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STRONGLY ELLIPTIC OPERATORS AND EXPONENTIATION OF OPERATOR LIE ALGEBRAS

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Abstract

An intriguing feature which is often present in theorems regarding the exponentiation of Lie algebras of unbounded linear operators on Banach spaces is the assumption of hypotheses on the Laplacian operator associated with a basis of the operator Lie algebra. The main objective of this work is to show that one can substitute the Laplacian by an arbitrary operator in the enveloping algebra and still obtain exponentiation, as long as its closure generates a strongly continuous one-parameter semigroup satisfying certain norm estimates, which are typical in the theory of strongly elliptic operators.

1. Introduction

The influential paper *Analytic vectors* [13], of E. Nelson, contains one of the first exponentiation theorems in the context of infinite-dimensional representation theory of Lie algebras. More specifically, Theorem 5 of this reference considers a Lie algebra of skew-symmetric (or skew-hermitian, if one prefers) operators defined on a common, dense and invariant subspace of a Hilbert space. According to that theorem, if the Laplacian operator associated with a basis of the operator Lie algebra is essentially self-adjoint, then this Lie algebra exponentiates to a strongly continuous unitary representation of a Lie group. In §3 of [15], D. W. Robinson improved this result, extending it to Lie algebras of conservative operators defined on a common, dense and invariant domain of a Banach space (for more examples of exponentiation theorems and applications, see [9]).

Another paper dealing with the exponentiation issue of operator Lie algebras is [1]. There, characterizations of exponentiable Lie algebras of linear operators on Banach spaces were obtained in Theorem 3.9 (in the isometric case) and in Remark 3.10 (in the general case). As in [13] and [15], the hypotheses on the Laplacian operator are crucial ingredients of these theorems: in particular, it is required that the closure of the Laplacian must generate a strongly continuous

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one-parameter semigroup $t \mapsto S_t$ satisfying the estimates

$$\rho_1(S_t(x)) \leq Ct^{-\frac{1}{2}}\|x\|, \quad t \in (0,1],\ x \in \mathscr{X}, \tag{1.1}$$

where ρ_1 denotes the first-order differential norm with respect to a fixed basis of the Lie algebra (ρ_1 will be properly defined in Definition 2.3). D. W. Robinson showed in [16, Corollary 5.6, p. 44] that the estimates (1.1) are typical when dealing with second-order strongly elliptic generators (the definition of strongly elliptic operators is given in Definition 2.5). The studies performed in the early chapters of [16] were, in part, motivated by the work of R. P. Langlands, in his PhD thesis [11] (see [12] for more information; see also [4]).

A canonical example one should have in mind, in order to motivate the use of these generalized Laplacian operators, is the initial value problem for the heat operator on $\mathbb{R}^n \times \mathbb{R}$ [6, Chapter 4]. In this context, one typically obtains a (contractive) strongly continuous one-parameter semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, which acts via a convolution with the function known as the *heat kernel*. Then, the usual Laplacian operator defined on the Schwartz function space is a pregenerator of this semigroup [5, Chapter I, 2.13, p. 69]. The use of general strongly elliptic operators also has motivations from PDE theory; see, for example, [6, Chapter 7] and [14].

In [16, Theorem 5.1, p. 30] and [16, Theorem 2.2, p. 80] the following theorem was proved:

THEOREM 1.1 (Strongly elliptic operators). *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space, G a Lie group with Lie algebra $\mathfrak{g}$, $V\colon G \to \mathscr{L}(\mathscr{X})$ a strongly continuous representation of G by bounded operators and $H_m \in \mathfrak{U}(\partial V[\mathfrak{g}])_{\mathbb{C}}$ an element of order m which is strongly elliptic. Then, $-H_m$ is an infinitesimal pregenerator of a strongly continuous semigroup $t \mapsto S_t$ satisfying $S_t[\mathscr{X}] \subseteq C^\infty(V)$, for $t \in (0,1]$ and, for each $n \in \mathbb{N}$, there exists a constant $C_n > 0$ such that*

$$\rho_n(S_t(y)) \leq C_n t^{-\frac{n}{m}}\|y\|, \quad t \in (0,1], y \in \mathscr{X}. \tag{1.2}$$

More precisely, such constants are of the form $C_n = KL^n n!$, for some $K, L \geq 1$.

