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ASYMPTOTIC BEHAVIOR OF j-MULTIPLICITIES

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Abstract

Let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. Let $R_+ = \oplus_{n\in\mathbb{N}} R_n$ denote the irrelevant ideal of R and let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module. When $\dim(R_0) \leq 2$ and $\mathfrak{q}_0$ is an arbitrary ideal of R_0, we show that the j-multiplicity of the graded local cohomology module $j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n)$ has a polynomial behavior for all $n \ll 0$.

1. Introduction

Throughout this paper, let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. We also assume that R is standard which means that $R = R_0[R_1]$ is a finitely generated R_0-algebra by elements of degree one. Let $R_+ = \oplus_{n\in\mathbb{N}} R_n$ be the irrelevant ideal of R and $\mathfrak{m} := \mathfrak{m}_0 \oplus R_+$ denote the graded maximal graded ideal of R. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module.

It is well known that the study on the Hilbert-Samuel multiplicity of ideals on Noetherian rings is a relevant topic of interest in Commutative Algebra and Algebraic Geometry. In [1], Achilles and Manaresi introduced a extension of the Hilbert-Samuel multiplicity, called *j-multiplicity* and since then, much efforts have been made in order to extend the good properties of the usual multiplicity (e.g. [12], [13] and [10]).

One interesting property is the asymptotic behavior of the Hilbert-Samuel coefficients of the graded local cohomology modules. More precisely, when $\dim(R_0) \leq 2$, Brodmann, Fumasoli and Tajarod [3], Brodmann and Rohrer in [6], and Brodmann, Rohrer and Sazeedeh [7] have shown that, if $\mathfrak{q}_0$ is an $\mathfrak{m}_0$-primary ideal of R_0 and $n \ll 0$, the Hilbert-Samuel coefficients

$$e_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) \qquad \text{and} \qquad e_1(\mathfrak{q}_0, H^i_{R_+}(M)_n)$$

have a polynomial behavior, where $H^i_{R_+}(M) = \oplus_{n\in\mathbb{Z}} H^i_{R_+}(M)_n$ is the *ith graded local cohomology module of M with respect to R_+* [8].

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The main motivation to study graded modules $H^i_{R_+}(M)$ and their graded components comes from geometry. In geometric terms, it means to study the asymptotic behavior of Serre cohomology modules of schemes and coherent sheaves induced by a graded R-module M ([8], [4]). This motivates us to extend their computations to j-multiplicity and consider the following question:

QUESTION. Let $\mathfrak{q}_0$ be an arbitrary ideal of R_0. For $n \ll 0$, is the generalized Hilbert-Samuel coefficient $j_i(\mathfrak{q}_0, H^i_{R_+}(M)_n)$ a numerical polynomial for $i = 0, 1, \ldots, \dim(R_0)$?

From now on, let $\mathfrak{q}_0$ be an arbitrary ideal of R_0. Our main results (Theorem 3.3 and Theorem 3.6) show that in the cases (i) and (ii) below, there exists a numerical polynomial $Q \in \mathbb{Q}[x]$ of degree less than i such that

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \text{for all } n \ll 0 \text{ and } i \in \mathbb{N}_0.$$

The two cases are

(i) $\dim(R_0) \leq 1$;
(ii) $\dim(R_0) = 2$ and $\dim(H^i_{R_+}(M)_n) \geq 1$ for all $n \ll 0$.

A key element for the proof of theses results is Theorem 3.1, that shows the Artinianess of the graded R-modules

$$H^1_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right) \qquad \text{and} \qquad \Gamma_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right),$$

where $\mathfrak{p}_0 \subseteq \mathfrak{q}_0$ are arbitrary ideals of R_0, when $\dim R_0 \leq 1$.

We briefly describe the content of the paper. In Section 2, we fix our notations and we recall the definitions on the Hilbert-Samuel polynomials for Noetherian modules, Hilbert-Kirby polynomial for Artinian modules, Hilbert-Samuel coefficients, generalized Hilbert-Samuel multiplicities and, in particular, the j-multiplicity. We also recall key remarks for the rest of the paper. In Section 3, we start with an investigation concerning the Artinianess of the graded local cohomology modules previously cited. Further we investigate the polynomial behavior of the j-multiplicity $j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n)$ for $n \ll 0$.

2. Set-up

First we fix our terminology and we establish some preparatory facts that will be needed throughout the paper.

NOTATION 2.1. From now on, let $R = \oplus_{n \in \mathbb{N}_0} R_n$ denote a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. We also assume that R is standard which means that $R = R_0[R_1]$ is a finitely generated R_0-algebra by elements of degree one. Let $R_+ = \oplus_{n\in\mathbb{N}} R_n$ be the irrelevant ideal of R and $\mathfrak{m} := \mathfrak{m}_0 \oplus R_+$ denote the graded maximal graded ideal of R. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module.

Let $(\widehat{R_0}, \widehat{\mathfrak{m}_0 R})$ denote the $\mathfrak{m}_0$-adic completion of the local ring $(R_0, \mathfrak{m}_0)$. We also denote $\widehat{R} := \widehat{R_0} \otimes_{R_0} R$ and $\widehat{M}$ by the graded $\widehat{R}$-module $\widehat{R_0} \otimes_{R_0} M$.

The next facts can be found in [9] and ([3, Remark 2.1]).

