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ON $\mathscr{M}$-NORMAL EMBEDDED SUBGROUPS AND THE STRUCTURE OF FINITE GROUPS

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Abstract

Let G be a group and H be a subgroup of G. H is said to be $\mathscr{M}$-normal supplemented in G if there exists a normal subgroup K of G such that $G = HK$ and $H_1K < G$ for every maximal subgroup H_1 of H. Furthermore, H is said to be $\mathscr{M}$-normal embedded in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K = 1$ or $H \cap K$ is $\mathscr{M}$-normal supplemented in G. In this paper, some new criteria for a group to be nilpotent and p-supersolvable for some prime p are obtained.

1. Introduction

In this paper, all groups considered are finite groups.

In finite group theory, it has been an active field to investigate the influence of the properties of subgroups on the structure of a finite group, and many good results have been obtained in this field. Among the lots of properties of subgroups, a good method is to use the "complemented subgroup". Let G be a group and H be a subgroup of G. Then H is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. As a class of basic subgroups of a finite group, primary subgroups attract the interest of many group workers, and many classical results are obtained by applying the property of complemented subgroups to primary subgroups. In [6], Hall proved that a group G is solvable if and only if its every Sylow subgroup is complemented in it. Arad and Ward [1] proved that G is solvable if and only if its every Sylow 2-subgroup and Sylow 3-subgroup is complemented in it. A. Ballester-Bolinches and Guo in [2] proved that G is supersolvable with elementary abelian Sylow subgroup if and only if its every minimal subgroup is complemented in it.

In [7], Miao gave the definition of $\mathscr{M}$-supplemented subgroups of a group G: a subgroup H of G is called a $\mathscr{M}$-supplemented subgroup of G (or $\mathscr{M}$-supplemented in G) if there exists a subgroup B of G such that $G = HB$ and TB is a proper subgroup of G for every maximal subgroup T of H. In the

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same paper, Miao gave some conditions of supersolvability of a group G under the assumption that some primary subgroups are $\mathscr{M}$-supplemented in G.

On the other hand, the embedded property of some subgroups also have great influence on the structure of a group, see [3], [9] for example.

It is natural to ask the following question:

QUESTION. If we combine the property of $\mathscr{M}$-supplemented subgroups and the embedded property of subgroups together, what can be obtained about the property of a group?

In this paper, we first give the following definitions, and then we will proceed to give some conditions for a group to be nilpotent and p-supersolvable for some prime p.

DEFINITION 1.1. Let G be a group and H be a subgroup of G. H is said to be $\mathscr{M}$-normal supplemented in G if there exists a normal subgroup K of G such that $G = HK$ and $H_1K < G$ for every maximal subgroup H_1 of H.

DEFINITION 1.2. Let G be a group and H be a subgroup of G. H is said to be $\mathscr{M}$-normal embedded in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K = 1$ or $H \cap K$ is $\mathscr{M}$-normal supplemented in G.

Our main results are as follows.

THEOREM A. *Let G be a group. If every maximal subgroup of every Sylow subgroup of G is $\mathscr{M}$-normal embedded in G, then G is nilpotent.*

THEOREM B. *Let G be a group and P be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. If every maximal subgroup of P is $\mathscr{M}$-normal embedded in G, then G is p-supersolvable.*

Let $\mathscr{F}$ be a class of groups. $\mathscr{F}$ is said to be a *formation* provided that (1) if $G \in \mathscr{F}$ and H is a normal subgroup of G, then $G/H \in \mathscr{F}$, and (2) if G/M and G/N are in $\mathscr{F}$, then $G/(M \cap N) \in \mathscr{F}$. Furthermore, $\mathscr{F}$ is said to be *saturated* if $G \in \mathscr{F}$ whenever $G/\Phi(G) \in \mathscr{F}$. It is well known that the class of all p-supersolvable groups and the class of all nilpotent groups are saturated formations. Besides, let $\pi(G) = \{p \mid p \text{ is a prime and } p \text{ divides } |G|\}$ and G be a group with $\pi(G) = \{p_1, p_2, \ldots, p_m\}$, $p_1 > p_2 > \cdots > p_m$. If there exists a chain of subgroups

$$1 = G_0 \leq G_1 \leq \cdots \leq G_m = G$$

such that G_i is normal in G for $i = 0, 1, \ldots, m$ and for each i the quotient G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G. Then G is said to have *a Sylow tower of supersolvable type*. All other terminology and notation are standard, see [8].

