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SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In the paper we study the question of the solvability and unique solvability of systems of the higher order differential equations with the argument deviations

$$u_i^{(m_i)}(t) = p_i(t)u_{i+1}(\tau_i(t)) + q_i(t), \quad (i = \overline{1, n}), \quad \text{for } t \in I := [a, b],$$

and

$$u_i^{(m_i)}(t) = F_i(u)(t) + q_{0i}(t), \quad (i = \overline{1, n}), \quad \text{for } t \in I,$$

under the conjugate

$$u_i^{(j_1-1)}(a) = a_{ij_1}, \quad u_i^{(j_2-1)}(b) = b_{ij_2}, \quad j_1 = \overline{1, k_i}, \quad j_2 = \overline{1, m_i - k_i}, \quad i = \overline{1, n},$$

and the right-focal

$$u_i^{(j_1-1)}(a) = a_{ij_1}, \quad u_i^{(j_2-1)}(b) = b_{ij_2}, \quad j_1 = \overline{1, k_i}, \quad j_2 = \overline{k_i + 1, m_i}, \quad i = \overline{1, n},$$

boundary conditions, where $u_{n+1} = u_1$, $n \geq 2$, $m_i \geq 2$, $p_i \in L_\infty(I; R)$, $q_i, q_{0i} \in L(I; R)$, $\tau_i \colon I \to I$ are the measurable functions, F_i are the local Caratheodory's class operators, and k_i is the integer part of the number $m_i/2$.

In the paper are obtained the efficient sufficient conditions that guarantee the unique solvability of the linear problems and take into the account explicitly the effect of argument deviations, and on the basis of these results are proved new conditions of the solvability and unique solvability for the nonlinear problems.

1. Introduction

Consider on the interval $I = [a, b]$ the systems of the higher order linear differential equations with argument deviations

$$u_i^{(m_i)}(t) = p_i(t)u_{i+1}(\tau_i(t)) + q_i(t), \quad (i = \overline{1, n}), \tag{1.1}$$

and the system of the higher order nonlinear functional differential equations

$$u_i^{(m_i)}(t) = F_i(u)(t) + q_{0i}(t), \quad (i = \overline{1, n}), \tag{1.2}$$

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where $n \geq 2$, $m_i \geq 2$, $u_{n+1} := u_1$, $u := (u_i)_{i=1}^n$, $q_i, q_{0i} \in L(I; R)$, $p_i \in L_\infty(I, R)$, $\tau_i: I \to I$ are measurable functions, and $F_i \in K(C^{m_1,\dots,m_n}, L_\infty)$ (See Definitions 1.1 below).

In this work we study systems (1.1) and (1.2) under the conjugate

$$\begin{aligned} &u_i^{(j_1-1)}(a) = a_{ij_1}, \quad u_i^{(j_2-1)}(b) = b_{ij_2}, \\ &j_1 = \overline{1, k_i}, \quad j_2 = \overline{1, m_i - k_i}, \quad i = \overline{1, n}, \end{aligned} \tag{1.3$_1$}$$

and the right-focal

$$\begin{aligned} &u_i^{(j_1-1)}(a) = a_{ij_1}, \quad u_i^{(j_2-1)}(b) = b_{ij_2}, \\ &j_1 = \overline{1, k_i}, \quad j_2 = \overline{k_i + 1, m_i}, \quad i = \overline{1, n}, \end{aligned} \tag{1.3$_2$}$$

boundary conditions, where $k_i := [m_i/2]$ is the integer part of the number $m_i/2$.

Throughout the paper we use the following notations: N is the set of the natural numbers; $R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$; R^n is the space of the nth dimensional column vectors $x := (x_i)_{i=1}^n$ with the components $x_i \in R$, $(i = \overline{1, n})$, and the norm $\|x\| = \sum_{i=1}^n |x_i|$; $C_1^{n-1}(I; R)$ is the Banach space of the functions $u: I \to R$ which are continuous together with their $(n-1)th$ derivatives, with the norm

$$\|u\|_{C_1^{n-1}} = \max\left\{\sum_{i=1}^n |u^{(i-1)}(t)| : t \in I\right\};$$

$C^{m_1,\dots,m_n}(I; R^n)$ is the Banach space of the vector-functions $u := (u_i)_{i=1}^n: I \to R^n$, where $u_i \in C_1^{m_i-1}(I; R)$, $(i = \overline{1, n})$, with the norm

$$\|u\|_{C^{m_1,\dots,m_n}} = \max\left\{\sum_{i=1}^n \sum_{j=1}^{m_i} |u_i^{(j-1)}(t)| : t \in I\right\},$$

and for the case when $m_i = 1$, $(i = \overline{1, n})$, we will use the notations

$$C_n^0(I; R^n) := C^{1,\dots,1}(I; R^n), \quad \|u\|_{C_n^0} := \|u\|_{C^{1,\dots,1}};$$

$\widetilde{C}^{n-1}(I; R)$ is the set of the functions $u: I \to R$ which are absolutely continuous together with their $(n-1)$th derivatives; $L(I; R)$ is the Banach space of the Lebesgue integrable functions $p: I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|\, ds$; $L_\infty(I; R)$ is the space of the essentially bounded measurable functions $p: I \to R$ with the norm $\|p\|_\infty = \operatorname{ess\,sup}\{|p(t)| : t \in I\}$; $L_\infty(I; R^n)$ is the space of the n-dimensional vector-functions $p := (p_i)_{i=1}^n: I \to R^n$, with $p_i \in L_\infty(I; R)$,

$(i = \overline{1,n})$; $M^n(I)$ is the set of the vector-functions $\tau := (\tau_i)_{i=1}^n : I \to I^n$, with the measurable components $\tau_i : I \to I$, $(i = \overline{1,n})$. Also we define the functional $\rho : L(I; R) \times M^1(I) \to R_+$ by the equality

$$\rho(y,\tau) = \frac{2\pi}{b-a}\left(\int_a^b |y(s)||\tau(s) - s|\, ds\right)^{1/2},$$

where if $y \in L(I; R)$ and $\tau \in M^1(I)$, and for an arbitrary $x \in R$ we assume that

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

By a solution of problem (1.2), (1.3_r), $r \in \{1,2\}$, we understand a vector-function $u := (u_i)_{i=1}^n$ where $u_i \in \widetilde{C}^{m_i-1}(I; R)$, $(i = \overline{1,n})$, which satisfies system (1.2) almost everywhere on I and satisfies conditions (1.3_r).

DEFINITION 1.1. We will say that the operator $F : C^{m_1,\dots,m_n}(I; R^n) \to L_\infty(I; R)$ belongs to the Caratheodory's local class $K(C^{m_1,\dots,m_n}, L_\infty)$, if F is continuous operator, and for an arbitrary $r > 0$ the inclusion

$$\sup\big\{|F(x)(t)| : \|x\|_{C^{m_1,\dots,m_n}} \leq r,\, x \in C^{m_1,\dots,m_n}(I; R^n)\big\} \in L_\infty(I; R_+),$$

holds.

The boundary value problems for the functional differential equations attract the attention of many mathematicians and are intensively studied. At present the foundation of the general theory of such kind of problems are already laid and many of them are investigated in details (see [1]–[11], [15]–[22], and references therein). But many of mentioned results have something in common, for such an important class of functional differential equations as differential equations with argument deviations, the effect of argument deviations can not be taken into consideration explicitly. Also, there remains a wide class of boundary value problems on the solvability of which not much is known. Among them are the two point boundary value problems systems of higher order differential equations with argument deviations or in general of higher order functional differential equations.