(A) Hilbert polynomial for Noetherian modules. Let $N = \oplus_{n\in\mathbb{Z}} N_n$ be a Noetherian R-module. Then the following statements are equivalent:

(i) $\ell_{R_0}(N_n) < \infty$ for all $n \gg 0$.

(ii) There is some $m \in \mathbb{N}$ with $\mathfrak{m}_0^m N_n = 0$ for all $n \gg 0$.

Moreover, by a similar argument showed in [14], if N satisfies the statements (i) and (ii), we see that there exists an uniquely determined polynomial $P_N(n) \in \mathbb{Q}[n]$ of degree at most $\dim_{R_0}(R_1/\mathfrak{m}_0 R_1) - 1$ such that

$$\ell_{R_0}(N_n) = P_N(n) \qquad \text{for all } n \gg 0.$$

The polynomial $P_N(n)$ is called the *Hilbert-Samuel polynomial* of N.

(B) Hilbert-Kirby polynomial for Artinian modules. Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be an Artinian R-module. Then all the R_0-modules A_n are Artinian and the following statements are equivalent:

(i) The R_0-module A_n is finitely generated for all$n \ll 0$.

(ii) $\ell_{R_0}(A_n) < \infty$ for all $n \ll 0$.

(iii) There is some $m \in \mathbb{N}$ (which depends of n) with $\mathfrak{m}_0^m A_n = 0$ for all $n \ll 0$.

Moreover, by a similar argument as in [11, Theorem 2] and [7, Remark 2.1], if A satisfies the statements of (i)–(iii), there exists an uniquely determined polynomial $P_A(n) \in \mathbb{Q}[n]$ of degree at most $\dim_{R_0}(R_1/\mathfrak{m}_0 R_1) - 1$ such that

$$\ell_{R_0}(A_n) = P_A(n) \quad \text{for all } n \ll 0.$$

So, we say that A is an Artinian graded R-module with *Hilbert-Kirby polynomial $P_A(n)$*.

(C) Generalized Hilbert-Samuel coefficients. Let T be a finitely generated module over Noetherian local ring $(S, \mathfrak{m})$ and let I be an arbitrary ideal of S. Set $W = \bigoplus_{n=0}^{\infty} \Gamma_{\mathfrak{m}}(I^n T/I^{n+1}T)$. Since W is annihilated by a large power of $\mathfrak{m}$, the Hilbert-Samuel function

$$H_W(n) = \sum_{v=0}^{n} \ell_S(\Gamma_{\mathfrak{m}}(I^v T/I^{v+1}T))$$

is well defined. In [13], it is defined the *generalized Hilbert-Samuel function* of I on T by $H_{I,T}(n) := H_W(n)$ for every positive integer $n \geq 0$. Let d be a positive integer such that $d \geq \dim T$. Then, for n large, $H_{I,T}(n)$ is eventually a polynomial $Q_T^I(n)$ of degree at most d and it can be written as

$$Q_T^I(n) = \sum_{i=0}^{d} (-1)^i j_i(I,T) \binom{n+d-i}{d-i},$$

where $j_i(I,T)$, $0 \leq i \leq d$, are called the *generalized Hilbert-Samuel coefficients* of I with respect to T. The coefficient $j_0(I,T)$ is called the *j-multiplicity* of I with respect to T.

Note that, if I is $\mathfrak{m}$-primary ideal, the generalized Hilbert-Samuel function coincides with the usual Hilbert-Samuel function; in particular, the generalized Hilbert-Samuel coefficients $j_i(I,T)$ coincides with the usual Hilbert-Samuel coefficients $e_i(I,T)$, for all $i = 0, \dots, d$.

Recall that the Krull dimension of

$$\mathscr{F}(I,T) := G_I(T)/\mathfrak{m}G_I(T) = \bigoplus_{n=0}^{\infty} I^n T/\mathfrak{m} I^n T$$

is called the analytic spread of I with respect to T and denoted by $\ell(I,T)$. In general, $\ell(I,T) \leq \dim T$. By [12, Lemma 3.1], $j_0(I,T) \neq 0$ if and only if $\ell(I,T) = d$.

Similarly to the usual Hilbert-Samuel multiplicity, there is an associativity formula for j-multiplicity [12] given by

$$j_0(I,T) = \sum_{\mathfrak{p} \in \mathrm{Assh}(T)} \ell_{S_\mathfrak{p}}(T_\mathfrak{p}) j_0(I, S/\mathfrak{p}),$$

where $\mathrm{Assh}_S(T) := \{\mathfrak{p} \in \mathrm{Ass}_S(T) \mid \dim(S/\mathfrak{p}) = \dim(T)\}$. For these and other properties about the j-multiplicities see [12] and [13].

Also, recall that $x \in \mathfrak{m}$ is an *$\mathfrak{m}$-filter regular element with respect to an S-module T*, if it satisfies the following equivalent conditions:

(i) x is a nonzero divisor with respect to $T/\Gamma_\mathfrak{m}(T)$;

(ii) $(0 :_T x) \subseteq \Gamma_\mathfrak{m}(T)$;

(iii) $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_S(T) \setminus \{\mathfrak{m}\}} \mathfrak{p}$.