The objective of this paper is to prove a converse of this statement, in the same spirit as in [1, Theorem 2.8, Remark 2.9]. The main novelty of the proof presented here consists in showing that one can substitute the Laplacian operator employed in [1] by an arbitrary operator in the enveloping algebra of the operator Lie algebra and still obtain exponentiation, provided that its closure generates a strongly continuous one-parameter semigroup satisfying certain a priori norm estimates, which are similar to (1.2); see Theorems 3.1 and 3.8, below. The proof given here deals with an arbitrary dense core domain

$\mathscr{D}$ which is left invariant by the operator Lie algebra under consideration. As a corollary, a characterization of the exponentiation issue when $\mathscr{D}$ is the maximal C^∞ domain is also obtained (Theorem 3.9).

2. Preliminaries

A linear operator T on a Banach space $\mathscr{X}$ will always be assumed as being defined on a vector subspace $\operatorname{Dom} T$ of $\mathscr{X}$, called its *domain*. When its domain is dense, the operator will be called *densely defined*. Also, if S and T are two linear operators then their sum will be defined by

$$\operatorname{Dom}(S+T) := \operatorname{Dom} S \cap \operatorname{Dom} T,$$
$$(S+T)(x) := S(x) + T(x), \quad x \in \operatorname{Dom}(S+T)$$

and their composition by

$$\operatorname{Dom}(TS) := \{x \in \mathscr{X} : S(x) \in \operatorname{Dom} T\},$$
$$(TS)(x) := T(S(x)), \quad x \in \operatorname{Dom}(TS),$$

following the usual conventions of the classical theory of unbounded linear operators on Hilbert and Banach spaces. The *range* of T will be denoted by $\operatorname{Ran} T$.

The algebra of continuous linear operators on $\mathscr{X}$ (having $\mathscr{X}$ as their domains) will be denoted by $\mathscr{L}(\mathscr{X})$. Also, the following standard definitions will be employed:

DEFINITION 2.1 (One-parameter semigroups and groups). A one-parameter semigroup on a Banach space $(\mathscr{X}, \|\cdot\|)$ is a family $\{V_t\}_{t\geq 0}$ of linear operators in $\mathscr{L}(\mathscr{X})$ satisfying

$$V_0 = I,\, V_{s+t} = V_s V_t, \quad s, t \geq 0.$$

If, in addition, the semigroup satisfies the property that $\lim_{t\to t_0} \|V_t(x) - V_{t_0}(x)\| = 0$, for all $t_0 \geq 0$ and $x \in \mathscr{X}$, then $\{V_t\}_{t\geq 0}$ is called a *strongly continuous one-parameter semigroup*. In this case, as a consequence of the Uniform Boundedness Theorem, there always exist $M > 0$ and $w \geq 0$ such that $\|V_t(x)\| \leq M \exp(wt)\|x\|$, for all $t \geq 0$ and $x \in \mathscr{X}$ [5, Proposition 5.5, p. 39]. If w and M can be chosen equal to 0 and 1, respectively, such a semigroup will be called a *contractive semigroup*. These definitions are analogous for one-parameter groups, switching from "$t \geq 0$" to "$t \in \mathbb{R}$", "$\exp(wt)$" to "$\exp(w|t|)$" and "$[0, \infty)$" to "$\mathbb{R}$". Also, if $\|V_t(x)\| = \|x\|$, for every $t \in \mathbb{R}$ and $x \in \mathscr{X}$, then $t \mapsto V_t$ is called an *isometric group*.

If G is a Lie group with unit e, then a family $\{V_g\}_{g\in G}$ of linear operators in $\mathscr{L}(\mathscr{X})$ satisfying

$$V_e = I,\, V_{gh} = V_g V_h, \quad g, h \in G$$

and

$$\lim_{g\to h} V_g(x) = V_h(x), \quad x \in \mathscr{X}, h \in G,$$

is called a *strongly continuous representation* of G on $\mathscr{X}$.