The aim of our paper is a generalisation for the problems (1.1), (1.3_1) and (1.1), (1.3_2) of the results of paper [12] of I. Kiguradze and T. Kusano from 1999. In that article (see also [13]) the authors notice that if p is the constant sign function, then the periodic problem

$$u^{(n)}(t) = p(t)u(t) + q(t), \quad u^{(i-1)}(b) = u^{(i-1)}(a), \quad (i = 1,\dots,n),$$

is uniquely solvable if n is odd, or $n = 2m$ and $(-1)^m p(t) \leq 0$, but for the case $n = 2m$, when $(-1)^m p(t) \geq 0$, $p \not\equiv 0$, they proved that (See

Theorem 1.1 in [5]) the mentioned periodic problem is uniquely solvable if one of the following two conditions holds:

$$\int_a^b |p(s)|\,ds \le \frac{2}{b-a}\left(\frac{2\pi}{b-a}\right)^{n-2} \quad \text{or} \quad |p(t)| \le \left(\frac{2\pi}{b-a}\right)^{n} \quad \text{a.e. on } I, \tag{1.4}$$

where $|p| \not\equiv (2\pi/(b-a))^n$. The integral conditions of (1.4) still are not sharp, while the pointwise conditions of (1.4) are optimal (see Remark 1.1 in [12]).

Our generalisations of the mentioned results explicitly take into consideration argument deviations, and, using them, we find the efficient sufficient conditions of solvability and unique solvability for nonlinear problems (1.2), (1.3_1) and (1.2), (1.3_2), when system (1.2) in some sense is close to the mentioned linear system.

2. Main results

2.1. Linear problems

In this section we consider the question of the unique solvability of problems (1.1), (1.3_1) and (1.1), (1.3_2).

THEOREM 2.1. *Let the functions $\tau \in M^n(I)$, and $p \in L_\infty(I; R^n)$, be such that the conditions*

$$0 \le \sigma_i p_i(t), \quad (i = \overline{1,n}), \quad \text{for } t \in I, \tag{2.1}$$

where $\sigma_i \in \{-1, 1\}$, and

$$\prod_{i=1}^{n}\left(2\|p_i\|_\infty + \rho(p_i,\tau_i)\|p_i\|_\infty^{1/2}\right) < 4^{\sum_{i=1}^n k_i}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^n m_i}, \tag{2.2$_1$}$$

hold. Then problem (1.1), (1.3_1) *is uniquely solvable.*

THEOREM 2.2. *Let the functions $\tau \in M^n(I)$, and $p \in L_\infty(I; R^n)$ be such that conditions* (2.1)*, and*

$$\prod_{i=1}^{n}\left(4\|p_i\|_\infty + \rho(p_i,\tau_i)\|p_i\|_\infty^{1/2}\right) < 4^{n}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^n m_i}, \tag{2.2$_2$}$$

hold. Then problem (1.1), (1.3_2) *is uniquely solvable.*

From Theorems 2.1 and 2.2 follows

COROLLARY 2.3. *Let the functions $\tau \in M^n(I)$, $p \in L_\infty(I; R^n)$, and $\alpha_i \in L(I; R_+)$, $(i = \overline{1,n})$, be such that conditions* (2.1)*,*

$$\|p_i\|_\infty |p_i(t)||\tau_i(t) - t| \le \alpha_i(t), \quad (i = \overline{1,n}), \quad \text{a.e on } I, \tag{2.3}$$

and

$$\prod_{i=1}^{n}\left[\|p_i\|_\infty + \frac{\pi}{b-a}\|\alpha_i\|_L^{1/2}\right] < 2^{2\sum_{i=1}^{n} k_i - n}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^{n} m_i}$$

hold. Then problem (1.1)*,* (1.3_1) *is uniquely solvable.*

COROLLARY 2.4. *Let the functions* $\tau \in M^n(I)$*,* $p \in L_\infty(I; R^n)$*, and* $\alpha_i \in L(I; R_+)$*,* $(i = \overline{1, n})$*, be such that conditions* (2.1)*,* (2.3)*, and*

$$\prod_{i=1}^{n}\left[2\|p_i\|_\infty + \frac{\pi}{b-a}\|\alpha_i\|_L^{1/2}\right] < 2^{n}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^{n} m_i},$$

hold. Then problem (1.1)*,* (1.3_2) *is uniquely solvable.*

From the last corollaries with $\alpha_i \equiv 0$ immediately follows

COROLLARY 2.5. *Let the functions* $\tau \in M^n(I)$ *and* $p \in L_\infty(I; R^n)$ *be such that conditions* (2.1)*,*

$$|p_i(t)||\tau_i(t) - t| = 0, \quad (i = \overline{1, n}), \quad a.e \text{ on } I \tag{2.4}$$

and

$$\prod_{i=1}^{n}\|p_i\|_\infty < 2^{2\sum_{i=1}^{n} k_i - n}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^{n} m_i}$$

hold. Then problem (1.1)*,* (1.3_1) *is uniquely solvable.*

COROLLARY 2.6. *Let the functions* $\tau \in M^n(I)$ *and* $p \in L_\infty(I; R^n)$ *be such that conditions* (2.1)*,* (2.4)*, and*

$$\prod_{i=1}^{n}\|p_i\|_\infty < \left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^{n} m_i}$$

hold. Then problem (1.1)*,* (1.3_2) *is uniquely solvable.*

For the illustration of Theorems 2.1 and 2.2 we consider here the system

$$u_1''(t) = p_1(t)u_2(t) + q_1(t), \quad u_2''(t) = p_2(t)u_1(\tau(t)) + q_2(t), \tag{2.5}$$

with $p_1, p_2 \in L_\infty(I; R)$ and $\tau \in M^1(I)$, under boundary conditions (1.3_1) and (1.3_2), and the equation

$$u^{(m)}(t) = p(t)u(\tau(t)) + q(t), \tag{2.6}$$

where $m = 2k + 2$, $(k \in N)$, under boundary conditions

$$u^{(2j-2)}(a) = c_{1j}, \quad u^{(2j-2)}(b) = c_{2j}, \quad j = \overline{1, k+1}, \tag{2.7$_1$}$$

and

$$u^{(2j-2)}(a) = c_{1j}, \quad u^{(2j-1)}(b) = c_{2j}, \quad j = \overline{1, k+1}. \tag{2.7$_2$}$$

Then from Theorems 2.1 and 2.2 immediately follow the corollaries

COROLLARY 2.7. *Let $r \in \{1, 2\}$, the functions $p_1, p_2 \in L_\infty(I; R)$, and $\tau \in M^1(I)$, be such that conditions* (2.1) *with $n = 2$, and*

$$\|p_1\|_\infty\left(r\|p_2\|_\infty + \frac{\pi}{b-a}\big(\|p_2\|_\infty \int_a^b |p_2(s)||\tau(s) - s|\,ds\big)^{1/2}\right) < \frac{1}{2^{r+1}}\left(\frac{\pi}{b-a}\right)^4,$$

hold. Then problem (2.5), (1.3_r) *is uniquely solvable.*

COROLLARY 2.8. *Let $r \in \{1, 2\}$, and the functions $p \in L_\infty(I; R_+)$, $\tau \in M^1(I)$, be such that the conditions*

$$0 \le \sigma p(t) \quad \textit{for } t \in I,$$

where $\sigma \in \{-1, 1\}$, and

$$r\|p\|_\infty + \frac{\pi}{b-a}\left(\|p\|_\infty \int_a^b |p(s)||\tau(s) - s|\,ds\right)^{1/2} < 2^{(2-r)k+1}\left(\frac{\pi}{2(b-a)}\right)^m,$$

hold. Then problem (2.6), (2.7_r) *is uniquely solvable.*

2.2. Nonlinear problems

For the formulation of our main result we need the following definitions:

DEFINITION 2.9. Let $\tau \in M^n(I)$, then we will say that

$$h \in D_{1,\tau}(I), \tag{2.8$_1$}$$

if $h \in L_\infty(I; R^n_+)$, and for an arbitrary $p \in L_\infty(I; R^n)$, such that

$$0 \le \sigma_i p_i(t) \le h_i(t)(i = \overline{1, n}) \quad \text{for } t \in I, \tag{2.9}$$

where $\sigma_i \in \{-1, 1\}$, $(i = \overline{1, n})$, the homogeneous system

$$u_i^{(m_i)}(t) = p_i(t)u_{i+1}(\tau_i(t))(i = \overline{1, n}), \tag{2.10}$$

where $u_{n+1} = u_1$, under the boundary conditions

$$\begin{gathered} u_i^{(j_1-1)}(a) = 0, \quad u_i^{(j_2-1)}(b) = 0, \\ j_1 = \overline{1, k_i}, \quad j_2 = \overline{1, m_i - k_i}, \quad i = \overline{1, n}, \end{gathered} \tag{2.11$_1$}$$

has no nontrivial solution.

