{1/\ell_{tn}} = \delta_{tC}(K).$$

Similarly, if $t \in \mathbb{Q}$ we have $\delta_{tC}(K) = \delta_C(K)$. To verify the result for $t \in \mathbb{R}$, we proceed as follows. If $t_1 < t < t_2$, from the definitions of $M_k(\alpha),\ T_k(K,\alpha)$ and $\tau_C(K,\theta)$, we have the following:

(1) for $\theta \in t_1C^o$, $\tau_{t_1C}(K,\theta) \geq \tau_{tC}(K,\theta)$; and

(2) for $\theta \in tC^o$, $\tau_{tC}(K,\theta) \geq \tau_{t_2C}(K,\theta)$.

Taking a sequence $\{t_{1,j}\} \subset \mathbb{Q}$ with $t_{1,j} \uparrow t$ and a sequence $\{t_{2,j}\} \subset \mathbb{Q}$ with $t_{2,j} \downarrow t$, using the above inequalities together with (2.4) and (2.9),

$$\lim_{j\to\infty} \delta_{t_{1,j}C}(K) = \lim_{j\to\infty} \delta_{t_{2,j}C}(K) = \delta_{tC}(K).$$

We can use the Hausdorff metric on the family of our convex bodies C satisfying (2.1) considered as compact sets in $\mathbb{R}^d$. Using similar ideas from the previous proof, we verify the next result.

PROPOSITION 2.2. *Given $K \subset \mathbb{C}^d$, the mapping $C \to \delta_C(K)$ is continuous.*

PROOF. Taking a sequence $\{C_j\}$ of convex bodies satisfying (2.1) converging to C in the Hausdorff metric, we can find $\epsilon_j \to 0$ with

$$(1-\epsilon_j)C \subset C_j \subset (1+\epsilon_j)C, \quad j = 1,2,\ldots$$

As in the proof of Proposition 2.1, we have

(1) for $\theta \in (1-\epsilon_j)C^o$, $\tau_{(1-\epsilon_j)C}(K,\theta) \geq \tau_{C_j}(K,\theta)$; and

(2) for $\theta \in C_j^o$, $\tau_{C_j}(K,\theta) \geq \tau_{(1+\epsilon_j)C}(K,\theta)$.

Since $\epsilon_j \to 0$ implies $\mathrm{vol}(C_j) \to \mathrm{vol}(C)$ and $M_{C_j} \to M_C$, using the above inequalities together with (2.4) and (2.9), we find $\lim_{j\to\infty} \delta_{C_j}(K) = \delta_C(K)$.

For most of the subsequent sections, we work in $\mathbb{C}^2$. First, recall the definition of the Robin function $\rho_{\mathbf{u}}$ associated to $u \in L(\mathbb{C}^2)$:

$$\rho_{\mathbf{u}}(z) := \limsup_{|\lambda|\to\infty} [u(\lambda z) - \log|\lambda|].$$

For $z = (z_1, z_2) \neq (0,0)$ we define

$$\underline{\rho_{\mathbf{u}}}(z) := \limsup_{|\lambda|\to\infty}[u(\lambda z) - \log|\lambda z|] = \rho_{\mathbf{u}}(z) - \log|z|$$

so that $\underline{\rho_{\mathbf{u}}}(tz) = \underline{\rho_{\mathbf{u}}}(z)$ for $t \in \mathbb{C} \setminus \{0\}$. Here $|z|^2 = |z_1|^2 + |z_2|^2$. We can consider $\underline{\rho_{\mathbf{u}}}$ as a function on $\mathbb{P}^1 = \mathbb{P}^2 \setminus \mathbb{C}^2$ where to $p = (p_1, p_2)$ with $|p| = 1$ we associate the point where the complex line $\lambda \to \lambda p$ hits $\mathbb{P}^1$.

For a special class of convex bodies, there is a generalization of the notion of Robin function. Following [8], if we let C be the triangle $T_{a,b}$ with vertices $(0,0)$, $(b,0)$, $(0,a)$ where a, b are relatively prime positive integers, we have the following:

(1) $H_C(z_1, z_2) = \max[\log^+|z_1|^b, \log^+|z_2|^a]$ (note $H_C = 0$ on the closure of the unit polydisk $P^2 := \{(z_1, z_2) : |z_1|, |z_2| < 1\}$), and, indeed, $H_C = V_{C,P^2} = V_{C,T^2}$ where $T^2 := \{(z_1, z_2) : |z_1|, |z_2| = 1\}$;

(2) defining $\lambda \circ (z_1, z_2) := (\lambda^a z_1, \lambda^b z_2)$, we have

$$H_C(\lambda \circ (z_1, z_2)) = H_C(z_1, z_2) + ab\log|\lambda|$$

for $(z_1, z_2) \in \mathbb{C}^2 \setminus P^2$ and $|\lambda| \geq 1$.

Definition 2.3. Given $u \in L_C$, we define the C-Robin function of u:

$$\rho_u(z_1, z_2) := \limsup_{|\lambda|\to\infty}(u(\lambda \circ (z_1, z_2)) - ab\log|\lambda|)$$

for $(z_1, z_2) \in \mathbb{C}^2$.

Applying the transformation formula Theorem 4.1 of [8] in the case where $d = 2$; C is our triangle with vertices $(0,0)$, $(b,0)$, $(0,a)$; $C' = ab\Sigma$; and we consider the proper polynomial mapping

$$F(z_1, z_2) = (z_1^a, z_2^b),$$

we obtain

$$abV_{F^{-1}(K)}(z_1, z_2) = V_{C,K}(z_1^a, z_2^b)$$

so that

$$\begin{aligned} ab\rho_{\mathbf{V_{F^{-1}(K)}}}(z_1, z_2) &= \limsup_{|\lambda|\to\infty}[V_{C,K}(\lambda^a z_1^a, \lambda^b z_2^b) - ab\log|\lambda|] \\ &= \limsup_{|\lambda|\to\infty}[V_{C,K}(\lambda \circ (z_1^a, z_2^b)) - ab\log|\lambda|] \\ &= \rho_{V_{C,K}}(z_1^a, z_2^b) = \rho_{V_{C,K}}(F(z_1, z_2)). \end{aligned}$$

More generally, letting

$$\zeta = (\zeta_1, \zeta_2) = F(z) = F(z_1, z_2) = (z_1^a, z_2^b),$$

for $u \in L_C$, we have

$$\widetilde{u}(z) := u(F(z_1, z_2)) = u(\zeta) \in abL \tag{2.10}$$

and

$$\rho_u(\zeta) = \rho_u(F(z_1, z_2)) = ab\rho_{\widetilde{\mathbf{u}}/\mathbf{ab}}(z), \tag{2.11}$$

where $\rho_{\widetilde{\mathbf{u}}/\mathbf{ab}}$ is the standard Robin function of $\widetilde{u}/ab \in L$. Note that if $u \in L_C^+$ then $\widetilde{u} \in abL^+$. We apply these results in the next section.