So, in view of (iii), each $\mathfrak{m}$-filter regular element $x \in \mathfrak{m}$ is a parameter with respect to T. Let I is an ideal of S. We say that $x \in I$ is a superficial element of I with respect to T, if there exists $c \in \mathbb{N}$ such that for all $k \geq c$, we should have $(I^{k+1}T :_T x) \cap I^c T = I^k T$ (see [9, Definition 8.5.1]).

The following remark describes some well known properties in the literature (e.g. [7]).

REMARK 2.2. (a) By the graded Flat Base Change Theorem for local cohomology modules and functor extension, there are the following isomorphisms

$$\widehat{R}_0 \otimes_{R_0} H^i_{R_+}(M)_n \cong H^i_{\widehat{R}_+}(\widehat{M})_n$$

and

$$\left(0 :_{H^i_{R_+}(M)_n} \mathfrak{q}_0\right) \otimes_{R_0} \widehat{R}_0 \cong \left(0 :_{H^i_{\widehat{R}_+}(\widehat{M})_n} \mathfrak{q}_0 \widehat{R}_0\right)$$

for all $n \in \mathbb{Z}$, and for any ideal $\mathfrak{q}_0 \subset R_0$.

(b) Let $(S, \mathfrak{m})$ be a Noetherian local ring and let T be a finite S-module. So we have that $\dim_{\widehat{S}}(\widehat{S} \otimes_S T) = \dim_S(T)$.

(c) Without loss of generality we can move from finite residual field to infinite as follows. For a Noetherian local ring $(S, \mathfrak{m})$, let X be a variable over S. Set $S' = S[X]_{\mathfrak{m}S[X]}$. Then $S \subseteq S'$ is a faithfully flat extension of Noetherian local rings with the same Krull dimension. The residue field $S'/\mathfrak{m}S'$ of S contains the residue field $S/\mathfrak{m}$ of S and is an infinite field. Since S is a faithfully flat extension of a Noetherian local S-algebra, we get the same properties of (a) and (b) for the ring S.

(d) Let $(S, \mathfrak{m})$ be a Noetherian local ring and let T be a finite S-module with $\dim T = 1$. Let $N = H^0_{\mathfrak{m}}(T)$ and $\overline{T} = T/N$. Note that $H^0_{\mathfrak{m}}(\overline{T}) = 0$ and $\dim \overline{T} = \dim T = 1$, which implies that $\operatorname{depth} \overline{T} = 1$. Therefore $\overline{T}$ is a Cohen-Macaulay S-module.

3. Artinianess and asymptotic behavior

The next result is a key ingredient for the rest of the paper.

THEOREM 3.1. *Let* $\dim R_0 \leq 1$ *and let* $\mathfrak{p}_0$, $\mathfrak{q}_0$ *be ideals of* R_0. *Then, for all* $i \in \mathbb{N}_0$,

(i) $H^1_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right)$ *and* $\Gamma_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right)$ *are Artinian R-modules provided* $\mathfrak{p}_0 \subseteq \mathfrak{q}_0$.

(ii) $\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 \mathfrak{q}_0 H^i_{R_+}(M)}$ *is an Artinian R-module, provided* $\mathfrak{p}_0$, $\mathfrak{q}_0$ *are* $\mathfrak{m}_0$*-primary ideals of* R_0.

PROOF. (i) The short exact sequence

$$0 \longrightarrow \mathfrak{p}_0 H^i_{R_+}(M) \longrightarrow \mathfrak{q}_0 H^i_{R_+}(M) \longrightarrow \mathfrak{q}_0 H^i_{R_+}(M)/\mathfrak{p}_0 H^i_{R_+}(M) \longrightarrow 0$$

induces the long exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(\mathfrak{p}_0 H^i_{R_+}(M)\big) \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(\mathfrak{q}_0 H^i_{R_+}(M)\big) \longrightarrow \Gamma_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right)$$

$$\longrightarrow H^1_{\mathfrak{m}_0 R}\big(\mathfrak{p}_0 H^i_{R_+}(M)\big) \longrightarrow H^1_{\mathfrak{m}_0 R}\big(\mathfrak{q}_0 H^i_{R_+}(M)\big) \longrightarrow H^1_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right)$$

$$\longrightarrow H^2_{\mathfrak{m}_0 R}\big(\mathfrak{p}_0 H^i_{R_+}(M)\big) \longrightarrow H^2_{\mathfrak{m}_0 R}\big(\mathfrak{q}_0 H^i_{R_+}(M)\big) \longrightarrow H^2_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{p}_0 H^i_{R_+}(M)}\right).$$

Since $\dim R_0 \leq 1$, we may choose $x \in \mathfrak{m}_0$ such that $\sqrt{xR_0} = \mathfrak{m}_0$. In fact, if $\dim R_0 = 0$, choose $x = 0$, and if $\dim R_0 = 1$, choose $x \in \mathfrak{m}_0 \setminus \bigcup_1^s \tilde{\mathfrak{p}_i}$, where $\tilde{\mathfrak{p}_1}, \ldots, \tilde{\mathfrak{p}_s}$ denote all minimal primes of R_0. Also, $\sqrt{\mathfrak{m}_0 R} = \sqrt{xR}$; whence $H^2_{\mathfrak{m}_0 R}(\mathfrak{p}_0 H^i_{R_+}(M)) = 0$.