2. Preliminaries

In this section, we will give some lemmas which are useful in the proof of our main results.

LEMMA 2.1 ([8, Theorem 1.6.10]). *If H is a subgroup of G with $|G : H| = p$, where p is the smallest primer divisor of $|G|$. Then $H \trianglelefteq G$.*

LEMMA 2.2. *Let P be a Sylow p-subgroup of a group G, where p is the smallest primer divisor of $|G|$. Then G is p-nilpotent if and only if P is $\mathscr{M}$-normal embedded in G.*

PROOF. If G is p-nilpotent, then it is trivial that P is $\mathscr{M}$-normal embedded in G.

Conversely, suppose that P is $\mathscr{M}$-normal embedded in G. Then there exists a normal subgroup L of G such that $G = PL$ and $P \cap L = 1$ or $P \cap L$ is $\mathscr{M}$-normal supplemented in G. If $P \cap L = 1$, then G is p-nilpotent. Suppose that $P \cap L \neq 1$. Then $P \cap L \in \mathrm{Syl}_p(L)$ and it follows from Lemma 2.1 of [7] that $P \cap L$ is $\mathscr{M}$-supplemented in L. Therefore, L is p-nilpotent by Lemma 2.9 of [7], and thus G is p-nilpotent, as claimed.

LEMMA 2.3. *Let G be a group. Then the following hold:*

(1) *If $H \leq L \leq G$ and H is $\mathscr{M}$-normal embedded in G, then H is $\mathscr{M}$-normal embedded in L;*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. If H is $\mathscr{M}$-normal embedded in G, then H/N is $\mathscr{M}$-normal embedded in G/N;*

(3) *If H is a π-subgroup and N is a normal π-subgroup of G, then HN/N is $\mathscr{M}$-normal embedded in G/N if and only if H is $\mathscr{M}$-normal embedded in G.*

PROOF. The proof is trivial and is left to the reader.

LEMMA 2.4 ([10, Lemma 2.6]). *Let P be a Sylow subgroup of a group G and N a normal subgroup of G. If $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

3. Proof of main results

In this section, we will give the proof of our main results. We first give the following lemma, which is useful for the proof of Theorem A.

LEMMA 3.1. *Let G be a group and p be the smallest primer divisor of $|G|$. If every maximal subgroup of every Sylow p-subgroup of G is $\mathscr{M}$-normal embedded in G, then G is p-nilpotent.*

PROOF. Suppose that the theorem is false and let G be a counterexample of minimal order.

Let D be a maximal subgroup of some Sylow p-subgroup of G. Then D is $\mathcal{M}$-normal embedded in G. Therefore, there exists a normal subgroup K of G such that $G = DK$ and $D \cap K = 1$ or $D \cap K$ is $\mathcal{M}$-normal embedded in G.

If $|D| = p$, then $D \cap K = 1$ or D. Suppose that $D \cap K = 1$. Let H be a Sylow p-subgroup of K. Then $|H| = p$ and H is a maximal subgroup of some Sylow p-subgroup of G. By the hypothesis, H is $\mathcal{M}$-normal embedded in G, and thus H is $\mathcal{M}$-normal embedded in K by Lemma 2.3. Since G is a counterexample of minimal order, we have that K is p-nilpotent, and thus G is p-nilpotent, which is a contradiction. Suppose that $D \cap K = D$. Then D is $\mathcal{M}$-normal supplemented in G. Therefore, there exists a normal subgroup T of G such that $G = DT$ and $T < G$. By arguing similarly as above, we have that T is p-nilpotent, and thus G is p-nilpotent, another contradiction.