Given a strongly continuous one-parameter semigroup $t \mapsto V_t$ on $(\mathscr{X}, \|\cdot\|)$, consider the subspace of vectors $x \in \mathscr{X}$ such that the limit $\lim_{t\to 0} t^{-1}(V_t(x) - x)$ exists. Then, as usual, the linear operator T defined by

$$\operatorname{Dom} T := \left\{x \in \mathscr{X} : \lim_{t\to 0} t^{-1}(V_t(x) - x) \text{ exists in } \mathscr{X}\right\}$$

and $T(x) := \lim_{t\to 0} t^{-1}(V_t(x) - x)$ is called the *infinitesimal generator* (or, simply, the generator) of the semigroup $t \mapsto V_t$ (the definition for groups is analogous). It is always a densely defined and closed linear operator [5, Theorem 1.4, p. 51]. If a closable linear operator is such that its closure is the generator of a strongly continuous one-parameter (semi)group $t \mapsto V_t$, then it is called an *infinitesimal pregenerator* (or, simply, a pregenerator) of $t \mapsto V_t$. Also, if G is a Lie group then, for each fixed element X of its Lie algebra $\mathfrak{g}$, the infinitesimal generator of the one-parameter group $t \mapsto V_{\exp tX}$ will be denoted by $dV(X)$ (exp denotes the exponential map of the Lie group G). A vector $x \in \mathscr{X}$ is called a C^∞ *vector for* V, or a *smooth vector for* V, if the map $G \ni g \mapsto V_g(x)$ is of class C^∞. The subspace of smooth vectors for V will be denoted by $C^\infty(V)$.

For every strongly continuous Lie group representation $g \mapsto V_g$, the map $\partial V : X \mapsto \partial V(X) := dV(X)|_{C^\infty(V)}$ is a Lie algebra representation which extends to a representation of (the complexification of) the universal enveloping algebra of $\mathfrak{g}$; to see this, adapt the argument in [18, Proposition 10.1.6, p. 263] to the Banach space context, for example. This motivates the following definition:

Definition 2.2 (Representation by closed linear operators). Let $\mathscr{X}$ be a Banach space, $\mathscr{D}$ a linear subspace of $\mathscr{X}$ and let $\operatorname{End}(\mathscr{D})$ denote the algebra of all endomorphisms on $\mathscr{D}$ or, in other words, the linear operators defined on $\mathscr{D}$ with ranges contained in $\mathscr{D}$. Consider a real finite-dimensional Lie algebra $\mathfrak{g}$ and a function η defined on $\mathfrak{g}$ assuming values on (not necessarily continuous) closed linear operators on $\mathscr{X}$ which leave $\mathscr{D}$ invariant; in other words, $\eta(X)$ is a closed linear operator on $\mathscr{X}$ such that $\eta(X)[\mathscr{D}] \subseteq \mathscr{D}$, for all $X \in \mathfrak{g}$. Assume that $\mathscr{D}$ is dense in $\mathscr{X}$. Assume, also, that $\mathscr{D}$ is a core for all the operators in $\eta[\mathfrak{g}]$, so that $\mathscr{D}$ will be called a *core domain* for $\eta[\mathfrak{g}]$, and that

$\eta_{\mathscr{D}}\colon \mathfrak{g} \to \mathscr{L} \subseteq \mathrm{End}(\mathscr{D})$, defined by $\eta_{\mathscr{D}}(X) := \eta(X)|_{\mathscr{D}}$ is, without loss of generality, a faithful (or injective, if one prefers) Lie algebra representation onto $\mathscr{L}$. Then, the triple $(\mathscr{D}, \mathfrak{g}, \eta)$ is said to be a *representation of* $\mathfrak{g}$ *by closed linear operators on* $\mathscr{X}$. It is said to *exponentiate*, or to be *exponentiable*, if there exists a simply connected Lie group G having $\mathfrak{g}$ as its Lie algebra and a strongly continuous representation $V\colon G \to \mathscr{L}(\mathscr{X})$ such that $\mathscr{D} \subseteq C^\infty(V)$ and $dV(X) = \overline{\eta(X)|_{\mathscr{D}}} = \eta(X)$, for all $X \in \mathfrak{g}$. In this case, it will sometimes be used the terminology that the operator Lie algebra $\mathscr{L}$ *exponentiates*. If, moreover, every $dV(X)$, $X \in \mathfrak{g}$, is the generator of an isometric group, it will be said that $(\mathscr{D}, \mathfrak{g}, \eta)$ exponentiates to an isometric representation.