As $\Gamma_{\mathfrak{m}_0 R}(\mathfrak{q}_0 H^i_{R_+}(M))$ is a submodule of $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$, which is an Artinian R-module [3, Theorem 2.5], we have that $\Gamma_{\mathfrak{m}_0 R}(\mathfrak{q}_0 H^i_{R_+}(M))$ is Artinian. Therefore, to prove the statement it suffices to show that $H^1_{\mathfrak{m}_0 R}(\mathfrak{p}_0 H^i_{R_+}(M))$ is Artinian.

Let $N = H^i_{R_+}(M)$. Let $\mu = \mu(\mathfrak{p}_0)$ denote the minimal generating set of $\mathfrak{p}_0$. Consider the natural epimorphism $R_0^\mu \to \mathfrak{p}_0$. By tensorizing, we have another epimorphism

$$f \colon N \otimes_{R_0} R_0^\mu \longrightarrow N \otimes_{R_0} \mathfrak{p}_0,$$

or even better, the following exact sequence

$$0 \longrightarrow \operatorname{Ker}(f) \longrightarrow N \otimes_{R_0} R_0^\mu \longrightarrow N \otimes_{R_0} \mathfrak{p}_0 \longrightarrow 0,$$

which leads to a long one:

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}(\operatorname{Ker}(f)) \longrightarrow \Gamma_{\mathfrak{m}_0 R}(N \otimes_{R_0} R_0^\mu) \longrightarrow \Gamma_{\mathfrak{m}_0 R}(N \otimes_{R_0} \mathfrak{p}_0)$$

$$\longrightarrow H^1_{\mathfrak{m}_0 R}(\operatorname{Ker}(f)) \longrightarrow H^1_{\mathfrak{m}_0 R}(N \otimes_{R_0} R_0^\mu) \longrightarrow H^1_{\mathfrak{m}_0 R}(N \otimes_{R_0} \mathfrak{p}_0)$$

$$\longrightarrow H^2_{\mathfrak{m}_0 R}(\operatorname{Ker}(f)) \longrightarrow H^2_{\mathfrak{m}_0 R}(N \otimes_{R_0} R_0^\mu) \longrightarrow H^2_{\mathfrak{m}_0 R}(N \otimes_{R_0} \mathfrak{p}_0).$$

Since $H^2_{\mathfrak{m}_0 R}(\operatorname{Ker}(f)) = 0$, one obtains an epimorphism

$$H^1_{\mathfrak{m}_0 R}(N \otimes_{R_0} R_0^\mu) \longrightarrow H^1_{\mathfrak{m}_0 R}(N \otimes_{R_0} \mathfrak{p}_0).$$

Once $H^1_{\mathfrak{m}_0 R}(N)$ is Artinian by [3, Theorem 2.5], it follows that $H^1_{\mathfrak{m}_0 R}(N \otimes_{R_0} \mathfrak{p}_0)$ is Artinian.

Now, by doing the same above steps done from the epimorphism f but now performed for the epimorphism $g\colon N \otimes_{R_0} \mathfrak{p}_0 \to \mathfrak{p}_0 N$ defined by $n \otimes j \mapsto jn$, one would end up in an epimorphism

$$H^1_{\mathfrak{m}_0 R}(N \otimes_{R_0} \mathfrak{p}_0) \longrightarrow H^1_{\mathfrak{m}_0 R}(\mathfrak{p}_0 N),$$

in a way that $H^1_{\mathfrak{m}_0 R}(\mathfrak{p}_0 N)$ will be Artinian, as desired.

(ii) By tensorizing the exact sequence

$$0 \longrightarrow \mathfrak{q}_0 H^i_{R_+}(M) \longrightarrow H^i_{R_+}(M) \longrightarrow H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M) \longrightarrow 0$$

by $R_0/\mathfrak{p}_0$, we obtain the long exact sequence

$$\begin{aligned}\cdots \longrightarrow \operatorname{Tor}_1^{R_0}\big(R_0/\mathfrak{p}_0, H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M)\big) &\longrightarrow \mathfrak{q}_0 H^i_{R_+}(M)/\mathfrak{p}_0\mathfrak{q}_0 H^i_{R_+}(M) \\ \longrightarrow H^i_{R_+}(M)/\mathfrak{p}_0 H^i_{R_+}(M) &\longrightarrow H^i_{R_+}(M)/(\mathfrak{p}_0 + \mathfrak{q}_0) H^i_{R_+}(M) \longrightarrow 0.\end{aligned}$$

Now, the result follows by [3, Lemma 2.2] and [3, Corollary 2.6].

In the next result, let ${}^*\widehat{D}(-)$ denote the *Matlis dual functor of graded $\widehat{R}$-modules (see [7, Remark 2.3] for more details).

Proposition 3.2. *Let T be a graded R-module such that each R_0-module T_n is finitely generated. Suppose that the R-module $L := \Gamma_{\mathfrak{m}_{\mathscr{W}} R}(T)$ is Artinian, and that*

$$\ell_{R_0}(\Gamma_{\mathfrak{m}_{\mathscr{W}}}(T_n)) < \infty \qquad \text{for all } n \in \mathbb{Z}.$$

Then there exists a polynomial $P_L \in \mathbb{Q}[x]$ such that $\ell_{R_0}(\Gamma_{\mathfrak{m}_{\mathscr{W}}}(T_n)) = P_L(n)$ for all $n \ll 0$ and

$$\deg(P_L) = \dim_{\widehat{R}} {}^*\widehat{D}(0 :_T \mathfrak{m}_0) - 1.$$

Proof. Since $L_n = \Gamma_{\mathfrak{m}_0}(T_n)$ is an Artinian R_0-module finitely generated, we get $\ell(\Gamma_{\mathfrak{m}_0}(T_n)) < \infty$ for all $n \in \mathbb{Z}$.