Therefore, $|D| > p$. If $O_{p'}(G) \neq 1$. Then $G/O_{p'}(G)$ satisfies the hypothesis of this theorem by Lemma 2.3 and $G/O_{p'}(G)$ is p-nilpotent by the choice of G, whence G is p-nilpotent, which is a contradiction. So, we may assume that $O_{p'}(G) = 1$. If $D \cap K = 1$, then $D_1 K < G$ for every maximal subgroup D_1 of D, and thus D is $\mathcal{M}$-normal supplemented in G. Since $G/K \cong D$, there exists a normal series $K \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_s \trianglelefteq G$ and $|G : K_s| = p$. Let T be an arbitrary Sylow p-subgroup of K_s. Then T is a maximal subgroup of some Sylow p-subgroup of G, and thus T is $\mathcal{M}$-normal embedded in G. It follows from Lemma 2.3 that T is $\mathcal{M}$-normal embedded in K_s. Now by Lemma 2.2, K_s is p-nilpotent, which gives the contradiction that G is p-nilpotent. If $D \cap K \neq 1$, then $D \cap K$ is $\mathcal{M}$-normal supplemented in G. Write $\overline{D} = D \cap K$ and let T be a normal subgroup of G such that $G = \overline{D}T$, and $\overline{D}_1 T < G$ for every maximal subgroup $\overline{D}_1$ of $\overline{D}$. Since G/T is a p-group, there exists a normal subgroup T_1/T of G/T and $|G/T : T_1/T| = p$. It follows that $T_1 \trianglelefteq G$ and $|G : T_1| = p$, by arguing similarly as above, we have a contradiction, completing the proof.

Proof of Theorem A. Suppose that the theorem is false and let G be a counterexample of minimal order.

It follows from Lemma 3.1 that G has a Sylow tower of supersolvable type. Therefore, let p be the largest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then P is normal in G. We will get a contradiction by distinguishing the following two cases.

Case 1: $P \cap \Phi(G) \neq 1$. Let L be a minimal normal subgroup of G contained in $P \cap \Phi(G)$. Suppose that L is not a maximal subgroup of P, then by Lemma 2.3 we have that G/L satisfies the hypothesis of this theorem, and thus G/L is nilpotent by the choice of G. It follows that G is nilpotent, which is a contradiction. Therefore, L is a maximal subgroup of P. Then $L = P \cap \Phi(G)$. By the hypothesis of this theorem, we have that L is $\mathcal{M}$-normal embedded in

G. Therefore, there exists a normal subgroup K of G such that $G = LK$ and $L \cap K = 1$ or $L \cap K$ is $\mathscr{M}$-normal supplemented in G. As $L \leq \Phi(G)$, we have that $K = G$, and thus $L = P \cap \Phi(G)$ is $\mathscr{M}$-normal supplemented in G. Let B be a normal subgroup of G such that $G = (P \cap \Phi(G))B$. Then $G = B$ and for every maximal subgroup P_1 of $P \cap \Phi(G)$, $P_1 B = B = G$, which contradicts that $P \cap \Phi(G)$ is $\mathscr{M}$-normal supplemented in G.