3. C-Rumely formula for $C = T_{a,b}$

In this section, we let $C = T_{a,b}$. We begin with some integral formulas associated to functions in $L^+(\mathbb{C}^2)$. The integral formula Theorem 5.5 of [2] in this setting is the following.

THEOREM 3.1 (Bedford-Taylor). *Let $u, v, w \in L^+(\mathbb{C}^2)$. Then*

$$\int_{\mathbb{C}^2} (u\, dd^c v - v\, dd^c u) \wedge dd^c w = \int_{\mathbb{P}^1} (\underline{\rho_{\mathbf{u}}} - \underline{\rho_{\mathbf{v}}})(dd^c \underline{\rho_{\mathbf{w}}} + \omega)$$

where ω is the standard Kähler form on $\mathbb{P}^1$.

Next, following the arguments in [7], we get a symmetrized integral formula involving Robin functions $\rho_{\mathbf{u}}, \rho_{\mathbf{v}}$ for $u, v \in L^+(\mathbb{C}^2)$ and their projectivized versions $\underline{\rho_{\mathbf{u}}}, \underline{\rho_{\mathbf{v}}}$:

$$\begin{aligned}\int_{\mathbb{P}^1} (\underline{\rho_{\mathbf{u}}} - \underline{\rho_{\mathbf{v}}})\big[(dd^c \underline{\rho_{\mathbf{u}}} + \omega) + (dd^c \underline{\rho_{\mathbf{v}}} + \omega)\big] \\ = \int_{\mathbb{C}^1} \rho_{\mathbf{u}}(1,t)\, dd^c \rho_{\mathbf{u}}(1,t) + \rho_{\mathbf{u}}(0,1) \\ - \left[\int_{\mathbb{C}^1} \rho_{\mathbf{v}}(1,t)\, dd^c \rho_{\mathbf{v}}(1,t) + \rho_{\mathbf{v}}(0,1)\right].\end{aligned} \tag{3.1}$$

From (2.10), if $u, v, w \in L_C^+$,

$$ab \int_{\mathbb{C}^2} (u\, dd^c v - v\, dd^c u) \wedge dd^c w = \int_{\mathbb{C}^2} (\widetilde{u}\, dd^c \widetilde{v} - \widetilde{v}\, dd^c \widetilde{u}) \wedge dd^c \widetilde{w}.$$

We apply Theorem 3.1 to the right-hand-side, multiplying by factors of ab since $\widetilde{u}, \widetilde{v}, \widetilde{w} \in abL^+$, to obtain, with the aid of (2.11), the desired integral formula (cf., (6.3) in [8]):

$$\int_{\mathbb{C}^2} (u\, dd^c v - v\, dd^c u) \wedge dd^c w = (ab)^2 \int_{\mathbb{P}^1} (\underline{\rho_{\widetilde{\mathbf{u}}/\mathbf{ab}}} - \underline{\rho_{\widetilde{\mathbf{v}}/\mathbf{ab}}})(dd^c \underline{\rho_{\widetilde{\mathbf{w}}/\mathbf{ab}}} + \omega). \tag{3.2}$$

Next, for $u, v \in L_C^+$, we define the mutual energy

$$\mathscr{E}(u, v) := \int_{\mathbb{C}^2} (u - v)[(dd^c u)^2 + dd^c u \wedge dd^c v + (dd^c v)^2].$$

(cf., (3.1) in [1]). We connect this notion with C-transfinite diameter by recalling the following formula from [1].

THEOREM 3.2. *Let $K \subset \mathbb{C}^2$ be compact and nonpluripolar. Then*

$$\log \delta_C(K) = \frac{-1}{c} \mathscr{E}(V_{C,K}^*, H_C)$$

where $c = 3!M_C$ with $M_C := \iint_C (x + y)\, dx\, dy$.

REMARK 3.3. This formula is actually valid in $\mathbb{C}^d$ for $d > 1$ for any convex body $C \subset (\mathbb{R}^+)^d$ satisfying (2.1) with the appropriate definitions of $\mathscr{E}$ and c.

Our goal in this section is to rewrite $\mathscr{E}(V_{C,K}^*, H_C)$ using the integral formulas in order to get a formula relating $\delta_C(K)$ and $\rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}$ more in the spirit of Proposition 3.1 in [7]. This will be used in the next section to prove a formula for the C-transfinite diameter $\delta_C(K)$ of a product set $K = E \times F$.

PROPOSITION 3.4. *We have*

$$\mathscr{E}(V_{C,K}^*, H_C) = (ab)^2 \left[\int_{\mathbb{C}^1} \rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}(1, t) dd^c \rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}(1, t) - \rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}(0, 1) \right].$$

Hence from Theorem 3.2

$$-3!M_C \log \delta_C(K) = (ab)^2 \left[\int_{\mathbb{C}^1} \rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}(1, t) dd^c \rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}(1, t) - \rho_{\widetilde{\mathbf{V}}_{\mathbf{C,K}}/\mathbf{ab}}(0, 1) \right]. \tag{3.3}$$

PROOF. Applying the formula (3.2) with $w = u$ and $w = v$ and adding, we obtain

$$\int_{\mathbb{C}^2} (u\, dd^c v - v\, dd^c u) \wedge dd^c (u + v) = (ab)^2 \int_{\mathbb{P}^1} (\underline{\rho_{\widetilde{\mathbf{u}}/\mathbf{ab}}} - \underline{\rho_{\widetilde{\mathbf{v}}/\mathbf{ab}}})[(dd^c \underline{\rho_{\widetilde{\mathbf{u}}/\mathbf{ab}}} + \omega) + (dd^c \underline{\rho_{\widetilde{\mathbf{v}}/\mathbf{ab}}} + \omega)].$$