As each L_n is a finitely generated R_0-module, it follows that $\mathfrak{m}_0^k L = 0$ for some k. In fact, since L is Artinian, $\mathfrak{m}_0^{k+1} L = \mathfrak{m}_0^k L$ for some k so that $\mathfrak{m}_0^k L_n = 0$ for each n by Nakayama Lemma. Hence, $L = (0 :_T \mathfrak{m}_0^k)$. The result follows then by [7, Proposition 2.4].

Now we are prepared to show our first main result.

Theorem 3.3. *Assume that* $\dim(R_0) \leq 1$. *Let $i \in \mathbb{N}_0$ and let $\mathfrak{q}_0$ be an arbitrary ideal of R_0. Then*

(i) *there exists a numerical polynomial $P \in \mathbb{Q}[x]$ such that*

$$\ell_{R_0}\left(\Gamma_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{q}_0^2 H^i_{R_+}(M)}\right)_n\right) = P(n) \qquad \textit{for all } n \ll 0$$

and

$$\deg(P) = \dim^*_{\widehat{R}} \widehat{D}\left(0 :_{\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{q}_0^2 H^i_{R_+}(M)}} \mathfrak{m}_0\right) - 1.$$

(ii) *there exists a numerical polynomial $Q \in \mathbb{Q}[x]$ of degree less than i such that*

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \textit{for all } n \ll 0.$$

PROOF. (i) By Theorem 3.1, the R-module $\Gamma_{\mathfrak{m}_0 R}\left(\frac{\mathfrak{q}_0 H^i_{R_+}(M)}{\mathfrak{q}_0^2 H^i_{R_+}(M)}\right)$ is Artinian. Since the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $i \in \mathbb{N}_0$ ([8, Proposition 15.1.5]), the result follows by Proposition 3.2.

(ii) First, by Remark 2.2, we may assume that $R_0/\mathfrak{m}_0$ is an infinite residue field.

Note that if $i = 0$, the result is clear because $H^0_{R_+}(M)_n = 0$ for $n \ll 0$. So, suppose that $i > 0$ and let the short exact sequence

$$0 \longrightarrow \Gamma_{R_+}(M) \longrightarrow M \longrightarrow M/\Gamma_{R_+}(M) \longrightarrow 0.$$

This sequence induces a long exact sequence

$$\cdots \longrightarrow H^i_{R_+}(\Gamma_{R_+}(M))_n \longrightarrow H^i_{R_+}(M)_n \longrightarrow H^i_{R_+}(M/\Gamma_{R_+}(M))_n \longrightarrow \cdots$$

of R_0-modules. By mean of the isomorphism $H^i_{R_+}(M) \simeq H^i_{R_+}(M/\Gamma_{R_+}(M))$, we may assume $\Gamma_{R_+}(M) = 0$. By [3, Theorem 2.5], the R-module $R_0/\mathfrak{m}_0 \otimes H^i_{R_+}(M)$ is Artinian for any i. Thus, by [3, Lemma 3.3], there exists an M-regular element $x \in R_1$ such that $H^i_{R_+}(M)_n \xrightarrow{\cdot x} H^i_{R_+}(M)_{n+1}$ is surjective for $n \ll 0$.

Therefore, if we apply cohomology to the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M(1) \longrightarrow [M/xM](1) \longrightarrow 0,$$

we get a short exact sequence of R_0-modules

$$0 \longrightarrow H^{i-1}_{R_+}(M/xM)_{n+1} \longrightarrow H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_{n+1} \longrightarrow 0,$$

and by additivity of j-multiplicity [12, Theorem 3.11], we may conclude that

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) - j_0(\mathfrak{q}_0, H^i_{R_+}(M)_{n+1}) = j_0(\mathfrak{q}_0, H^{i-1}_{R_+}(M/xM)_{n+1})$$

for all $n \ll 0$. Now, the statement (ii) follows by induction on i.

COROLLARY 3.4 ([3, Theorem 3.5 (b)]). *Assume that* $\dim(R_0) \leq 1$. *Let* $i \in \mathbb{N}_0$ *and let* $\mathfrak{q}_0$ *be an* $\mathfrak{m}_0$*-primary ideal of* R_0. *Then there exists a numerical polynomial* $Q \in \mathbb{Q}[x]$ *of degree less than* i *such that*

$$e_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \text{for all } n \ll 0.$$

In the following remark we recall some useful properties for the rest of the paper.

REMARK 3.5. Choose an uncountable extension field L of $K = R_0/\mathfrak{m}_0$ (for instance, the field of fractions of a polynomial ring in uncountably many indeterminates over K). Suppose R_0 is a complete ring. By [5, Proposition 6], there is a diagram of Noetherian local rings

$$R_0'' \xleftarrow{g_0} R_0' \xrightarrow{f_0} R_0,$$

such that $(R_0', \mathfrak{m}_0')$ and $(R_0'', \mathfrak{m}_0'')$ are complete with residue fields K' and $K'' \cong L$ respectively, f_0 is surjective and g_0 turns R_0'' into an unramified flat R_0'-algebra (that is, $\mathfrak{m}_0' R_0''$ is the maximal ideal of R_0'').