The faithfulness requirement in the definition above is imposed to guarantee that, if $\{X_k\}_{1\leq k\leq d}$ is a basis for $\mathfrak{g}$, then $\{\eta_{\mathscr{D}}(X_k)\}_{1\leq k\leq d}$ is a basis for $\eta_{\mathscr{D}}[\mathfrak{g}]$. This will facilitate the use of some future notations; see also Definition 2.5.

When dealing with $C^\infty(V)$, or a subspace $\mathscr{D}$ of $\mathscr{X}$ which is left invariant by the generators $dV(X)$, $X \in \mathfrak{g}$, it is sometimes useful to deal with finer "differential" norms:

Definition 2.3 (Projective C^∞-topology on the space of smooth vectors). Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $g \mapsto V_g$ be a strongly continuous representation of G on a Banach space $(\mathscr{X}, \|\cdot\|)$. Fix an ordered basis $\mathscr{B} := (X_k)_{1\leq k\leq d}$ for $\mathfrak{g}$ and equip $C^\infty(V)$ with the topology defined by the family

$$\{\rho_n : n \in \mathbb{N}\}$$

of norms, where

$$\rho_0(x) := \|x\|, \quad dV(X_0) := I$$

and

$$\rho_n(x) := \max\bigl\{\|dV(X_{i_1})\dots dV(X_{i_n})x\| : 0 \leq i_j \leq d\bigr\};$$

this is called the *projective C^∞-topology* on $C^\infty(V)$, and it does not depend upon the fixed basis $\mathscr{B}$. When equipped with this topology, $C^\infty(V)$ becomes a Fréchet space; to see this, just adapt the proof of [7, Corollary 1.1] to the realm of Banach spaces.

If $\mathscr{D} \subseteq \mathscr{X}$ is a linear subspace of $\mathscr{X}$ and $\mathscr{L} \subseteq \mathrm{End}(\mathscr{D})$ is a real finite-dimensional Lie algebra of linear operators on $\mathscr{X}$ having an ordered basis $(B_k)_{1\leq k\leq d}$ consisting of closable operators, then the projective C$^\infty$-topology on $\mathscr{D}$ is defined in an analogous way, substituting the operators $dV(X_{i_k})$ by the operators $\overline{B_{i_k}}$, $1 \leq i_k \leq d$.

A few words about the use of some future notations must be said. Consider a Lie algebra $\mathfrak{g}$ with an ordered basis $(B_k)_{1\leq k\leq d}$. Throughout the whole manuscript, whenever a monomial B^{u} of *size* $n \geq 1$ in the elements of $(B_k)_{1\leq k\leq d}$ is considered, it is understood that $\mathsf{u} = (\mathsf{u}_j)_{j\in\{1,\dots,n\}}$ is a function from $\{1,\dots,n\}$ to $\{1,\dots,d\}$ and $B^{\mathsf{u}} := B_{\mathsf{u}_1}\dots B_{\mathsf{u}_n}$ is an element of the complexified universal enveloping algebra $(\mathfrak{U}[\mathfrak{g}])_{\mathbb{C}} := \mathfrak{U}[\mathfrak{g}] + i\mathfrak{U}[\mathfrak{g}]$ (when $(B_k)_{1\le k\le d}$ consists of closable operators, the word "monomial" will also be used to designate an element $B^{\mathsf{u}} := \overline{B_{\mathsf{u}_1}}\dots\overline{B_{\mathsf{u}_n}}$). Therefore, B^{u} can be thought of as an unordered monomial, or a noncommutative monomial (the expression "unordered" refers to the fact that the sequence $(\mathsf{u}_1,\dots,\mathsf{u}_n)$ of numbers is not necessarily non-decreasing). The size of the monomial is defined to be $|\mathsf{u}| = n$ so, roughly speaking, it is the size of the "word" $B_{\mathsf{u}_1}\dots B_{\mathsf{u}_n}$. For a matter of convenience, the size of a linear combination of monomials is defined to be the size of the "biggest" monomial composing the sum. However, the following question arises: could this sum be written in another way, so that the associated size is different? It may very well be the case that the basis elements share an extra relation, thus causing this kind of ambiguity: for example, if X and Y are elements of a Lie algebra with basis $\{X, Y, [X, Y]\}$, then $[X, Y] = XY - YX$ has size 2, but also size 1, depending on how it is written. Therefore, the concept of size is not an intrinsic one, and is intimately related to how the sum was written. The same cannot be said about the definition of *order* of an element in $(\mathfrak{U}[\mathfrak{g}])_{\mathbb{C}}$: by the Poincaré-Birkhoff-Witt Theorem [10, Theorem 3.8, p. 217] one sees that every linear combination of monomials P can be written in a unique way as a polynomial on the variables $(B_k)_{1\le k\le d}$ of the form