DEFINITION 2.10. Let $\tau \in M^n(I)$, then we will say that

$$h \in D_{2,\tau}(I), \tag{2.8$_2$}$$

if $h \in L_\infty(I; R_+^n)$ and for an arbitrary $p \in L_\infty(I; R^n)$, for which conditions (2.9) holds, the homogeneous system (2.10) under the boundary conditions

$$\begin{gathered} u_i^{(j_1-1)}(a) = 0, \quad u_i^{(j_2-1)}(b) = 0, \\ j_1 = \overline{1, k_i}, \quad j_2 = \overline{k_i + 1, m_i}, \quad i = \overline{1, n}, \end{gathered} \tag{2.11$_2$}$$

has no nontrivial solution.

THEOREM 2.11. *Let $r \in \{1, 2\}$ and the functions $\tau \in M^n(I)$, $h \in L_\infty(I; R^n)$, be such that inclusion (2.8_r) holds. Moreover, let the continuous operators $V_{1i}, V_{2i}\colon C^{m_1,\dots,m_n}(I; R^n) \to L_\infty(I; R_+)$, and the numbers $\sigma_i \in \{-1, 1\}$, $(i = \overline{1, n})$, $r_0 > 0$, be such that the conditions*

$$\begin{aligned} V_{2i}(x)(t) \le \min\{V_{1i}(x)(t),\ h_i(t) - V_{1i}(x)(t) : t \in I\} \\ \text{for } \|x\|_{C^{m_1,\dots,m_n}} \ge r_0,\ t \in I,\ (i = \overline{1, n}), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |F_i(x)(t) - \sigma_i V_{1i}(x)(t) x_{i+1}(\tau_i(t))| \le V_{2i}(x)(t)|x_{i+1}(\tau_i(t))| \\ + \eta_i(t, \|x\|_{C^{m_1,\dots,m_n}}) \quad \text{for } \|x\|_{C^{m_1,\dots,m_n}} \ge r_0,\ t \in I,\ (i = \overline{1, n}), \end{aligned} \tag{2.13}$$

with $x := (x_i)_{i=1}^n \in C^{m_1,\dots,m_n}(I; R^n)$, $x_{n+1} = x_1$, hold, where the functions $\eta_i\colon I \times R_+ \to R_+$ are summable in the first argument, nondecreasing in the second one, and admit to the conditions

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \eta_i(s, \rho)\, ds = 0, \quad (i = \overline{1, n}). \tag{2.14}$$

Then problem (1.2), (1.3_r) has at least one solution.

COROLLARY 2.12. *Let the number $r_0 > 0$, the functions $\tau \in M^n(I)$, $h \in L_\infty(I; R_+^n)$, and the continuous operators*

$$V_{1i}, V_{2i}\colon C^{m_1,\dots,m_n}(I; R^n) \longrightarrow L_\infty(I; R_+)$$

be such that the conditions

$$\prod_{i=1}^{n}\big(2\|h_i\|_\infty+\rho(h_i,\tau_i)\|h_i\|_\infty^{1/2}\big) < 4^{\sum\limits_{i=1}^{n}k_i}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^{n}m_i}, \qquad (2.15_1)$$

and (2.12)–(2.14) *hold, where the functions $\eta_i : I\times R_+\to R_+$ are summable in the first argument and nondecreasing in the second one. Then problem* (1.2), (1.3_1) *has at least one solution.*

COROLLARY 2.13. *Let the number $r_0>0$, the functions $\tau\in M^n(I)$, $h\in L_\infty(I;R_+^n)$, and the continuous operators*

$$V_{1i},V_{2i} : C^{m_1,\dots,m_n}(I;R^n)\longrightarrow L_\infty(I;R_+)$$

be such that the conditions

$$\prod_{i=1}^{n}\big(4\|h_i\|_\infty+\rho(h_i,\tau_i)\|h_i\|_\infty^{1/2}\big) < 4^{n}\left(\frac{\pi}{2(b-a)}\right)^{\sum_{i=1}^{n}m_i}, \qquad (2.15_2)$$

and (2.12)–(2.14) *hold, where the functions $\eta_i\colon I\times R_+\to R_+$ are summable in the first argument and nondecreasing in the second one. Then problem* (1.2), (1.3_2) *has at least one solution.*

For the case when (1.2) is the system of the higher order differential equations with the argument deviations of the form

$$u_i^{(m_i)}(t)=f_i(t,u_{i+1}(\tau_i(t)))+q_{0i}(t),\quad (i=\overline{1,n}), \qquad (2.16)$$

where $u_{n+1}=u_1$, and $f_i : I\times R\to R$, $(i=\overline{1,n})$, are the functions from the Caratheodory's class, the following Corollaries are true

COROLLARY 2.14. *Let $r\in\{1,2\}$, and the functions $\tau\in M^n(I)$, $h\in L_\infty(I;R_+^n)$, be such that inclusion (2.8_r) holds. Moreover, let the Caratheodory's class functions $f_{1i}, f_{2i}\colon I\times R\to R_+$, $(i=\overline{1,n})$, and the numbers $\sigma_i\in\{-1,1\}$, $(i=\overline{1,n})$, $r_0>0$, be such that the conditions*

$$f_{2i}(t,x)\le\min\{f_{1i}(t,x),h_i(t)-f_{1i}(t,x):t\in I\}\quad \textit{for}\ x\in R, t\in I, \qquad (2.17)$$

and

$$|f_i(t,x)-\sigma_i f_{1i}(t,x)x|\le f_{2i}(t,x)|x|+\eta_i(t,|x|)\quad \textit{for}\ |x|\ge r_0, t\in I, \qquad (2.18)$$

hold for all $i\in\{\overline{1,n}\}$, where the functions $\eta_i\colon I\times R_+\to R_+$ are summable in the first argument, nondecreasing in the second one, and satisfy the conditions (2.14). *Then problem* (2.16), (1.3_r) *has at least one solution.*

EXAMPLE 2.15. As example consider the system of the differential equations

$$u_i^{(m_i)}(t) = \sigma_i h_i(t) \sin \|u\|_{C^{m_1,\dots,m_n}} u_{i+1}(\tau_i(t)) + q_{0i}(t), \quad (i = \overline{1,n}), \quad (2.19)$$

where $\sigma \in \{-1, 1\}$, $u_{n+1} = u_1$, and the functions $\tau \in M^n(I)$, $h \in L_\infty(I, R_+^n)$, are such that inclusion (2.8_r) holds where $r \in \{1, 2\}$. Then from Theorem 2.11 with $r_0 = 0$, $\eta_i \equiv 0$, and $V_{1i}(u)(t) = h_i(t) \sin \|u\|_{C^{m_1,\dots,m_n}}$, $V_{2i} \equiv 0$, it follows the solvability of problem (2.19), (1.3_r).

THEOREM 2.16. *Let $r \in \{1, 2\}$, and the functions $\tau \in M^n(I)$, $h \in L_\infty(I; R_+^n)$, be such that inclusion* (2.8_r) *holds. Moreover, let the numbers $\sigma_i \in \{-1, 1\}$ be such that the conditions*

$$0 \le \sigma_i(f_i(t,x) - f_i(t,y)) \operatorname{sgn}(x-y) \le h_i(t)|x-y|$$
$$\text{for } x, y \in R,\ t \in I,\ (i = \overline{1,n}), \quad (2.20)$$

hold, where the functions $\eta_i \colon I \times R_+ \to R_+$ are summable in the first argument, nondecreasing in the second one, and satisfy conditions (2.14). *Then problem* (2.16), (1.3_r) *is uniquely solvable.*

From Theorem 2.16 and Corollaries 2.12, 2.13 follows

COROLLARY 2.17. *Let $r \in \{1, 2\}$, and the functions $\tau \in M^n(I)$, and $h \in L_\infty(I; R_+^n)$, be such that conditions* (2.15_r) *and* (2.20) *hold. Then problem* (2.16), (1.3_r) *is uniquely solvable.*

EXAMPLE 2.18. As example consider on I the system of the differential equations

$$u_i^{(m_i)}(t) = \sigma_i h_i(t) \frac{u_{i+1}(\tau_i(t))}{(1 + |u_{i+1}(\tau_i(t))|)^{\alpha_i}} + q_{0i}(t), \quad (i = \overline{1,n}), \quad (2.21)$$

where $\sigma \in \{-1, 1\}$, $\alpha_i \in (0, 1)$, $u_{n+1} = u_1$, and the functions $\tau \in M^n(I)$, $h \in L_\infty(I, R_+^n)$, $q_{0i} \in L(I; R^n)$, are such that inclusion (2.8_r) holds where $r \in \{1, 2\}$. Then from Theorem 2.16 follows the unique solvability of problem (2.21), (1.3_r).