We claim from the definition of $\mathscr{E}(u, v)$, it follows that

$$\begin{aligned}\mathscr{E}(u, v) = &\int_{\mathbb{C}^2} [u(dd^c v)^2 - v(dd^c u)^2] \\ &+ (ab)^2 \int_{\mathbb{P}^1} (\underline{\rho_{\tilde{\mathbf{u}}/\mathbf{ab}}} - \underline{\rho_{\tilde{\mathbf{v}}/\mathbf{ab}}})[(dd^c \underline{\rho_{\tilde{\mathbf{u}}/\mathbf{ab}}} + \omega) + (dd^c \underline{\rho_{\tilde{\mathbf{v}}/\mathbf{ab}}} + \omega)]. \qquad (3.4)\end{aligned}$$

To see this, using the previous formula it suffices to show

$$\mathscr{E}(u, v) - \int_{\mathbb{C}^2} u(dd^c u)^2 + \int_{\mathbb{C}^2} v(dd^c v)^2 = \int_{\mathbb{C}^2} (u\, dd^c v - v\, dd^c u) \wedge dd^c(u + v).$$

In verifying this, all integrals are over $\mathbb{C}^2$. We write

$$\begin{aligned}\mathscr{E}(u, v) &= \int (u - v)[(dd^c u)^2 + dd^c v \wedge dd^c(u + v)] \\ &= \int u(dd^c u)^2 - \int v(dd^c u)^2 + \int (u - v) dd^c v \wedge dd^c(u + v) \\ &= \int u(dd^c u)^2 - \int v(dd^c v)^2 + \int (u - v) dd^c v \wedge dd^c(u + v) \\ &\qquad + \int v[(dd^c v)^2 - (dd^c u)^2].\end{aligned}$$

We finish this proof by working with the sum of the last two integrals:

$$\begin{aligned}&\int (u - v)\, dd^c v \wedge dd^c(u + v) + \int v[(dd^c v)^2 - (dd^c u)^2] \\ &\quad = \int (u - v)\, dd^c v \wedge dd^c(u + v) + \int v[dd^c(v - u) \wedge dd^c(u + v)] \\ &\quad = \int (u\, dd^c v - v\, dd^c u) \wedge dd^c(u + v)\end{aligned}$$

as desired.

Letting $u = V_{C,K_1}$ and $v = V_{C,K_2}$ in (3.4) where K_1, K_2 are regular compact sets in $\mathbb{C}^2$,

$$\begin{aligned}&\mathscr{E}(V_{C,K_1}, V_{C,K_2}) = (ab)^2 \\ &\quad \times \int_{\mathbb{P}^1} \Big(\underline{\rho_{\tilde{\mathbf{V}}_{\mathbf{C,K_1}}/\mathbf{ab}}} - \underline{\rho_{\tilde{\mathbf{V}}_{\mathbf{C,K_2}}/\mathbf{ab}}}\Big)\Big[\Big(dd^c \underline{\rho_{\tilde{\mathbf{V}}_{\mathbf{C,K_1}}/\mathbf{ab}}} + \omega\Big) + \Big(dd^c \underline{\rho_{\tilde{\mathbf{V}}_{\mathbf{C,K_2}}/\mathbf{ab}}} + \omega\Big)\Big].\end{aligned}$$

In particular, since $H_C = V_{C,P^2} = V_{C,T^2}$ where T^2 is the unit torus in $\mathbb{C}^2$,

$$\mathscr{E}(V_{C,K_1}, H_C) = (ab)^2 \times \int_{\mathbb{P}^1} \Big(\rho_{\underline{\widetilde{\mathbf{V}}_{\mathbf{C},\mathbf{K_1}}/\mathbf{ab}}} - \rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}\Big)\Big[\Big(dd^c \rho_{\underline{\widetilde{\mathbf{V}}_{\mathbf{C},\mathbf{K_1}}/\mathbf{ab}}} + \omega\Big) + \Big(dd^c \rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}} + \omega\Big)\Big].$$

The result will follow from (3.1) once we verify

$$\int_{\mathbb{C}^1} \rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(1,t)\, dd^c \rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(1,t) + \rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(0,1) = 0. \tag{3.5}$$

To verify (3.5), we begin by observing that since

$$H_C(z_1,z_2) = \max\big(\log^+|z_1|^b, \log^+|z_2|^a\big), \quad \widetilde{H}_C(z_1,z_2) := H_C(z_1^a, z_2^b),$$

for $(z_1,z_2) \in \mathbb{C}^2 \setminus (P^2)^o$,

$$\rho_{H_C}(z_1^a, z_2^b) = H_C(z_1^a, z_2^b) = ab\rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(z_1,z_2).$$

In particular,

$$\rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(0,1) = \frac{1}{ab} H_C(0,1) = 0$$

and

$$\rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(1,t) = \frac{1}{ab} H_C(1,t^b) = \frac{1}{ab}\max(0, ab\log|t|).$$

Thus $dd^c\rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(1,t)$ is supported on $|t| = 1$ where it is (normalized) arclength measure. On this set, we have $\rho_{\underline{\widetilde{\mathbf{H}}_{\mathbf{C}}/\mathbf{ab}}}(1,t) = 0$ and (3.5) follows.

4. Product formula for C a triangle

We first use (3.3) to prove a formula for the C-transfinite diameter of a product set when C is a triangle $T_{a,b}$ with vertices $(0,0)$, $(b,0)$, $(0,a)$. Then in the next section we give a (conceptually) simpler proof that is valid for general convex bodies.

THEOREM 4.1. *Let $K = E \times F$ where $E, F \subset \mathbb{C}$ are compact. Then for $C = T_{a,b}$,*

$$-\log \delta_C(K) = \frac{ab}{a+b} \cdot \left(\frac{-\log D(E)}{a} + \frac{-\log D(F)}{b}\right);$$

i.e.

$$\delta_C(K) = D(E)^{b/(a+b)} D(F)^{a/(a+b)},$$

where $D(E)$, $D(F)$ are the univariate transfinite diameters of E, F.