In the proof of [5, Lemma 7] it is shown that

$$H^i_{R_+}(M)_n \otimes_{R_0} R_0'' \cong H^i_{R_+ \otimes_{R_0} R_0''}(M \otimes_{R_0} R_0'')_n.$$

Thus, tensorizing it by $(f_0^{-1}(\mathfrak{q}_0)R_0'')^m/(f_0^{-1}(\mathfrak{q}_0)R_0'')^{m+1}$, where $\mathfrak{q}_0$ is an arbitrary ideal of R_0, we obtain

$$\frac{\mathfrak{q}_0^m H^i_{R_+}(M)_n}{\mathfrak{q}_0^{m+1} H^i_{R_+}(M)_n} \otimes_{R_0} R_0'' \cong \frac{(f_0^{-1}(\mathfrak{q}_0)R_0'')^m H^i_{R_+ \otimes_{R_0} R_0''}(M \otimes_{R_0} R_0'')_n}{(f_0^{-1}(\mathfrak{q}_0)R_0'')^{m+1} H^i_{R_+ \otimes_{R_0} R_0''}(M \otimes_{R_0} R_0'')_n}$$

for all $n \in \mathbb{Z}$ and $i, m \in \mathbb{N}_0$. By using [8, Theorem 13.1.8] it yields

$$\begin{aligned}\Gamma_{\mathfrak{m}_0'}&\left(\frac{\mathfrak{q}_0^m H^i_{R_+}(M)_n}{\mathfrak{q}_0^{m+1} H^i_{R_+}(M)_n}\right) \otimes_{R_0} R_0'' \\ &\cong \Gamma_{\mathfrak{m}_0' R_0''}\left(\frac{(f_0^{-1}(\mathfrak{q}_0)R_0'')^m H^i_{R_+ \otimes_{R_0} R_0''}(M \otimes_{R_0} R_0'')_n}{(f_0^{-1}(\mathfrak{q}_0)R_0'')^{m+1} H^i_{R_+ \otimes_{R_0} R_0''}(M \otimes_{R_0} R_0'')_n}\right).\end{aligned}$$

By flatness and unramifiedness, it may be verified that $\ell_{R_0}(T) = \ell_{R_0''}(T \otimes_{R_0} R_0'')$ for an R_0-module T of finite length. Besides, $\Gamma_{\mathfrak{m}_0}(T) = \Gamma_{\mathfrak{m}_0'}(T)$. With these remarks and the definition of j-multiplicity we get

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = j_0(f_0^{-1}(\mathfrak{q}_0)R_0'', H^i_{R_+ \otimes_{R_0} R_0''}(M \otimes_{R_0} R_0'')_n).$$

THEOREM 3.6. *Assume that* $\dim(R_0) = 2$. *Let* $i \in \mathbb{N}_0$ *and let* $\mathfrak{q}_0$ *be an arbitrary ideal of* R_0. *If* $\dim(H^i_{R_+}(M)_n) \geq 1$ *for all* $n \ll 0$, *there exists a numerical polynomial* $Q \in \mathbb{Q}[x]$ *of degree less than* i *such that*

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \textit{for all } n \ll 0.$$

PROOF. First, consider $\dim(H^i_{R_+}(M)_n) = 2$ for infinitely many $n \in \mathbb{Z}$. Because of it, $\mathscr{K} := \bigcup_{n\in\mathbb{Z}} \operatorname{Assh}(H^i_{R_+}(M)_n) \cap \operatorname{Assh}(R_0)$ is a non-empty finite set and $\operatorname{Assh}(H^i_{R_+}(M)_n) \cap \operatorname{Assh}(R_0) \neq \emptyset$ for infinitely many integers n (see proof [7, Corollary 3.12]). So there exists $\mathfrak{q} \in \mathscr{K}$ such that $\mathfrak{q} \in \operatorname{Assh}(H^i_{R_+}(M)_n)$ for infinitely many integers n.

For any $\mathfrak{p} \in \mathscr{K}$, the ring $(R_0)_\mathfrak{p}$ is Artinian. Since $R_\mathfrak{p}$ is a graded ring with an local base ring $(R_0)_\mathfrak{p}$, by the Flat Base Change Theorem for local cohomology modules and [8, Exercise 7.1.4], we deduce that $H^i_{R_+}(M)_\mathfrak{p} \cong H^i_{(R_\mathfrak{p})_+}(M_\mathfrak{p}) \cong H^i_\mathfrak{n}(M_\mathfrak{p})$ is Artinian, where $\mathfrak{n}$ denotes the maximal ideal of $R_\mathfrak{p}$. Hence, [2, Lemma 4.3] provides $(H^i_{R_+}(M)_n)_\mathfrak{p} \cong (H^i_{R_+}(M)_\mathfrak{p})_n$ is non zero for infinitely many $n < 0$ if and only if it is non zero for $n \ll 0$.