CASE 2: $P \cap \Phi(G) = 1$. By Theorem 1.8.7 of [5], $P = L_1 \times L_2 \times \cdots \times L_t$, where each L_i is a minimal normal subgroup of G for $i = 1, 2, \ldots, t$. If $t = 1$, then P is a minimal normal subgroup of G. Let E be a maximal subgroup of P. Then there exists a normal subgroup K of G such that $G = EK$ and $E \cap K = 1$ or $E \cap K$ is $\mathscr{M}$-normal supplemented in G. As $P \cap K \trianglelefteq G$, we have that $P \cap K = P$ or $P \cap K = 1$. If $P \cap K = P$, then $E \cap K = E$, whence E is $\mathscr{M}$-normal supplemented in G. If $P \cap K = 1$, then $E \cap K = 1$, and thus $E_1 K < G$ for every maximal subgroup E_1 of E, which also shows that E is $\mathscr{M}$-normal supplemented in G. Now let B be a normal subgroup of G such that $G = EB$ and $E_1 B < G$ for every maximal subgroup E_1 of E. If $P \cap B = 1$, then $E \cap B = 1$. Therefore, $E = P$, which is a contradiction. If $P \cap B = P$, then $E \leq P \leq B$, and thus $G = EB = B$, whence $E_1 B = B = G$ for every maximal subgroup E_1 of E, which is also a contradiction. Therefore, $t \geq 2$. Write $L = L_1$. If L is not a maximal subgroup of P, then G/L satisfies the hypothesis of this theorem by Lemma 2.3, and thus G/L is nilpotent by the choice of G. If $t > 2$, then L_2 is not a maximal subgroup of P, and thus G/L_2 is nilpotent similarly. It follows that $G \cong G/(L_1 \cap L_2)$ is nilpotent, which is a contradiction. Therefore, $t = 2$ and L_2 is a maximal subgroup of P and $|L_2| > p$. Now by the hypothesis of this theorem, L_2 is $\mathscr{M}$-normal embedded in G. Therefore, there exists a normal subgroup K of G such that $G = L_2 K$ and $L_2 \cap K = 1$ or $L_2 \cap K$ is $\mathscr{M}$-normal supplemented in G. Let B be a normal subgroup of G such that $G = L_2 B$ and $TB < G$ for every maximal subgroup T of L_2. It follows that $G = L_2(TB)$. As $L_2 \cap TB \trianglelefteq G$ and L_2 is minimal normal in G, we have that $L_2 \cap TB = 1$. Therefore, $|L_2| = |G : TB| = p$ by Lemma 2.3 of [7], which is a contradiction. Therefore, L is a maximal subgroup of P and $t = 2$. Similarly arguing as above, we have that L_2 is also a maximal subgroup of P. It follows that $|L_1| = |L_2| = p$. By hypothesis, $L = L_1$ is $\mathscr{M}$-normal embedded in G. Then there exists a normal subgroup K of G such that $L \cap K = 1$ or $L \cap K$ is $\mathscr{M}$-normal supplemented in G. If $L \cap K = 1$, then K is nilpotent by Lemma 2.3 and the choice of G. For every $q \in \pi(G)$ and $q \neq p$ and $Q \in \mathrm{Syl}_q(G)$. It follows that $Q \in \mathrm{Syl}_q(K)$ and $Q \trianglelefteq K$, and thus $Q \trianglelefteq G$. Noticing that the Sylow p-subgroup of G is also normal, we have that G is nilpotent, which is a contradiction. If $L \cap K \neq 1$, then $L \cap K = L$ and L is $\mathscr{M}$-normal supplemented in G. By arguing similarly as above, we can get a contradiction, completing the proof.

The following two lemmas are useful in the proof of Theorem B.

LEMMA 3.2. *Let G be a group and p be an odd prime divisor of $|G|$. If every Sylow p-subgroup of G is $\mathcal{M}$-normal embedded in G, then G is p-supersolvable.*

PROOF. Suppose that the theorem is false and let G be a counterexample of minimal order. Then by Lemma 2.3, we may assume that $O_{p'}(G) = 1$.

Let P be a Sylow p-subgroup of G. Then there exists a normal subgroup K of G such that $G = PK$ and $P \cap K = 1$ or $P \cap K$ is $\mathcal{M}$-normal supplemented in G. If $P \cap K = 1$ or $P \cap K = P$, then P is $\mathcal{M}$-normal supplemented in G and $P_i K < G$ for every maximal subgroup P_i of P. As $|G : P_i K| = p$ by Lemma 2.3 of [7], we have that $G/(P_i K)_G$ is isomorphic to a subgroup of S_p. Let $T = \cap(P_i K)_G$. Then $T \cap P \leq \Phi(P)$, and T is p-nilpotent by Lemma 2.4. Since $O_{p'}(G) = 1$, we have that $T = 1$. Therefore, $G = G/T$ is isomorphic to a subgroup of $S_p \times S_p \times \cdots \times S_p$, and thus $\Phi(P) = 1$. It follows that $P \cap K \leq \Phi(P) = 1$, and thus K is the normal Hall p'-subgroup of G. Therefore, $K = O_{p'}(G) = 1$, which is a contradiction.

Suppose that $1 < P \cap K < P$. Then $P \cap K$ is a Sylow p-subgroup of K, and $P \cap K$ is $\mathcal{M}$-normal supplemented in K. We have that $O_{p'}(K) \leq O_{p'}(G) = 1$. Now by arguing similarly the pair $(P \cap K, K)$ as the pair (P, G) in the above paragraph, we can also get a contradiction, completing the proof.