PROOF. We first assume a, b are positive integers and use Proposition 3.4. To this end, we compute $\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}$ for $K = E \times F$. We can assume E, F are regular compact sets in $\mathbb{C}$ and we let $\rho_E = -\log D(E)$ and $\rho_F = -\log D(F)$ be the Robin constants of these sets. From Proposition 2.4 of [5],

$$V_{C,K}(z_1, z_2) = \max(bg_E(z_1), ag_F(z_2))$$

where g_E, g_F are the Green functions for E, F. Note that

$$\rho_E = \lim_{|z_1|\to\infty}[g_E(z_1) - \log|z_1|] \quad \text{and} \quad \rho_F = \lim_{|z_2|\to\infty}[g_F(z_2) - \log|z_2|].$$

Thus from Definition 2.3

$$\begin{aligned}\rho_{V_{C,K}}(z_1, z_2) &= \limsup_{|\lambda|\to\infty}\big[\max[bg_E(\lambda^a z_1), ag_F(\lambda^b z_2)] - ab\log|\lambda|\big]\\ &= \max[b(\rho_E + \log|z_1|), a(\rho_F + \log|z_2|)]\end{aligned}$$

so that

$$\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(z_1, z_2) = \frac{1}{ab}\rho_{V_{C,K}}(z_1^a, z_2^b) = \max\left[\frac{1}{a}\rho_E + \log|z_1|, \frac{1}{b}\rho_F + \log|z_2|\right].$$

Hence

$$\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(1, t) = \max\left(\frac{1}{a}\rho_E, \frac{1}{b}\rho_F + \log|t|\right)$$

so that $dd^c\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(1, t)$ is normalized arclength measure on a circle where the value of the function $\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(1, t) = \frac{1}{a}\rho_E$. Finally, $\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(0, 1) = \frac{1}{b}\rho_F$ and the result when a, b are positive integers follows from Proposition 3.4 since

$$\begin{aligned}(ab)^2\left(\int_{\mathbb{C}^1}\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(1, t)\, dd^c\rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(1, t) - \rho_{\tilde{\mathbf{V}}_{\mathbf{C},\mathbf{K}}/\mathbf{ab}}(0, 1)\right)&\\ = (ab)^2\left(\frac{1}{a}\rho_E + \frac{1}{b}\rho_F\right)&\end{aligned}$$

and a calculation shows that $M_C = (ab/6)(a+b)$ so that $3!M_C = (ab)(a+b)$. If $a, b \in \mathbb{Q}$, the result follows from Proposition 2.1; finally, the general case when $a, b \in \mathbb{R}$ follows from Proposition 2.2.

5. Product formula for general C

In this section, we give an alternate proof of Theorem 4.1 which is applicable in a much more general setting. We assume that C is a convex body satisfying (2.1) which is a *lower set*: whenever $(j_1, j_2) \in nC \cap \mathbb{N}^2$ we have $(k_1, k_2) \in nC \cap \mathbb{N}^2$

for all $k_\ell \leq j_\ell,\ \ell = 1, 2$. For example, the triangles $T_{a,b}$ are lower sets. This proof is modeled on that of Bloom-Calvi in [4]. As in the previous section, we take $K = E \times F$ where E, F are compact sets in $\mathbb{C}$. Let μ_E, μ_F be *Bernstein-Markov measures* for E, F: recall ν is a Bernstein-Markov measure for E if for any $\epsilon > 0$, there exists a constant c_ϵ so that

$$\|p_n\|_E \leq c_\epsilon (1+\epsilon)^n \|p_n\|_{L^2(\nu)}, \quad n = 1, 2, \ldots$$

where p_n is any polynomial of degree n. If E, F are regular, one can take, e.g., μ_E and μ_F to be the distributional Laplacians of the Green functions g_E and g_F. Let $\mu := \mu_E \otimes \mu_F$. Let $\{p_j(z)\}_{j=0,1,2,\ldots}$ be monic orthogonal polynomials for $L^2(\mu_E)$ and let $\{q_k(z)\}_{k=0,1,2,\ldots}$ be monic orthogonal polynomials for $L^2(\mu_F)$; then $\{p_j(z)q_k(w)\}_{j,k=0,1,2,\ldots}$ are orthogonal in $L^2(\mu)$. Using the grlex ordering $\prec$ on $\mathbb{N}^2$ and the lower set property of C, it is easy to see that each $L^2(\mu)$-orthogonal polynomial $p_j(z)q_k(w)$ is in a class $M_\ell(\alpha)$ (recall (2.5)) where $\alpha = (j, k)$ and $\ell = \deg_C(z^j w^k)$. Here and below j, k are nonnegative integers.

We want to use (2.8): the asymptotics of V_n and $\prod_{j=1}^{d_n} T_n(K, \alpha(j))^n$ are the same; i.e., the limits of their nd_n-th roots coincide. If μ is a Bernstein-Markov measure on K, it follows readily that one can replace the sup-norm minimizers $T_k(K, \alpha)$ by $L^2(\mu)$-norm minimizers

$$\widetilde{T}_k(K, \alpha) := \inf\{\|p\|_{L^2(\mu)} : p \in M_k(\alpha)\}^{1/k}.$$

In our setting, for $\alpha = (j, k)$ the polynomial $p_j(z)q_k(w)$ is the minimizer and

$$\|p_j q_k\|_{L^2(\mu)} = \|p_j\|_{L^2(\mu_E)} \cdot \|q_k\|_{L^2(\mu_F)}.$$

Moreover, we know from the univariate theory that

$$\lim_{j\to\infty} \left\|p_j\right\|_{L^2(\mu_E)}^{1/j} = D(E) \quad \text{and} \quad \lim_{k\to\infty} \left\|q_k\right\|_{L^2(\mu_F)}^{1/k} = D(F).$$

For simplicity, we write

$$p_j := \|p_j\|_{L^2(\mu_E)} \quad \text{and} \quad q_k := \|q_k\|_{L^2(\mu_F)}.$$

In this notation, to utilize (2.8), we consider

$$\left(\prod_{(j,k)\in nC} p_j q_k\right)^{1/nd_n}.$$

We suppose that $(b, 0)$ and $(0, a)$ are extreme points of C and that the outer face F_C of C; i.e., the portion of the topological boundary of C outside of the coordinate axes, can be written both as a graph $\{(x, f(x)) : 0 \le x \le b\}$ and as a graph $\{(g(y), y) : 0 \le y \le a\}$.