In particular, $\dim H^i_{R_+}(M)_n = 2$ for $n \ll 0$ and $\operatorname{Assh}(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$. Now as $(R_0)_\mathfrak{p}$ is Artinian, by [3, Theorem 3.5 (a)] there exists a numerical polynomial $P := P_{H^i_{R_+}(M)_\mathfrak{p}}(x) \in \mathbb{Q}[X]$ such that $\deg(P) < i$ and $P_{H^i_{R_+}(M)_\mathfrak{p}}(n) = \ell_{(R_0)_\mathfrak{p}}((H^i_{R_+}(M)_\mathfrak{p})_n)$ for all $n \ll 0$. So by the associativity formula for j-multiplicity (see in the introduction, item (D)), for $n \ll 0$ we get

$$\begin{aligned} j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) &= \sum_{\mathfrak{p}\in\operatorname{Assh}(H^i_{R_+}(M)_n)} \ell_{(R_0)_\mathfrak{p}}((H^i_{R_+}(M)_n)_\mathfrak{p})\, j_0(\mathfrak{q}_0, R_0/\mathfrak{p}) \\ &= \sum_{\mathfrak{p}\in\operatorname{Assh}(H^i_{R_+}(M)_n)} P_{H^i_{R_+}(M)_\mathfrak{p}}(n)\, j_0(\mathfrak{q}_0, R_0/\mathfrak{p}). \end{aligned}$$

Therefore, there exists a numerical polynomial $Q \in \mathbb{Q}[X]$ with $\deg(Q) < i$, such that

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \text{for all } n \ll 0.$$

Now we may assume that $\dim(H^i_{R_+}(M)_n) = 1$ for $n \ll 0$. By Remark 2.2, $(a) - (c)$, we may assume that $(R_0, \mathfrak{m}_0)$ is complete. Consider the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}(M) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$$

where $\overline{M} = M/\Gamma_{\mathfrak{m}_0 R}(M)$. This sequence induces the long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \longrightarrow H^i_{R_+}(M) & \\ \xrightarrow{\mu} H^i_{R_+}(\overline{M}) \longrightarrow H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \longrightarrow \cdots. & \end{aligned}$$

Since $H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M))$ is Artinian for all $i \in \mathbb{Z}$ [3, Lemma 2.3], it follows that $\mathrm{Ker}(\mu)$ and $\mathrm{Coker}(\mu)$ is Artinian. Now consider the short exact sequences

$$0 \longrightarrow \mathrm{Ker}(\mu) \longrightarrow H^i_{R_+}(M) \longrightarrow \mathrm{Im}(\mu) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}(\mu) \longrightarrow H^i_{R_+}(\overline{M}) \longrightarrow \mathrm{Coker}(\mu) \longrightarrow 0.$$

Also, since $\dim_{R_0}(\mathrm{Ker}(\mu)_n) = \dim_{R_0}(\mathrm{Coker}(\mu)_n) = 0$ and $j_0(\mathfrak{q}_0, -)$ is additive we deduce $j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = j_0(\mathfrak{q}_0, H^i_{R_+}(\overline{M})_n)$ for all $n \ll 0$. We may then assume that $\Gamma_{\mathfrak{m}_0 R}(M) = 0$.

By Remark 3.5, we may replace R and M by $R \otimes_{R_0} R_0''$ and $M \otimes_{R_0} R_0''$ respectively and assume that the residue field K of the base ring R_0 is uncountable.

Consider now the following sets:

$$\mathscr{S} := \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(T_n) \cup \bigcup_{k\in\mathbb{Z}} \mathrm{Ass}_{R_0}(M_k) \cup \mathrm{Min}(R_0) \subseteq \mathrm{Spec}(R_0)$$

and

$$\mathscr{P} := \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{G_{\mathfrak{q}_0}(R_0)}(G_{\mathfrak{q}_0}(T_n)) \backslash \mathrm{Var}(G_{\mathfrak{q}_0}(R_0)_+) \subseteq \mathrm{Spec}(G_{\mathfrak{q}_0}(R_0))$$

where $T_n = H^i_{R_+}(M)_n / \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)_n)$, $G_{\mathfrak{q}_0}(R_0)$ and $G_{\mathfrak{q}_0}(T_n)$ is the associated graded ring of R_0 with respect to $\mathfrak{q}_0$ and the associated graded module of T_n with respect to $\mathfrak{q}_0$, respectively. Note that $\mathscr{S}$ and $\mathscr{P}$ are countable sets of ideals of R_0 and $G_{\mathfrak{q}_0}(R_0)$ respectively, since $G_{\mathfrak{q}_0}(R_0)$ is Noetherian and $G_{\mathfrak{q}_0}(T_n)$ is finitely generated for all $n \in \mathbb{Z}$. Since $\Gamma_{\mathfrak{m}_0 R}(M) = 0$, no ideal in $\mathscr{S}$ contains $\mathfrak{q}_{\mathscr{W}}$, and no ideal in $\mathscr{P}$ contains $G_{\mathfrak{q}_{\mathscr{W}}}(R_0)_+$. Therefore, as K is uncountable, we may apply [5, Proposition 4] to obtain an element $x \in \mathfrak{q}_{\mathscr{W}} \backslash (\mathfrak{m}_0\mathfrak{q}_0 \cup \bigcup \mathscr{S})$ such that $x + \mathfrak{q}_0^2 \in G_{\mathfrak{q}_0}(R_0)_1 \backslash \bigcup \mathscr{P}$.