3. Auxiliary propositions

First we consider Propositions 3.1–3.3 which are contained in [9] (Theorems 256–258)

PROPOSITION 3.1. *Let* $x \in \widetilde{C}^0(I; R)$, $x' \in L_\infty(I; R)$, $\int_a^b x(s)\,ds = 0$, *and* $x(a) = x(b)$, *then*

$$\int_a^b x^2(s)\,ds \le \left(\frac{b-a}{2\pi}\right)^2 \int_a^b x'^2(s)\,ds.$$

PROPOSITION 3.2 (Wirtinger). *Let* $v \in \widetilde{C}^0(I; R)$, $v' \in L_\infty(I; R)$, *and* $v(b) = v(a) = 0$, *then*

$$\int_a^b v^2(s)\,ds \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b v'^2(s)\,ds. \tag{3.1}$$

PROPOSITION 3.3. *Let* $v \in \widetilde{C}^0(I; R)$, $v' \in L_\infty(I; R)$, *and* $v(a) = 0$ *or* $v(b) = 0$, *then*

$$\int_a^b v^2(s)\,ds \le 4\left(\frac{b-a}{\pi}\right)^2 \int_a^b v'^2(s)\,ds.$$

LEMMA 3.4. *Let* $n_0 \in N$, $v \in \widetilde{C}^{n_0-1}(I; R)$, $v^{(n_0)} \in L_\infty(I; R)$, *and* $v^{(i)}(b) = v^{(i)}(a)$, $(i = \overline{0, n_0 - 1})$. *Then*

$$\int_a^b (v^{(k_0)}(s))^2\,ds \le \left(\frac{b-a}{2\pi}\right)^{2(n_0-k_0)} \int_a^b (v^{(n_0)}(s))^2\,ds, \quad (k_0 = \overline{1, n_0}). \tag{3.2}$$

Moreover if the function v *changes sign, then*

$$\int_a^b v^2(s)\,ds \le 4\left(\frac{b-a}{2\pi}\right)^{2n_0} \int_a^b (v^{(n_0)}(s))^2\,ds, \tag{3.3}$$

and if $\int_a^b v(s)\,ds = 0$, *then*

$$\int_a^b v^2(s)\,ds \le \left(\frac{b-a}{2\pi}\right)^{2n_0} \int_a^b (v^{(n_0)}(s))^2\,ds. \tag{3.4}$$

PROOF. Validity of (3.2) ((3.4)) immediately follows from Proposition 3.1 with $x = v'(x = v)$, because $v(a) = v(b)$ and therefore $\int_a^b v'(s)\,ds = 0$. Let now the function v changes the sign, then there exists the point $c \in\,]a, b[$ such that $v(c) = 0$, and if

$$x(t) = \begin{cases} v(t) & \text{for } c \le t \le b, \\ v(t-(b-a)) & \text{for } b < t \le c+(b-a), \end{cases}$$

then $x(c) = x(c+(b-a)) = 0$, $x \in \widetilde{C}([c, c+(b-a)]; R)$, $x' \in L_\infty([c, c+(b-a)]; R)$, and in view of Proposition 3.2 we get

$$\int_c^{c+(b-a)} x^2(\eta)\,d\eta \le \left(\frac{b-a}{\pi}\right)^2 \int_c^{c+(b-a)} x'^2(\eta)\,d\eta.$$

On the other hand in view of the definition of the function x we have

$$\int_b^{c+(b-a)} x^2(\eta)\,d\eta = \int_a^c v^2(s)\,ds, \qquad \int_b^{c+(b-a)} x'^2(\eta)\,d\eta = \int_a^c v'^2(s)\,ds,$$

and $v(t) = x(t)$ for $t \in [c, b]$. The last three expressions results into the inequality

$$\int_a^b v^2(s)\,ds \le 4\left(\frac{b-a}{2\pi}\right)^2 \int_a^b v'^2(s)\,ds,$$

from which by (3.2) (with $k_0 = 1$) inequality (3.3) follows.

Lemma 3.5. *Let $m \in N$, $v \in \widetilde{C}^{m-1}(I; R)$ be a nonzero function, $v^{(m)} \in L_\infty(I; R)$, $k = [m/2]$, and*

$$v^{(j_1-1)}(a) = 0, \quad v^{(j_2-1)}(b) = 0, \quad j_1 = \overline{1, k}, \quad j_2 = \overline{1, m-k}. \tag{3.5}$$

Then the following inequalities

$$\int_a^b (v^{(j)}(s))^2\,ds \le 4^{m-2k+1}\left(\frac{b-a}{\pi}\right)^{2(m-j)} \int_a^b (v^{(m)}(s))^2\,ds, \quad (j = 0, 1), \tag{3.6}$$

hold.

Proof. In view of conditions (3.5) it is clear that the function $v^{(m-k)} \in \widetilde{C}^{k-1}(I; R)$ at least $\min\{m-k, k\}$ times changes the sign in the open interval $]a, b[$, and then all the functions $v^{(m-k+r)}$, $(r = \overline{0, k-1})$, change sign at least once in $]a, b[$. Therefore for an arbitrary $r \in \{\overline{0, m-k-1}\}$ there exist $c_r \in\,]a, b[$, $\sigma_r \in \{-1, 1\}$, and the sets of the positive measure $I_{1r} \subset\,]a, c_r[$, $I_{2r} \subset\,]c_r, b[$, such that

$$v^{(k+r)}(c_r) = 0 \quad \text{and} \quad (-1)^i \sigma_r v^{(k+r)}(t) > 0 \quad \text{for } t \in I_{ir}, (i = 1, 2).$$

Then for an arbitrary $r \in \{\overline{0, n-k-1}\}$, due to Proposition 3.3 we get the inequalities

$$\int_a^{c_r} (v^{(k+r)}(s))^2\,ds \le 4\left(\frac{c_r - a}{\pi}\right)^2 \int_a^{c_r} (v^{(k+r+1)}(s))^2\,ds,$$

$$\int_{c_r}^{b} (v^{(k+r)}(s))^2\,ds \le 4\left(\frac{b - c_r}{\pi}\right)^2 \int_{c_r}^{b} (v^{(k+r+1)}(s))^2\,ds,$$

from which it is clear that

$$\int_a^{b} (v^{(k+r)}(s))^2\,ds \le 4\left(\frac{b - a}{\pi}\right)^2 \int_a^{b} (v^{(k+r+1)}(s))^2\,ds, \quad (r = \overline{0, m-k-1}).$$

Now by iteration of the least inequalities we obtain

$$\int_a^{b} (v^{(k)}(s))^2\,ds \le 4^{m-k}\left(\frac{b - a}{\pi}\right)^{2(m-k)} \int_a^{b} (v^{(m)}(s))^2\,ds. \tag{3.7}$$

On the other hand, from conditions (3.5) follows that inequalities (3.1) and (3.2) with $n_0 = k$ and $k_0 = 1$ hold. Now if we substitute inequality (3.7) in (3.2) with ($n_0 = k$ and $k_0 = 1$) we get that (3.6) for $j = 1$ is valid. Also if we substitute (3.6) with $j = 1$ in (3.1) we obtain (3.6) for $j = 0$.