THEOREM 5.1. *Let $K = E \times F$ where E, F are compact subsets of $\mathbb{C}$. Then*

$$\delta_C(K) = D(E)^{A/(A+B)} \cdot D(F)^{B/(A+B)}, \tag{5.1}$$

where $A = \int_0^b uf(u)\,du$ and $B = \int_0^a ug(u)\,du$. Hence for any convex body C with $A = B$ we obtain

$$\delta_C(K) = [D(E)D(F)]^{1/2}.$$

PROOF. The outer face F_{nC} of nC can be written as

$$\{(x, y) : y = f_n(x) := nf(x/n),\ 0 \le x \le nb\}$$
$$= \{(x, y) : x = g_n(y) := ng(y/n),\ 0 \le y \le na\}.$$

Then the product $\prod_{(j,k)\in nC} p_j q_k$; i.e., the product over the integer lattice points in nC, is asymptotically given by

$$p_1^{f_n(1)} p_2^{f_n(2)} \cdots p_{nb}^{f_n(nb)} q_1^{g_n(1)} q_2^{g_n(2)} \cdots q_{na}^{g_n(na)}$$
$$= \left(p_1^{f(1/n)} p_2^{f(2/n)} \cdots p_{nb}^{f(nb/n)} q_1^{g(1/n)} q_2^{g(2/n)} \cdots q_{na}^{g(na/n)}\right)^n.$$

For simplicity in the calculation, we concentrate on the product

$$p_1^{f(1/n)} p_2^{f(2/n)} \cdots p_{nb}^{f(nb/n)}$$

Then using the fact that $p_j \asymp D(E)^j$,

$$\begin{aligned} p_1^{f(1/n)} p_2^{f(2/n)} \cdots p_{nb}^{f(nb/n)} &\asymp D(E)^{f(1/n)+2f(2/n)+\cdots+nbf(nb/n)} \\ &= D(E)^{n^2\cdot(1/n)[1/nf(1/n)+2/nf(2/n)+\cdots+nb/nf(nb/n)]} \\ &\asymp D(E)^{n^2\int_0^b uf(u)du}. \end{aligned}$$

Similarly, since $q_k \asymp D(F)^k$, we have

$$q_1^{g(1/n)} q_2^{g(2/n)} \cdots q_{na}^{g(na/n)} \asymp D(F)^{n^2\int_0^a ug(u)du}.$$

Hence

$$\prod_{(j,k)\in nC} p_j q_k \asymp D(E)^{n^3\int_0^b uf(u)du} \cdot D(F)^{n^3\int_0^a ug(u)du}.$$

Using (2.8) and (2.9), since

$$nd_nA_C \asymp nn^2\,\mathrm{area}(C) \cdot \frac{M_C}{\mathrm{area}(C)} = n^3 M_C,$$

and

$$\begin{aligned} M_C &= \iint_C x\,dy\,dx + \iint_C y\,dx\,dy \\ &= \int_0^b \int_0^{f(x)} x\,dy\,dx + \int_0^a \int_0^{g(y)} y\,dx\,dy \\ &= \int_0^b x f(x)\,dx + \int_0^a y g(y)\,dy = A + B, \end{aligned}$$

(5.1) follows.

REMARK 5.2. Note that $A = B$ occurs whenever $a = b$ and $f = g$; i.e., the convex body is symmetric about the line $y = x$. As special cases of this, we can take

$$C = C_p := \{(x, y) : x, y \geq 0,\ x^p + y^p \leq 1\}, \quad 1 \leq p < \infty \tag{5.2}$$

as well as

$$C_\infty := \{(x, y) : 0 \leq x, y \leq 1\}.$$

REMARK 5.3. We can use Theorem 5.1 to verify our result in Theorem 4.1 for $C = T_{a,b}$. Here $y = f(x) = a(1 - x/b),\ 0 \leq x \leq b$ and $x = g(y) = b(1 - y/a),\ 0 \leq y \leq a$. Then

$$\int_0^b x f(x)\,dx = \frac{ab^2}{6}; \int_0^a y g(y)\,dy = \frac{ba^2}{6};$$

and

$$M_{T_{a,b}} = \int_0^b \int_0^{a(1-x/b)} (x + y)\,dy\,dx = (ab/6)(a + b).$$

Hence

$$\delta_{T_{a,b}}(K) = D(E)^{b/(a+b)} D(F)^{a/(a+b)}.$$

Moreover, the calculations in Theorem 5.1 – and the resulting formula – are valid (and much easier) in a special case where F_C cannot be written as a graph $\{(x, f(x)) : 0 \leq x \leq b\}$ (nor as a graph $\{(g(y), y) : 0 \leq y \leq a\}$); namely, the rectangle $C = R_{a,b}$ with vertices $(0, 0)$, $(b, 0)$, $(0, a)$ and (b, a). Here we take $y = f(x) = a,\ 0 \leq x \leq b$ and $x = g(y) = b,\ 0 \leq y \leq a$ and the calculations in the proof of Theorem 5.1 yield

$$\int_0^b x f(x)\,dx = \frac{ab^2}{2}; \quad \int_0^a y g(y)\,dy = \frac{ba^2}{2};$$

and

$$M_{R_{a,b}} = \int_0^b \int_0^a (x+y)\,dy\,dx = (ab/2)(a+b).$$

Hence, for this rectangle we recover the same product formula as for $T_{a,b}$:

$$\delta_{R_{a,b}}(K) = D(E)^{b/(a+b)} D(F)^{a/(a+b)}.$$

6. The case of d-circled sets

One might wonder, given Remark 5.2, whether we *always* have equality of $\delta_C(K)$ for all convex bodies C that are symmetric about the line $y = x$ (e.g., C_p for $1 \le p \le \infty$), i.e., for *any* compact set K, not just product sets. This is not the case as we will illustrate for

$$\mathbb{B} := \{(z_1, z_2) : |z_2|^2 + |z_2|^2 \le 1\},$$

the closed Euclidean unit ball in $\mathbb{C}^2$. This is an example of a 2-*circled* set. We say a set $E \subset \mathbb{C}^d$ is d-*circled* if

$$(z_1, \dots, z_d) \in E \text{ implies } (e^{i\beta_1} z_1, \dots, e^{i\beta_d} z_d) \in E, \text{ for all real } \beta_1, \dots, \beta_d.$$