In particular, $x \in \mathfrak{m}_0$ is a parameter element of R_0 which is $\mathfrak{m}_0$-filter regular with respect to M and $H^i_{R_+}(M)_n$ for all $n \ll 0$. So, x is a non zero divisor with respect to the 1-dimensional R_0-module $T_n := H^i_{R_+}(M)_n / \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)$ for all $n \ll 0$. Besides, $x \in \mathfrak{q}_0$ is a superficial element respect to $H^i_{R_+}(M)_n$ for all $n \ll 0$, so one has $j_0(\mathfrak{q}_0, T_n) = j_0\big(\mathfrak{q}_0, \frac{T_n}{xT_n}\big)$ (see [12, Lemma 3.2]). Since $\dim T_n / xT_n = 0$ we conclude that

$$\begin{aligned} j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) &= j_0(\mathfrak{q}_0, T_n) = j_0\left(\mathfrak{q}_0, \frac{T_n}{xT_n}\right) \\ &= \ell_{R_0}\left(\frac{T_n}{xT_n}\right) = \ell_{R_0}\left(\frac{H^i_{R_+}(M)_n}{xH^i_{R_+}(M)_n + \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)}\right). \end{aligned}$$

From [7, Lemma 5.6] and the exact sequence

$$0 \longrightarrow \frac{\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)}{xH^i_{R_+}(M)_n \cap \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)} \longrightarrow \frac{H^i_{R_+}(M)_n}{xH^i_{R_+}(M)_n} \longrightarrow \frac{H^i_{R_+}(M)_n}{xH^i_{R_+}(M)_n + \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)} \longrightarrow 0,$$

it suffices to show that $\ell_{R_0}(H^i_{R_+}(M)_n/xH^i_{R_+}(M)_n)$ is a numerical polynomial of degree less than i for all $n \ll 0$.

Since x is a non-zero divisor with respect to M, one has

$$\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n/xH^i_{R_+}(M)_n) \subseteq \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M/xM)_n).$$

Since $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M/xM))$ is Artinian, $\ell_{R_0}(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M/xM)_n))$ is a numerical polynomial of degree less than i [7, Lemma 5.2]. As

$$H^i_{R_+}(M)_n/xH^i_{R_+}(M)_n$$

is $\mathfrak{m}_0$-torsion for all $n \ll 0$, we obtain the statement.

COROLLARY 3.7 ([7, Theorem 5.7]). *Let* $\dim(R_0) = 2$*, and let* $\mathfrak{q}_0$ *be an* $\mathfrak{m}_0$*-primary ideal of* R_0*. If* $\dim(H^i_{R_+}(M)_n) \geq 1$ *for all* $n \ll 0$*, then there exists a numerical polynomial* $Q \in \mathbb{Q}[x]$ *of degree less than* i *such that*

$$e_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \text{for all } n \ll 0.$$

For arbitrary dimension of R_0 we obtain the following result.

THEOREM 3.8. *Assume that* R_0 *is either a finite integral extension of a domain or essentially of finite type over a field. Let* $\mathfrak{q}_0$ *be an arbitrary ideal of* R_0*. For* $i \in \mathbb{N}_0$*, suppose that* $\dim(H^i_{R_+}(M)_n) \geq \dim(R_0) - 1$ *for infinitely many integers* $n \in \mathbb{Z}$*. Then there is a polynomial* $Q \in \mathbb{Q}[x]$ *of degree less than* i *such that*

$$j_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \text{for all } n \ll 0.$$

PROOF. If $\dim(H^i_{R_+}(M)_n) = \dim(R_0)$, the proof follows analogously to the first step of Theorem 3.6. If $\dim(H^i_{R_+}(M)_n) = \dim(R_0) - 1$, by [7, Theorem 5.9] there is a non-empty set $\mathscr{W} \in \operatorname{Spec}(R_0)$ with $\operatorname{Assh}(H^i_{R_+}(M)_n) = \mathscr{W}$ for all $n \ll 0$. So, for every $\mathfrak{p}_0 \in \mathscr{W}$ one has $\dim(R_0/\mathfrak{p}_0) = \dim(R_0) - 1$, so that $\dim((R_0)_{\mathfrak{p}_0}) \leq 1$.

By the Flat Base Change Theorem, we obtain

$$H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \cong (H^i_{R_+}(M)_n)_{\mathfrak{p}_0},$$

which is an $(R_0)_{\mathfrak{p}_0}$-module of finite length. In this way,

$$H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n = \Gamma_{\mathfrak{p}_0}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n)$$

for all $n \ll 0$. By [3, Theorem 3.5 (d)], there exists a numerical polynomial $P \in \mathbb{Q}[X]$ such that $\deg(P) < i$ and $\ell_{(R_0)_{\mathfrak{p}_0}}(\Gamma_{\mathfrak{p}_0}(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n)) = P(n)$ for all $n \ll 0$. The result follows by the associativity formula for j-multiplicity [12].

COROLLARY 3.9 ([7, Theorem 5.9]). *Let $i \in \mathbb{N}_0$ and assume that R_0 is either a finite integral extension of a domain or essentially of finite type over a field. Suppose that* $\dim(H^i_{R_+}(M)_n) \geq \dim(R_0) - 1$ *for infinitely many integers $n \in \mathbb{Z}$ and let $\mathfrak{q}_0$ be an $\mathfrak{m}_0$-primary ideal of R_0. Then, there is a polynomial $Q \in \mathbb{Q}[x]$ of degree less than i such that*

$$e_0(\mathfrak{q}_0, H^i_{R_+}(M)_n) = Q(n) \qquad \textit{for all } n \ll 0.$$

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