Also note that from Proposition 3.3 follows

Lemma 3.6. *Let $m \in N$, $v \in \widetilde{C}^{m-1}(I; R)$, $v^{(m)} \in L_\infty(I; R)$, $k = [m/2]$, and*

$$v^{(j_1-1)}(a) = 0, \quad v^{(j_2-1)}(b) = 0, \quad j_1 = \overline{1, k}, \quad j_2 = \overline{k+1, m}.$$

Then the following inequalities

$$\int_a^{b} (v^{(j)}(s))^2\,ds \le 4^{m-j}\left(\frac{b - a}{\pi}\right)^{2(m-j)} \int_a^{b} (v^{(m)}(s))^2\,ds, \quad (j = 0, 1),$$

hold.

Now for an arbitrary $\lambda \in (0, 1)$, and the function $h \in L_\infty(I; R_+^n)$, consider the systems of the nonlinear functional differential equations

$$u_i^{(m_i)}(t) = \lambda\sigma_i h_i(t) u_{i+1}(\tau_i(t)) + (1-\lambda)[F_i(u)(t) + q_{0i}(t)], \quad (i = \overline{1, n}), \tag{3.8}$$

where $u := (u_i)_{i=1}^n$, $u_{n+1} = u_1$, and $\sigma_i \in \{-1, 1\}$, $(i = \overline{1, n})$, under the boundary conditions

$$\begin{aligned} &u_i^{(j_1-1)}(a) = \lambda a_{ij_1}, \quad u_i^{(j_2-1)}(b) = \lambda b_{ij_2}, \\ &j_1 = \overline{1, k_i}, \quad j_2 = \overline{1, m_i - k_i}, \quad i = \overline{1, n}, \end{aligned} \tag{3.9$_1$}$$

and

$$\begin{aligned} &u_i^{(j_1-1)}(a) = \lambda a_{ij_1}, \quad u_i^{(j_2-1)}(b) = \lambda b_{ij_2}, \\ &j_1 = \overline{1, k_i}, \quad j_2 = \overline{k_i + 1, m_i}, \quad i = \overline{1, n}, \end{aligned} \tag{3.9$_2$}$$

with $k_i = [m_i/2]$.

It is not difficult to verify that the following proposition is true.

PROPOSITION 3.7. *Let $r \in \{-1, 1\}$, $m = \sum_{j=1}^n m_j$, $\alpha_0 = 0$, and $\alpha_i = \sum_{j=1}^i m_j (i = \overline{1, n})$. Then for an arbitrary $\lambda \in (0, 1)$, problems* (3.8), (3.9$_1$), *and* (2.10), (2.11$_1$) *are equivalent respectively to the problems*

$$w_i'(t) = \lambda\widetilde{\sigma}_i\widetilde{h}_i(t)w_{i+1}(\tau_i(t)) + (1 - \lambda)[\widetilde{F}_i(w)(t) + \widetilde{g}_{0i}(t)], \quad (i = \overline{1, m}),$$

$$w_{\alpha_{i-1}+j_1+1}(a) = \lambda\widetilde{a}_{ij_1}, \quad w_{\alpha_{i-1}+j_2+1}(b) = \lambda\widetilde{b}_{ij_2},$$

$$j_1 = \overline{0, k_i - 1}, \quad j_2 = \overline{0, m_i - k_i - 1}, \quad i = \overline{1, n},$$

and

$$w_i'(t) = \widetilde{p}_i(t)w_{i+1}(\tau_i(t)),$$

$$w_{\alpha_{i-1}+j_1+1}(a) = 0, \quad w_{\alpha_{i-1}+j_2+1}(b) = 0,$$

$$j_1 = \overline{0, k_i - 1}, \quad j_2 = \overline{0, m_i - k_i - 1}, \quad i = \overline{1, n},$$

where $w := (w_j)_{j=1}^m$, $w_{m+1} = w_1 = u_1$, and for all $i \in \overline{1, n}$ the following identities hold

$$\widetilde{a}_{ij_1} = a_{ij_1}, \quad \widetilde{b}_{ij_2} = b_{ij_2} \quad (j_1 = \overline{0, k_i - 1}, \quad j_2 = \overline{0, m_i - k_i - 1}),$$

$$w_{\alpha_{i-1}+1} \equiv u_i, \quad \widetilde{h}_{\alpha_i} \equiv h_i, \quad \widetilde{g}_{0\alpha_i} \equiv q_{0i},$$

$$\widetilde{\sigma}_{\alpha_i} \equiv \sigma_i, \quad \widetilde{p}_{\alpha_i} \equiv p_i, \quad \widetilde{F}_{\alpha_i}(w) \equiv F_i(u),$$

and

$$\sigma_j\widetilde{h}_j \equiv 1, \quad \sigma_j\widetilde{p}_j \equiv 1, \quad \widetilde{F}_j(w) \equiv w_{j+1}, \quad \tilde{g}_{0j} \equiv 0,$$

if $j \notin \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Moreover, it is clear that $\|w\|_{C_m^0} = \|u\|_{C^{m_1,\dots,m_n}}$.

REMARK 3.8. It is not difficult to verify that the propositions analogous to Proposition 3.7 are valid for problems (3.8), (3.9$_2$), and (2.10), (2.11$_2$).

Taking into account Proposition 3.7 and Remark 3.8 we get the following modifications of Corollary 2 of paper [15], which is formulated for the system of the first order equations.

PROPOSITION 3.9. *Let $r \in \{1, 2\}$, the functions $\tau \in M^n(I)$, and $h \in L_\infty(I; R^n)$, be such that inclusion (2.8_r) holds, and there exists a positive number ρ_1 such, that for an arbitrary $\lambda \in (0, 1)$ every solution u of problem* (3.8), (3.9_r) *admits to the estimate*

$$\|u\|_{C^{m_1,\dots,m_n}} \le \rho_1. \tag{3.10}$$

Then problem (1.2), (1.3_r), *has at least one solution.*

LEMMA 3.10. *Let $r \in \{1, 2\}$ and the functions $\tau \in M^n(I)$, $h \in L_\infty(I; R^n)$, be such that inclusion (2.8_r) holds. Then there exist a positive number $\rho_0 > 0$, such that an arbitrary solution $u = (u_i)_{i=1}^n$ of problem* (2.10), (1.3_r) *admits to the estimate*

$$\|u\|_{C^{m_1,\dots,m_n}} \le \rho_0 \sum_{i=1}^{n} (\gamma_{ri} + \|q_i\|_L), \tag{3.11$_r$}$$

where $\gamma_{1i} = \sum_{j=1}^{k_i} |a_{ij}| + \sum_{j=1}^{m_i-k_i} |b_{ij}|$, and $\gamma_{2i} = \sum_{j=1}^{k_i} |a_{ij}| + \sum_{j=k_i+1}^{m_i} |b_{ij}|$, provided that inequalities (2.9) *hold, for given $\sigma_i \in \{-1, 1\}$, $(i = \overline{1, n})$.*

To prove this lemma, we need Lemma 3.11, which can be proved analogously as Lemma 1.1 of [14], in which $z, z_k \in C^0(I; R)$, $(k \in N)$.