For a compact, d-circled set K, it is easy to see from the Cauchy estimates that

$$\inf\{\|p\|_K : p \in M_k(\alpha)\} = \|z^\alpha\|_K$$

where recall

$$M_k(\alpha) := \left\{ p \in \mathrm{Poly}(kC) : p(z) = z^\alpha + \sum_{\beta \in kC \cap \mathbb{N}^d,\ \beta \prec \alpha} c_\beta z^\beta \right\}.$$

Then for any convex body C satisfying (2.1), and any $\theta = (\theta_1, \dots, \theta_d) \in C^o$, we have

$$\begin{aligned} \tau_C(K, \theta) &:= \lim_{k \to \infty,\ \alpha/k \to \theta} T_k(K, \alpha) \\ &= \lim_{k \to \infty,\ \alpha/k \to \theta} \|z^\alpha\|_K^{1/k} = \max_{z \in K} |z_1|^{\theta_1} \cdots |z_d|^{\theta_d}. \end{aligned}$$

Thus, for a given d-circled set K, if we can explicitly determine these values, we can use (2.4) to compute $\delta_C(K)$.

Indeed, an elementary calculation for $K = \mathbb{B} \subset \mathbb{C}^2$ shows that

$$\tau_C(\mathbb{B}, \theta) = \left(\frac{\theta_1}{\theta_1 + \theta_2}\right)^{\theta_1/2} \left(\frac{\theta_2}{\theta_1 + \theta_2}\right)^{\theta_2/2}. \tag{6.1}$$

It follows readily from (2.4) that $\delta_{C_1}(\mathbb{B}) = e^{-1/4}$.

We next show that the main result in [11], specifically, equation (2.4) in our §1, remains valid even for certain *nonconvex sets C* and all d-circled sets K. To this end, let $C \subset (\mathbb{R}^+)^d$ be the closure of an open, connected set satisfying (2.1). As examples, one can take C_p as in (5.2) for $0 < p < 1$. Here, the definition of

$$\mathrm{Poly}(nC) = \Big\{ p(z) = \sum_{J \in nC \cap \mathbb{N}^d} c_J z^J,\ c_J \in \mathbb{C} \Big\}, \quad n = 1, 2, \ldots$$

makes sense; and we have $\mathrm{Poly}(nC) = \mathrm{span}\{e_1, \ldots, e_{d_n}\}$ where $e_j(z) := z^{\alpha(j)}$ are the standard basis monomials in $\mathrm{Poly}(nC)$ and d_n is the dimension of $\mathrm{Poly}(nC)$. Using the same notation

$$\mathrm{VDM}(\zeta_1, \ldots, \zeta_{d_n}) := \det[e_i(\zeta_j)]_{i,j=1,\ldots,d_n}$$

as in the convex setting, for a compact and d-circled set $K \subset \mathbb{C}^d$, we have the same notions of maximal Vandermonde $V_n = V_n(K)$; C-transfinite diameter $\delta_C(K)$; "monic" polynomial classes $M_k(\alpha)$ and corresponding Chebyshev constants $T_k(K, \alpha)$; and directional Chebyshev constants $\tau_C(K, \theta)$ for $\theta \in C^o$ as in (2.2), (2.3), (2.5), (2.6) and (2.7).

Proposition 6.1. *For $C \in (\mathbb{R}^+)^d$ the closure of an open, connected set satisfying (2.1) and for any d-circled set $K \subset \mathbb{C}^d$, we have:*

(1) *for $\theta \in C^o$,*

$$\tau_C(K, \theta) := \lim_{k \to \infty,\ \alpha/k \to \theta} T_k(K, \alpha),$$

i.e., the limit exists; and

(2) $\lim_{n\to\infty} V_n^{1/nd_n}$ *exists and equals* $\lim_{n\to\infty}\big[\prod_{j=1}^{d_n} T_n(K, \alpha(j))^n\big]^{1/nd_n}$;

(3) $\delta_C(K) = \big(\exp\big(\frac{1}{\mathrm{vol}(C)} \int_{C^o} \log \tau_C(K, \theta)\, dm(\theta)\big)\big)^{1/A_C}$ *where A_C is a positive constant defined in (2.9).*

Proof. Because $\inf\{\|p\|_K : p \in M_k(\alpha)\} = \|z^\alpha\|_K$, all the arguments in Lemmas 4.4 and 4.5 of [11] work to show

$$\prod_{j=1}^{d_n} T_n(K, \alpha(j))^n \le V_n \le d_n! \cdot \prod_{j=1}^{d_n} T_n(K, \alpha(j))^n.$$

The only other ingredients needed to complete the rest of the proof are simply to observe that even though the polynomial classes $M_k(\alpha)$ are not submultiplicative, the *monomials* z^α themselves are; i.e., $z^\alpha z^\beta = z^{\alpha+\beta} \in M_k(\alpha + \beta)$. This is all that is needed to show (1); then the proof in [11] gives (2) and (3).

From the general formula

$$\tau_C(K,\theta) = \max_{z\in K} |z_1|^{\theta_1} \cdots |z_d|^{\theta_d}$$

for any d-circled set $K \subset \mathbb{C}^d$,

$$\delta_C(K) = \left(\exp\left(\frac{1}{\operatorname{vol}(C)} \int_{C^o} \log\left(\max_{z\in K} |z_1|^{\theta_1} \cdots |z_d|^{\theta_d}\right) dm(\theta)\right)\right)^{1/A_C}.$$

Using (6.1), for $K = \mathbb{B} \subset \mathbb{C}^2$,

$$\delta_C(\mathbb{B}) = \left(\exp\left[\frac{1}{\operatorname{area}(C)} \int_{C^o} \log\left[\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{\theta_1/2} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{\theta_2/2}\right] dm(\theta)\right]\right)^{1/A_C}. \tag{6.2}$$

Note that for $C = C_p$ this gives a formula for the C_p-transfinite diameter of the ball $\mathbb{B}$ in $\mathbb{C}^2$ valid for all $0 < p \leq \infty$. We return to this approach to computing C-transfinite diameter using directional Chebyshev constants in Proposition 6.5.