LEMMA 3.3. *Let G be a group and P be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. If every minimal subgroup of P is $\mathcal{M}$-normal embedded in G, then G is p-supersolvable.*

PROOF. Let L be a minimal subgroup of P. Then there exists a normal subgroup K of G such that $G = LK$ and $L \cap K = 1$ or $L \cap K$ is $\mathcal{M}$-normal supplemented in G. As L is of order p, we have that $L \cap K = 1$ or $L \cap K = L$. In each case, we have that L is $\mathcal{M}$-normal supplemented in G. It follows from Theorem 4 of [4] that G is p-supersolvable, as claimed.

PROOF OF THEOREM B. Suppose that the theorem is false and let G be a counterexample of minimal order. If $|P| = p^2$, then the theorem is true by Lemma 3.3. Therefore, we may assume that $|P| > p^2$. Since $G/O_{p'}(G)$ satisfies the hypothesis of this theorem by Lemma 2.3, if $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is p-supersolvable by the choice of G, and thus G is p-supersolvable. Therefore, we may assume that $O_{p'}(G) = 1$.

We will get a contradiction by the following steps.

STEP 1: $O_p(G) \neq 1$. Let E be a maximal subgroup of P. Then there exists a normal subgroup K of G such that $G = EK$ and $E \cap K = 1$ or $E \cap K$ is $\mathcal{M}$-

normal supplemented in G. Since $G/K \cong E/E \cap K$ is a p-group, there exists a normal subgroup K_1 of G such that $K \leq K_1$ and $|G/K_1| = p$. Now let P_1 be a Sylow p-subgroup of K_1. Then P_1 must be a maximal subgroup of some Sylow p-subgroup of G and P_1 is $\mathcal{M}$-normal embedded in G by the hypothesis. It follows from Lemma 2.3 that P_1 is $\mathcal{M}$-normal embedded in K_1. Then by Lemma 3.2 one has that K_1 is p-supersolvable. Since $O_{p'}(K_1) \leq O_{p'}(G) = 1$, we have that $O_p(K_1) \neq 1$, and thus $O_p(G) \neq 1$.

STEP 2: $O_p(G) \cap \Phi(G) = 1$. If $O_p(G) \cap \Phi(G) \neq 1$, then we can choose a minimal normal subgroup L of G such that $L \leq O_p(G) \cap \Phi(G)$. If L is not a maximal subgroup of P, then the choice of G leads to that G/L is p-supersolvable, and thus G is p-supersolvable, which is a contradiction. Therefore, $O_p(G) \cap \Phi(G)$ is a maximal subgroup of P. Let $P_1 = O_p(G) \cap \Phi(G)$. Then there exists a normal subgroup K of G such that $G = P_1K$ and $P_1 \cap K$ is $\mathcal{M}$-normal supplemented in G. As $P_1 \leq \Phi(G)$, we have that $G = K$, and thus P_1 is $\mathcal{M}$-normal supplemented in G. Therefore, there exists a normal subgroup B of G such that $G = P_1B$ and $TB < G$ for every maximal subgroup T of P_1. As $P_1 \leq \Phi(G)$, we can also have that $G = B$, which contradicts that $TB < G$. So we conclude that $O_p(G) \cap \Phi(G) = 1$.

STEP 3: final contradiction. By the above claim and Theorem 1.8.7 of [5], we have that $O_p(G) = L_1 \times L_2 \times \cdots \times L_t$, where each L_i is a minimal normal subgroup of G for $i = 1, 2, \ldots, t$.