LEMMA 3.11. *Let $y, y_k \in L(I; R)$, $z, z_k \in L_\infty(I; R)$, $(k \in N)$,*

$$\lim_{k\to+\infty} \|z_k - z\|_\infty = 0, \quad \limsup_{k\to+\infty} \|y_k\|_L < +\infty,$$

and $\lim_{k\to+\infty} \int_a^t y_k(s)\,ds = \int_a^t y(s)\,ds$ uniformly on I. Then

$$\lim_{k\to+\infty} \int_a^t y_k(s) z_k(s)\,ds = \int_a^t y(s) z(s)\,ds \quad \textit{uniformly on } I.$$

PROOF. Assume that the Lemma 3.10 is not true for $r = 1$. Then for all $k \in N$ there exist the functions $p_k := (p_{ik})_{i=1}^n$, $q_k := (q_{ik})_{i=1}^n \in L_\infty(I; R^n)$, and the numbers $a_{ij_1k}, b_{ij_2k} \in R$, such that

$$0 \le \sigma_i p_{ik}(t) \le h_i(t), \int_a^b p_{ik}(s)\,ds \ne 0, \quad (i = \overline{1, n}), \tag{3.12}$$

and the problem

$$u_{ik}^{(m_i)}(t) = p_{ik}(t)u_{i+1k}(\tau_i(t)) + q_{ik}(t), \quad (i = \overline{1,n}),$$
$$u_{ik}^{(j_1-1)}(a) = a_{ij_1k}, \quad u_{ik}^{(j_2-1)}(b) = b_{ij_2k},$$
$$j_1 = \overline{1,k_i}, \quad j_2 = \overline{1,m_i-k_i}, \quad i = \overline{1,n},$$

where $u_{n+1k} = u_{1k}$, has such a solution $u_k := (u_{ik})_{i=1}^n$ that

$$\|u_k\|_{C^{m_1,\dots,m_n}} \geq k\sum_{i=1}^{n}(\gamma_{1ik} + \|q_{ik}\|_L),$$

where $\gamma_{1ik} = \sum_{j=1}^{k_i}|a_{ijk}| + \sum_{j=1}^{m_i-k_i}|b_{ijk}|$. Then if we suppose that

$$v_{ik}(t) = u_{ik}(t)/\|u_k\|_{C^{m_1,\dots,m_n}}, \quad \widetilde{q}_{ik}(t) = \sum_{i=1}^{n} q_{ik}(t)/\|u_k\|_{C^{m_1,\dots,m_n}},$$
$$\widetilde{a}_{ij_1k} = a_{ij_1k}/\|u_k\|_{C^{m_1,\dots,m_n}}, \quad \widetilde{b}_{ij_2k} = b_{ij_1k}/\|u_k\|_{C^{m_1,\dots,m_n}},$$
$$\widetilde{\gamma}_{ik} = \gamma_{1ik}/\|u_k\|_{C^{m_1,\dots,m_n}},$$

and $v_k := (v_{ik})_{i=1}^n$, we obtain

$$\|v_k\|_{C^{m_1,\dots,m_n}} = 1, \sum_{i=1}^{n}(\widetilde{\gamma}_{ik} + \|\widetilde{q}_{ik}\|_L) \leq \frac{1}{k}, \tag{3.13}$$

and the equalities

$$\begin{gathered} v_{ik}^{(m_i)}(t) = p_{ik}(t)v_{i+1k}(\tau_i(t)) + \widetilde{q}_{ik}(t), \quad (i = \overline{1,n}), \quad \text{for } t \in I, \\ v_{ik}^{(j_1-1)}(a) = \widetilde{a}_{ij_1k}, \quad v_{ik}^{(j_2-1)}(b) = \widetilde{b}_{ij_2k}, \\ j_1 = \overline{1,k_i}, \quad j_2 = \overline{1,m_i-k_i}, \quad i = \overline{1,n}, \end{gathered} \tag{3.14}$$

where $v_{n+1k} = v_{1k}$, from which due to (3.12) and (3.13) we have

$$|v_{ik}^{(m_i)}(t)| \leq h_i(t) + |\widetilde{q}_{ik}(t)|, \quad (i = \overline{1,n}), \quad \text{for } t \in I. \tag{3.15}$$

According to equalities $\|v_k\|_{C^{m_1,\dots,m_n}} = 1$, conditions (3.14), and (3.15) for an arbitrarily fixed $i_0 \in \{\overline{1,n}\}$, sequences $(v_{i_0k}^{(j)})_{k=1}^{+\infty}$, $(j = \overline{0,m_i-1})$, are uniformly bounded and equicontinuous on I. By the Arzela-Ascoli lemma, without loss of generality it can be assumed that these sequences are uniformly convergent on I. Therefore for arbitrary $i_0 \in \{\overline{1,n}\}$, there exists such function

$v_{i_0} \in \widetilde{C}^{m_{i_0}-1}(I; R)$ that $v_{i_0}^{(j)} = \lim\limits_{k\to+\infty} v_{i_0k}^{(j)}$, $(j = \overline{0, m_i - 1})$. Then from (3.13) and (3.14), follows that the function $v := (v_i)_{i=1}^n \in C^{m_1,\dots,m_n}(I; R^n)$, admits to the conditions (2.11_1), and

$$\lim_{k\to+\infty} \|v_k - v\|_{C^{m_1,\dots,m_n}} = 0, \quad \|v\|_{C^{m_1,\dots,m_n}} = 1. \tag{3.16}$$

Set $P_{ik}(t) = \int_a^t p_{ik}(s)\,ds$, $(i = \overline{1, n})$, then from (3.12) we get

$$P_{ik}(a) = 0, 0 \le \sigma_i(P_{ik}(t_2) - P_{ik}(t_1)) \le \int_{t_1}^{t_2} h_i(s)\,ds, \tag{3.17}$$

for $a \le t_1 \le t_2 \le b$, and then the sequences $(P_{ik})_{k=1}^{+\infty}$, $(i = \overline{1, n})$, are uniformly bounded and equicontinuous on I. Then by the Arzela-Ascoli lemma, without loss of generality it can be assumed that these sequences uniformly converge. Therefore if we denote the limits of these sequences by P_i, we get

$$\lim_{k\to+\infty} P_{ik}(t) = P_i(t), \quad (i = \overline{1, n}), \quad \text{uniformly on } I, \tag{3.18}$$

and then from (3.17) follows that

$$0 \le \sigma_i(P_i(t_2) - P_i(t_1)) \le \int_{t_1}^{t_2} h_i(s)\,ds, \quad (i = \overline{1, n}).$$

Consequently the functions P_i are absolutely continuous, there exist the functions $p_i \in L(I; R)$ such that $P_i(t) = \int_a^t p_i(s)\,ds$, and inequalities

$$0 \le \sigma_i p_i(t) \le h_i(t), \quad (i = \overline{1, n}), \quad \text{for } t \in I \tag{3.19}$$

hold. Then due to (3.16), and (3.18), by Lemma 3.11 with $y_k(t) = p_{ik}(t)$, $z_k(t) = v_{i+1k}(\tau_i(t))$, we obtain

$$\lim_{k\to+\infty} \int_a^t p_{ik}(s) v_{i+1k}(\tau_i(s))\,ds = \int_a^t p_i(s) v_{i+1}(\tau_i(s))\,ds, \quad (i = \overline{1, n}), \tag{3.20}$$

uniformly on I. Therefore if we integrate the equations of system (3.14) from a to t, and pass to the limit as $k \to +\infty$, due to conditions (3.13), (3.16), and (3.20), we find that v is a solution of problem (2.10), (2.11_1). Then from the inclusion $h \in P_\tau(I)$, and conditions (3.19) follows that problem (2.10), (2.11_1) has only the zero solution $v \equiv 0$, and we get the contradiction with the second equality of (3.16). I.e., our assumption is invalid and estimation (3.11_r) holds. For $r = 2$ the proof of our lemma is analogous.

4. Proof of main results

PROOF OF THEOREM 2.1. It is known from the general theory of boundary value problems for functional differential equations that problem (1.1), (1.3_1) has the Fredholm's property (see [15]), and therefore problem (1.1), (1.3_1) is uniquely solvable if and only if the homogeneous problem (2.10), (2.11_1) has only the trivial solution. Assume the contrary that problem (2.10), (2.11_1) has the nontrivial solution $(v_i)_{i=1}^n$, and introduce notations

$$m_{n+1} := m_1, \quad v_{n+k} := v_k, \ (k = 1, 2), \quad p_{n+1} := p_1, \quad \tau_{n+1} := \tau_1,$$
$$m_0 := m_n, \quad v_0 := v_n, \quad p_0 := p_n, \quad \tau_0 := \tau_n. \tag{4.1}$$

Let now there exists $r \in \{\overline{1, n}\}$ such that $v_{r+1} \equiv 0$, then from system (2.10) we get that $v_r^{(m_r)} \equiv 0$, from which in view of conditions (2.11_1) (in the point a) we have

$$v_r^{(k)}(t) = \sum_{j=1}^{m_r-k_r} \frac{v_r^{(m_r-j)}(a)(t-a)^{m_r-k-j}}{(m_r-k-j)!}, \quad (k = \overline{0, m_r - k_r - 1}). \tag{4.2}$$