We can also use orthogonal polynomials as in §5 to compute $\delta_C(\mathbb{B})$ for general C as in Proposition 6.1; this we do next.

PROPOSITION 6.2. *For C as in Proposition 6.1, the C-transfinite diameter of the ball $\mathbb{B}$ in $\mathbb{C}^2$ is equal to*

$$\delta_C(\mathbb{B}) = \exp\left(\frac{1}{2I_4}\left(I_1 + I_2 - I_3 - \frac{\log 2\pi}{2} \operatorname{area}(C)\right)\right), \tag{6.3}$$

where

$$I_1 = \iiint_{C\times[0,1]} \log\Gamma(x+z)\, dx\, dy\, dz, \tag{6.4}$$

$$I_2 = \iiint_{C\times[0,1]} \log\Gamma(y+z)\, dx\, dy\, dz, \tag{6.5}$$

$$I_3 = \iiint_{C\times[0,1]} \log\Gamma(x+y+z)\, dx\, dy\, dz, \tag{6.6}$$

and

$$I_4 = M_C = \iint_C (x+y)\, dx\, dy. \tag{6.7}$$

PROOF. Let μ be normalized surface area on ∂B. Then the monomials $z^a w^b$, a, b nonnegative integers, are orthogonal and

$$\|z^a w^b\|^2_{L^2(\mu)} = \frac{a!b!}{(a+b+1)!},$$

see [13, Propositions 1.4.8 and 1.4.9]. Let us estimate

$$\mathcal{Q}_n = \log \prod_{(a,b)\in nC} \frac{a!b!}{(1+a+b)!}.$$

We have

$$\log \prod_{(a,b)\in nC} a! = \sum_{(a,b)\in nC} \log\Gamma(a+1).$$

Recall the multiplication formula for the Gamma function. For $\mathrm{Re}(z) > 0$, we have

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{(2nz-1)/2}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right).$$

Applying the formula with $z = (a+1)/n$, we get

$$\log \prod_{(a,b)\in nC} a! = \sum_{(a,b)\in nC}\sum_{k=1}^{n} \log\Gamma\left(\frac{a}{n}+\frac{k}{n}\right) + \sum_{(a,b)\in nC} \frac{1-n}{2}\log 2\pi + \sum_{(a,b)\in nC} \frac{2a-1}{2}\log n.$$

Recalling that d_n, the number of elements of $nC\cap\mathbb{N}^2$, is the dimension of $\mathrm{Poly}(nC)$,

$$\log \prod_{(a,b)\in nC} a! = \sum_{(a,b)\in nC}\sum_{k=1}^{n} \log\Gamma\left(\frac{a}{n}+\frac{k}{n}\right) - nd_n\frac{\log 2\pi}{2} + \frac{d_n}{2}\log 2\pi + n\log n \sum_{(a,b)\in nC}\left(\frac{a}{n}\right) - \frac{d_n}{2}\log n.$$

Interpreting the sums over the pairs (a, b) as Riemann sums, we get

$$\sum_{(a,b)\in nC}\sum_{k=1}^{n} \log\Gamma\left(\frac{a}{n}+\frac{k}{n}\right) = n^3 I_1 + \mathcal{O}(n^2), \qquad \sum_{(a,b)\in nC}\left(\frac{a}{n}\right) = n^2 I_2 + \mathcal{O}(n)$$

with I_1 and I_2 given in (6.4) and (6.5). Together with the estimate $d_n = n^2\,\text{area}(C) + \mathcal{O}(n)$, we get

$$\log \prod_{(a,b)\in nC} a! = (n^3 \log n) I_5 + n^3\left(I_1 - \frac{\log 2\pi}{2}\,\text{area}(C)\right) + \mathcal{O}(n^2\log n)$$

where $I_5 = \int_C x\,dx\,dy$. Similarly,

$$\log \prod_{(a,b)\in nC} b! = (n^3 \log n) I_6 + n^3\left(I_1 - \frac{\log 2\pi}{2}\,\text{area}(C)\right) + \mathcal{O}(n^2\log n)$$

where $I_6 = \int_C y\,dx\,dy$. Moreover,

$$\begin{aligned}
&\log \prod_{(a,b)\in nC} (1+a+b)! \\
&\quad = \sum_{(a,b)\in nC}\sum_{k=2}^{n+1} \log\Gamma\left(\frac{a+b+k}{n}\right) \\
&\qquad + \sum_{(a,b)\in nC} \frac{1-n}{2}\log 2\pi + \sum_{(a,b)\in nC} \frac{2a+2b+3}{2}\log n. \\
&\quad = \sum_{(a,b)\in nC}\sum_{k=2}^{n+1} \log\Gamma\left(\frac{a+b+k}{n}\right) + \left(\frac{1-n}{2}\right) d_n \log 2\pi \\
&\qquad + n\log n \sum_{(a,b)\in nC} \left(\frac{a+b}{n}\right) + \frac{3}{2} d_n \log n \\
&\quad = (n^3\log n) I_4 + n^3\left(I_3 - \frac{\log 2\pi}{2}\,\text{area}(C)\right) + \mathcal{O}(n^2\log n)
\end{aligned}$$

with I_3 and I_4 given in (6.6) and (6.7). Hence,

$$\mathcal{Q}_n = n^3\left(I_1 + I_2 - I_3 - \frac{\log 2\pi}{2}\,\text{area}(C)\right) + \mathcal{O}(n^2\log n).$$

Now,

$$\log \delta_C(\mathbb{B}) = \lim_{n\to\infty} \frac{\text{area}(C)}{2n d_n M_C}\mathcal{Q}_n$$

where $d_n = \dim \text{Poly}(nC) \simeq n^2\,\text{area}(C)$ and

$$M_C = \iint_C (x+y)\,dx\,dy = I_4.$$

Hence

$$\delta_C(\mathbb{B}) = \exp\left(\frac{1}{2I_4}\left(I_1 + I_2 - I_3 - \frac{\log 2\pi}{2}\,\mathrm{area}(C)\right)\right),$$

which is (6.3).