Let $L \in \{L_1, L_2, \ldots, L_t\}$. If $|P/L| \leq p$, then we can choose a maximal subgroup E of P such that $E \leq L$. Then there exists a normal subgroup K of G such that $G = EK$ and $E \cap K = 1$ or $E \cap K$ is $\mathcal{M}$-normal supplemented in G. As L is minimal normal in G, we have that $L \cap K = 1$ or $L \cap K = L$. In the former case, we have that $E \cap K = 1$, and thus $E_1K < G$ for every maximal subgroup E_1 of E, that is to say E is $\mathcal{M}$-normal supplemented in G. In the latter case, $E = E \cap K$ is also $\mathcal{M}$-normal supplemented in G. Now let B be a normal subgroup of G such that $G = EB$ and $E_1B < G$ for every maximal subgroup E_1 of E. Since $G = EB = E(E_1B) = L(E_1B)$, and $L \cap E_1B = 1$, we have that $|L| = |G : E_1K|$, which equals to p by Lemma 2.3 of [7], and this is a contradiction. Therefore, $|P/L| > p$ and there exists a maximal subgroup H of P such that $L < H$. By the hypothesis there exists a normal subgroup K of G such that $G = HK$ and $H \cap K = 1$ or $H \cap K$ is $\mathcal{M}$-normal supplemented in G.

If $t = 1$, then $O_p(G) = L_1 = L$ is a minimal normal subgroup of G. If $L \cap K = L$, then $L \leq H \cap K$. Write $T = H \cap K$. Then there exists a normal subgroup B of G such that $G = TB$ and $T_1B < G$ for every maximal subgroup T_1 of T. As $L \not\leq \Phi(G)$, we have that $L \not\leq O_p(T)$. Therefore, there exists a maximal subgroup T_1 of H such that $T = LT_1$. It follows that

$G = TB = LT_1B$, and thus $|L| = |G : T_1B| = p$ by Lemma 2.3 of [7]. On the other hand, G/L satisfies the hypothesis of this theorem by Lemma 2.3, the choice of G leads to that G/L is p-supersolvable, and thus G is p-solvable. It follows that $C_G(O_p(G)) = O_p(G)$ since $O_{p'}(G) = 1$. Therefore, $G/O_p(G) = G/C_G(O_p(G))$ is isomorphism to a subgroup of $\mathrm{Aut}(L)$, which is abelian, and thus is p-supersolvable. Therefore, G is p-supersolvable, a contradiction. Therefore, $L \cap K = 1$. It follows that $K \leq C_G(L)$. Similarly as in the former argument, we have that G is p-solvable, and thus $K \leq C_G(O_p(G)) \leq O_p(G)$, which is also a contradiction.

Therefore, $t \geq 2$, and neither L_1 nor L_2 is a maximal subgroup of P. It follows that both G/L_1 and G/L_2 satisfy the hypothesis of this theorem by Lemma 2.3, by the choice of G we have that both G/L_1 and G/L_2 are p-supersolvable, and thus $G \cong G/(L_1 \cap L_2)$ is p-supersolvable, which is a final contradiction, completing the proof.

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REFERENCES

1. Arad, Z., and Ward, M. B., *New criteria for the solvability of finite groups*, J. Algebra 77 (1982), no. 1, 234–246.
2. Ballester-Bolinches, A., and Guo, X. Y., *On complemented subgroups of finite groups*, Arch. Math. (Basel) 72 (1999), no. 3, 161–166.
3. Ballester-Bolinches, A., and Li, Y. M., *On S-permutably embedded subgroups of finite groups*, Monatsh. Math. 172 (2013), no. 3–4, 247–257.
4. Gao, B. J., Zhang, J., and Miao, L., *Then influence of $\mathcal{M}$-subnormal subgroups with given order on the structure of finite groups*, Acta Sci. Natur. Univ. Sunyatseni 55 (2016), no. 5, 27–30.
5. Guo, W. B., *The theory of classes of groups*, Kluwer Academic Publishers Group, Dordrecht; Science Press Beijing, Beijing 2000.
6. Hall, P., *A characteristic property of soluble groups*, J. London Math. Soc. 12 (1937), no. 3, 188–200.
7. Miao, L., *The influence of $\mathcal{M}$-supplemented subgroups on the structure of finite groups*, Bull. Braz. Math. Soc. (N.S.) 40 (2009), no. 4, 495–509.
8. Robinson, D. J. S., *A course in the theory of groups*, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1993.
9. Yu, H., *On S-semipermutable or S-permutably embedded subgroups of finite groups*, Acta Math. Hungar. 151 (2017), no. 1, 173–180.

10. Zhang, X. J., Li, X. H., and Miao, L., *Sylow normalizers and p-nilpotence of finite groups*, Comm. Algebra 43 (2015), no. 3, 1354–1363.

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