Therefore in view of conditions (2.11_1) (in the point b) we get the following system of algebraic linear equations

$$(b-a)^{k_r-k} \sum_{j=1}^{m_r-k_r} \beta_{kj} x_j = 0, \quad (k = \overline{0, m_r - k_r - 1}), \tag{4.3}$$

where $x_j = v_r^{(m_r-j)}(a)(b-a)^{m_r-k_r-j}$, $\beta_{kj} = 1/(m_r - k - j)!$, and it is not difficult to verify that $\det(\beta_{kj}) \neq 0$. Therefore $v_r^{(m_r-j)}(a) = 0$, $(j = \overline{1, m_r - k_r})$, and from (4.2) it is clear that $v_r \equiv 0$. After analogous $n-1$ steps we get that $v_i \equiv 0$ for all $i \in \{\overline{1, n}\}$, which is the contradiction with our assumption. Thus $v_i^{(m_i)} \not\equiv 0$, $(i = \overline{1, n})$, and then if we assume that $\delta_i := \int_a^b |p_i(s)| v_{i+1}^2(\tau_i(s))\, ds$, $(i = \overline{1, n+1})$, and $\delta_{n+1} = \delta_1$, then from (2.10) and (4.1) it follows that

$$0 < \int_a^b (v_i^{(m_i)}(s))^2\, ds \leq \|p_i\|_\infty \delta_i, \quad (i = \overline{1, n}),$$
$$0 < \int_a^b (v_{n+1}^{(m_{n+1})}(s))^2\, ds = \int_a^b (v_1^{(m_1)}(s))^2\, ds$$
$$\leq \|p_1\|_\infty \delta_1 = \|p_{n+1}\|_\infty \delta_{n+1}, \tag{4.4}$$

and therefore

$$\delta_i > 0, \quad (i = \overline{1, n+1}). \tag{4.5}$$

Also note that in view of conditions $\sigma_i p_i \geq 0$, representations

$$\delta_i = \sigma_i \int_a^b v_i^{(m_i)}(s) v_{i+1}(s)\, ds + \int_a^b |p_i(s)| v_{i+1}(\tau_i(s)) \left(\int_s^{\tau_i(s)} v'_{i+1}(\xi)\, d\xi \right) ds \tag{4.6}$$

are valid for all $i \in \{\overline{1,n}\}$. Now note that the Schwarz inequality, (3.6) with $k = k_{i+1}$, $m = m_{i+1}$, $j = 0$, and (4.4) imply the following estimations

$$\begin{aligned}
\sigma_i \int_a^b v_i^{(m_i)}(s) v_{i+1}(s)\, ds &\leq \left(\int_a^b (v_i^{(m_i)}(s))^2\, ds \int_a^b v_{i+1}^2(s)\, ds \right)^{1/2} \\
&\leq 2^{m_{i+1}-2k_{i+1}+1} ((b-a)/\pi)^{m_{i+1}} \\
&\quad \times \left(\int_a^b (v_i^{(m_i)}(s))^2\, ds \int_a^b (v_{i+1}^{(m_{i+1})}(s))^2\, ds \right)^{1/2} \\
&\leq 2^{m_{i+1}-2k_{i+1}+1} ((b-a)/\pi)^{m_{i+1}} \\
&\quad \times (\|p_i\|_\infty \|p_{i+1}\|_\infty)^{1/2} (\delta_i \delta_{i+1})^{1/2}
\end{aligned} \tag{4.7}$$

for all $i \in \{\overline{1,n}\}$. Also the Schwarz inequality, (3.6) with $k = k_{i+1}$, $m = m_{i+1}$, $j = 1$ and (4.4), imply the following estimations

$$\begin{aligned}
&\int_a^b |p_i(s)| v_{i+1}(\tau_i(s)) \left(\int_s^{\tau_i(s)} v'_{i+1}(\xi)\, d\xi \right) ds \\
&\leq \int_a^b |p_i(s)| v_{i+1}(\tau_i(s)) |\tau_i(s) - s|^{1/2} \left(\int_s^{\tau_i(s)} v'^2_{i+1}(\xi)\, d\xi \right)^{1/2} ds \\
&\leq 2^{m_{i+1}-2k_{i+1}+1} \left(\frac{b-a}{\pi} \right)^{m_{i+1}-1} \left(\int_a^b |p_i(s)| |\tau_i(s) - s|\, ds \right)^{1/2} \\
&\quad \times \left(\int_a^b |p_i(s)| v_{i+1}^2(\tau_i(s))\, ds \right)^{1/2} \left(\int_a^b (v_{i+1}^{(m_{i+1})}(s))^2\, ds \right)^{1/2} \\
&\leq 2^{m_{i+1}-2k_{i+1}} \left(\frac{b-a}{\pi} \right)^{m_{i+1}} \rho(p_i, \tau_i) (\|p_{i+1}\|_\infty \delta_i \delta_{i+1})^{1/2} \quad (i = \overline{1,n}).
\end{aligned} \tag{4.8}$$

Now if we substitute in representations (4.6) estimations (4.7) and (4.8) we obtain

$$\delta_i \leq 2^{m_{i+1}-2k_{i+1}} \left(\frac{b-a}{\pi} \right)^{m_{i+1}} \left[2\|p_i\|_\infty^{1/2} + \rho(p_i, \tau_i) \right] (\|p_{i+1}\|_\infty \delta_i \delta_{i+1})^{1/2}, \tag{4.9}$$

for all $i \in \{\overline{1,n}\}$, and if we multiply all inequalities (4.9), then by the notations $\delta_{n+1} = \delta_1$, $p_{n+1} = p_1$, and (4.5), we get the contradiction with condition (2.2_1). Therefore our assumption is invalid and $v_i \equiv 0$, $(i = \overline{1,n})$.

PROOF OF THEOREM 2.2. It is known from the general theory of boundary value problems for functional differential equations that problem (1.1), (1.3_2) has the Fredholm's property (see [15]), and therefore problem (1.1), (1.3_2) is uniquely solvable iff the homogeneous problem (2.10), (2.11_2) has only the trivial solution. Assume the contrary that problem (2.10), (2.11_2) has the nontrivial solution $(v_i)_{i=1}^n$, and introduce the notations (4.1). Then if there exists $r \in \{\overline{1,n}\}$ such, that $v_{r+1} \equiv 0$, analogously as in the previous proof, we obtain that system (4.3) where $k = \overline{k_r, m_r - 1}$, has only the zero solution, and we get the contradiction with our assumption. Therefore analogously as in the previous proof, but by using Lemma 3.6 instead of Lemma 3.5, we get the following estimations

$$\sigma_i \int_a^b v_i^{(m_i)}(s) v_{i+1}(s)\, ds \le 2^{m_{i+1}} \left(\frac{b-a}{\pi}\right)^{m_{i+1}} (\|p_i\|_\infty \|p_{i+1}\|_\infty)^{1/2} (\delta_i \delta_{i+1})^{1/2},$$

and

$$\int_a^b |p_i(s)| v_{i+1}(\tau_i(s)) \left(\int_s^{\tau_i(s)} v'_{i+1}(\xi)\, d\xi \right) ds$$
$$\le 2^{m_{i+1}-2} \left(\frac{b-a}{\pi}\right)^{m_{i+1}} \rho(p_i, \tau_i)(\|p_{i+1}\|_\infty \delta_i \delta_{i+1})^{1/2},$$

for all $i \in \{\overline{1,n}\}$. Now if we substitute these two estimations in formula (4.6), analogously as in the previous proof we obtain the contradiction with condition (2.2_2). Therefore our assumption is invalid and $v_i \equiv 0$, $(i = \overline{1,n})$.

PROOF OF COROLLARY 2.7. Validity follows from Theorem 2.1 if $r = 1$, and Theorem 2.2 if $r = 2$, with $n = 2$, $m_1 = m_2 = 2$, $\tau_1 \equiv t$, $\tau_2 \equiv \tau$.