REMARK 6.3. We have

$$\log \delta_{C_p}(\mathbb{B}) = \lim_{n\to\infty} \frac{\mathrm{area}(C_p)}{2nd_n M_p}\mathcal{Q}_{n,p}$$

where $d_n = \dim \mathrm{Poly}(nC_p) \simeq n^2\,\mathrm{area}(C_p)$ and

$$M_{C_p} = \iint_{C_p} (x+y)\,dx\,dy = 2I_2.$$

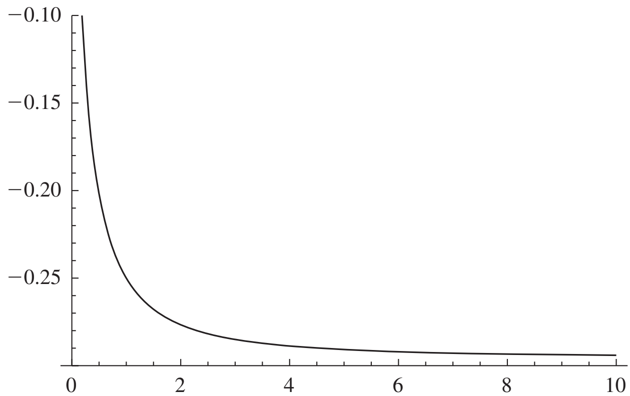

FIGURE 1. $\log \delta_{C_p}(\mathbb{B})$ as a function of p. When p goes to ∞, $\log \delta_{C_p}(\mathbb{B})$ tends to $(1 - 4\log 2)/6 \simeq -0.295$.

Hence

$$\delta_{C_p}(\mathbb{B}) = \exp\left(\frac{3p}{4B(1/p, 2/p)}\left(2I_1 - I_3 - \frac{\log 2\pi}{4p}B(1/p, 1/p)\right)\right),$$

where $B(x, y)$ denotes the Beta function.

REMARK 6.4. The integrals I_1, I_2 and I_3 can be simplified, eliminating the Gamma function from the integrand. We illustrate this with I_1. To this end, let

$$F(x) := \int_0^1 \log \Gamma(x+z)\,dz.$$

Then

$$F'(x) = \int_0^1 \frac{\Gamma'(x+z)}{\Gamma(x+z)}\,dz = \log\Gamma(x+1) - \log\Gamma(x) = \log x.$$

Thus $F(x) = x(\log x - 1) + c$ and it follows from the Raabe integral of the Gamma function that $c = \int_0^1 \log\Gamma(z)dz = \frac{1}{2}\log 2\pi$. Hence

$$I_1 = \iint_C F(x)\,dx\,dy = \iint_C \left(x(\log x - 1) + \frac{1}{2}\log 2\pi \right) dx\,dy.$$

In a similar fashion,

$$I_2 = \iint_C F(y)\,dx\,dy = \iint_C \left(y(\log y - 1) + \frac{1}{2}\log 2\pi \right) dx\,dy$$

and

$$I_3 = \iint_C F(x+y)\,dx\,dy = \iint_C \left((x+y)[\log(x+y)-1] + \frac{1}{2}\log 2\pi \right) dx\,dy.$$

Using these relations and (2.9), we recover (6.2).

Making use of (6.2), we get the following result for the case of $C = C_p$, $0 \le p < \infty$.

PROPOSITION 6.5. *We have*

$$\log\delta_{C_p}(\mathbb{B}) = \frac{3p}{2B(1/p, 2/p)} \left(\iint_{C_p} x\log x\,dx\,dy - \iint_{C_p} x\log(x+y)\,dx\,dy \right),$$

where $B(x, y)$ denotes the Beta function. In particular, for $p = 1$, $p = 2$ and $p = \infty$ we get

$$\delta_{C_1}(\mathbb{B}) = e^{-1/4}, \quad \delta_{C_2}(\mathbb{B}) = \sqrt{2}\,(\sqrt{2}-1)^{1/\sqrt{2}}, \quad \delta_{C_\infty}(\mathbb{B}) = 2^{-2/3}e^{1/6}.$$

PROOF. One has

$$\operatorname{area}(C_p) = \iint_{C_p} dx\,dy = \frac{1}{2p}B(1/p, 1/p), \qquad I_2 = \frac{1}{3p}B(1/p, 2/p),$$

and the given formula follows. The particular values for $p = 1, 2, \infty$ follow from computing the two integrals for these cases.

7. Final remarks

As noted in [8], the results given here in §§2 and 3 on C-Robin functions and C-transfinite diameter for triangles C in $\mathbb{R}^2$ with vertices $(0,0)$, $(b,0)$, $(0,a)$ where a, b are relatively prime positive integers should generalize to the case of a simplex C which is the convex hull of points

$$\{(0,\ldots,0), (a_1,0,\ldots,0), \ldots, (0,\ldots,0,a_d)\} \quad \text{in} \ \ (\mathbb{R}^+)^d$$

with $a_1, \ldots, a_d$ pairwise relatively prime (using the appropriate definition of the C-Robin function as defined in Remark 4.5 of [8]). For a product set $K = E_1 \times \cdots \times E_d$ in $\mathbb{C}^d$ where E_j are compact sets in $\mathbb{C}$, Proposition 2.4 of [5] gives that

$$V_{C,K}(z_1, \ldots, z_d) = \max[a_1 g_{E_1}(z_1), \ldots, a_d g_{E_d}(z_d)]$$

where g_{E_j} is the Green function for E_j. Hence a generalization of Theorem 4.1 will follow. However, unlike the standard ($C = \Sigma$) case, there is no known nor natural way to express a formula for the C-extremal function of a product set K when not all of the component sets are planar compacta; e.g., in the simplest such case, $K = E \times F \subset \mathbb{C}^3$ with $E \subset \mathbb{C}^2$ and $F \subset \mathbb{C}$. Nevertheless, it seems that the techniques adopted in §§5 and 6 using orthogonal polynomials and/or restricting to d-circled sets could likely be utilized to find more general product formulas for C-transfinite diameters.

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