PROOF OF COROLLARY 2.8. It is not difficult to verify that if the function u is the solution of problem (2.6), (2.7_r) then the vector function $(u_i)_{i=1}^k$ where $u_i = u^{(2i-2)}$ is the solution of problem (1.1), (1.3_r) with $n = k+1$, $p_i \equiv 1$, $q_i \equiv 0$, $\tau_i \equiv t$, $m_i = 2$ $(i = \overline{1,k})$, $p_{k+1} \equiv p$, $q_{k+1} \equiv q$, $m_{k+1} = 2$, $\tau_{k+1} = \tau$. Then from Theorem 2.1 if $r = 1$, and Theorem 2.2 if $r = 2$ follows validity of our corollary.

PROOF OF THEOREM 2.11. Let $\lambda \in (0,1)$ be an arbitrary fixed number, and $u := (u_i)_{i=1}^n$ is a solution of problem (3.8), (3.9_r). Let also ρ_0 and γ_{ri} be the numbers defined in Lemma 3.10, then due to condition (2.14) there exists the constant $\rho_1 > r_0$, such that the inequality

$$\rho_0 \sum_{i=1}^{n} \Big(\|\alpha_i\|_L + \gamma_{ri} + \int_a^b \eta_i(s, \rho)\, ds \Big) < \rho \quad \text{for } \rho \geq \rho_1 \tag{4.10}$$

holds where $\alpha_i(t) = 1 + |q_{0i}(t)|$. Now assume that $\|u\|_{C^{m_1,\dots,m_n}} \geq \rho_1$, and introduce the notation

$$\nu_i(t) := \frac{F_i(u)(t) - \sigma_i V_{1i}(u)(t) u_{i+1}(\tau_i(t)) + V_{2i}(u)(t)|u_{i+1}(\tau_i(t))| + \eta_i(t, \|u\|_{C^{m_1,\dots,m_n}}) + 1}{2(V_{2i}(u)(t)|u_{i+1}(\tau_i(t))| + \eta_i(t, \|u\|_{C^{m_1,\dots,m_n}}) + 1)}$$

for $t \in I$, then it is not difficult to verify that u is a solution also of system (1.1), with

$$\begin{aligned} p_i(t) &= \sigma_i \lambda h_i(t) \\ &\quad + \sigma_i(1-\lambda)\big(\sigma_i(2\nu_i(t) - 1)V_{2i}(u)(t) \operatorname{sgn} u_{i+1}(\tau_i(t)) + V_{1i}(u)(t)\big), \\ q_i(t) &= (1-\lambda)\big((2\nu_i(t) - 1)[1 + \eta_i(t, \|u\|_{C^{m_1,\dots,m_n}})] + q_{0i}(t)\big). \end{aligned}$$

On the other hand in view of conditions (2.13) the inequalities

$$0 < \nu_i(t) < 1, \quad (i = \overline{1, n}), \tag{4.11}$$

hold, and therefore the estimation

$$\begin{aligned} &\lambda h_i(t) + (1-\lambda)(V_{1i}(u)(t) - V_{2i}(u)(t)) \\ &\qquad \leq \sigma_i p_i(t) \leq \lambda h_i(t) + (1-\lambda)(V_{1i}(u)(t) + V_{2i}(u)(t)), \end{aligned}$$

is valid from which by (2.12) and (4.11) we obtain

$$0 \leq \sigma_i p_i(t) \leq h_i(t), \quad (i = \overline{1, n}), \quad \text{for } t \in I, \tag{4.12}$$

$$|q_i(t)| \leq 1 + |q_{0i}(t)| + \eta_i(t, \|u\|_{C^{m_1,\dots,m_n}}) \quad \text{for } t \in I. \tag{4.13}$$

Then for an arbitrary solution u of problem (1.1), (3.9_r) due to (4.12) and (4.13) from Lemma 3.10 we get the estimation

$$\|u\|_{C^{m_1,\dots,m_n}} \leq \rho_0 \sum_{i=1}^{n} \left(\|\alpha_i\|_L + \lambda\gamma_{ri} + \int_a^b \eta_i(s, \|u\|_{C^{m_1,\dots,m_n}})\, ds \right),$$

which in view of the assumption $\|u\|_{C^{m_1,\dots,m_n}} \geq \rho_1$, contradicts with inequality (4.10). Therefore our assumption is invalid and estimation (3.10) holds, and then from Proposition 3.9 follows the solvability of problem (1.2), (1.3_r).

PROOF OF COROLLARY 2.12 (2.13). In view of conditions (2.15_r) and $h \in L_\infty(I; R_+^n)$, $\tau \in M^n(I)$, for such $p_i \in L_\infty(I; R)$ that conditions (2.9) hold, in view of Theorem 2.1 if $r = 1$ and Theorem 2.2 if $r = 2$, problem (2.10), (2.11_r) has only the zero solution, and therefore inclusion (2.8_r) holds. Thus all the assumptions of Theorem 2.11 are fulfilled and then problem (1.2), (1.3_r) is solvable.

PROOF OF COROLLARY 2.14. Let $x \in C^{m_1,\dots,m_n}(I; R^n)$, and

$$g_i(t) = \sup\{|f_i(t, y) - \sigma_i f_{1i}(t, y)y| : |y| \le r_0\}(i = \overline{1, n}) \quad \text{for } t \in I.$$

Then in view of (2.18) and the fact that the functions η_i are nondecreasing in the second argument we obtain

$$\begin{aligned} &|f_i(t, x_{i+1}(\tau_i(t))) - \sigma_i f_{1i}(t, x_{i+1}(\tau_i(t)))x_{i+1}(\tau_i(t))| \\ &\qquad \le f_{2i}(t, x_{i+1}(\tau_i(t)))|x_{i+1}(\tau_i(t))| + g_i(t) + \eta_i(t, \|x\|_{C^{m_1,\dots,m_n}}) \\ &\qquad\qquad \text{for } \|x\|_{C^{m_1,\dots,m_n}} \ge r_0,\ t \in I,\ (i = \overline{1, n}), \end{aligned} \tag{4.14}$$

and due to (2.14) the equalities

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b (g_i(s) + \eta_i(s, \rho))\, ds = 0, \quad (i = \overline{1, n}), \tag{4.15}$$

are fulfilled. Therefore if we introduce the notations

$$\begin{aligned} F_i(x)(t) &= f_i(t, x_{i+1}(\tau_i(t))), \\ V_{ji}(x)(t) &= f_{ji}(t, x_{i+1}(\tau_i(t)))x_{i+1}(\tau_i(t)), \quad (j = 1, 2), \end{aligned}$$

due to (2.17), (4.14), and (4.15), we get that all the assumptions of Theorem 2.11 hold, from which the validity of our corollary immediately follows.

PROOF OF THEOREM 2.16. From condition (2.20) it is clear that all the conditions of Corollary 2.14 with $r_0 = 0$, $f_{1i} \equiv f_{2i} \equiv h_i/2$, and $\eta_i \equiv |f_i(\cdot, 0)|$ hold. Therefore it remains to prove that problem (2.16), (1.3_r) has no more then one solution.

Let $\widetilde{u}_1 := (u_{1i})_{i=1}^n$ and $\widetilde{u}_2 := (u_{2i})_{i=1}^n$ be an arbitrary solutions of problem (2.16), (1.3_r) and $u = \widetilde{u}_1 - \widetilde{u}_2$. Then the function u admits to conditions (2.11_r), and from condition (2.20) we obtain that

$$0 \le \sigma_i u_i^{(m_i)}(t) \operatorname{sgn} u_{i+1}(\tau_i(t)) \le h_i(t)|u_{i+1}(\tau_i(t))| \quad \text{for } t \in I.$$

Therefore if we define the functions p_i by the equalities

$$p_i(t) = \begin{cases} u_i^{(m_i)}(t)/u_{i+1}(\tau_i(t)) & \text{if } u_{i+1}(\tau_i(t)) \ne 0, \\ 0 & \text{if } u_{i+1}(\tau_i(t)) = 0, \end{cases}$$

we get that u is a solution of the linear homogeneous problem (2.10), (2.11$_r$), and the functions p_i satisfy inequalities (2.9). Thus from the inclusion $h \in D_{r,\tau}(I)$ it follows that $u \equiv 0$, and therefore $\widetilde{u}_1 \equiv \widetilde{u}_2$.

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