$$\sum_{|\alpha|\le m} c_\alpha B_1^{\alpha_1}\dots B_k^{\alpha_k}\dots B_d^{\alpha_d},$$

where $c_{\alpha'} \neq 0$ for some multi-index α' of order m. In this case, the order of P is defined to be $|\alpha'| = m$, the order of the multi-index α', and it does *not* depend on the particular choice of the basis. Note that the notations were carefully chosen, in order to avoid confusions. The symbols u, v and w will always be employed to indicate the use of an unordered monomial, while the greek letters α, β and γ will denote multi-indices, following the usual convention for ordered monomials. The use of the noncommutative monomial notation will occur in the main theorem of the paper. As an abbreviation, they will usually be referred to as "monomials", rather than "noncommutative monomials".

Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space, $\mathscr{D} \subseteq \mathscr{X}$ a linear subspace of $\mathscr{X}$ and $\mathscr{L} \subseteq \mathrm{End}(\mathscr{D})$ a real finite-dimensional Lie algebra of linear operators acting on $\mathscr{X}$, with an ordered basis $(B_k)_{1\le k\le d}$. Then, for each $1 \le i, j \le d$, one has

$$(\mathrm{ad}\, B_i)(B_j) := [B_i, B_j] = \sum_{k=1}^{d} c_{ij}^{(k)} B_k, \tag{2.1}$$

for some constants $c_{ij}^{(k)} \in \mathbb{R}$. Therefore, if B^{u} and B^{v} are two monomials in the variables $(B_k)_{1\leq k\leq d}$, applying recursively the identity above it is possible to prove that the element $(\operatorname{ad} B^{\mathsf{u}})(B^{\mathsf{v}}) := B^{\mathsf{u}}B^{\mathsf{v}} - B^{\mathsf{v}}B^{\mathsf{u}}$, which has size $|\mathsf{u}|+|\mathsf{v}|$, is actually a sum of terms of size at most $|\mathsf{u}|+|\mathsf{v}|-1$. To see this, one must proceed recursively: if $B^{\mathsf{u}} := B_{\mathsf{u}_1}\dots B_{\mathsf{u}_{|\mathsf{u}|}}$ and $B^{\mathsf{v}} := B_{\mathsf{v}_1}\dots B_{\mathsf{v}_{|\mathsf{v}|}}$, then switching the last element of B^{u} with the first element of B^{v}, and using relation (2.1), gives

$$B^{\mathsf{u}}B^{\mathsf{v}} = B_{\mathsf{u}_1}\dots B_{\mathsf{v}_1}B_{\mathsf{u}_{|\mathsf{u}|}}\dots B_{\mathsf{v}_{|\mathsf{v}|}} + B_{\mathsf{u}_1}\dots \operatorname{ad}(B_{\mathsf{u}_{|\mathsf{u}|}})(B_{\mathsf{v}_1})\dots B_{\mathsf{v}_{|\mathsf{v}|}}$$

$$= B_{\mathsf{u}_1}\dots B_{\mathsf{v}_1}B_{\mathsf{u}_{|\mathsf{u}|}}\dots B_{\mathsf{v}_{|\mathsf{v}|}} + (\text{sum of } d \text{ terms of size at most } |\mathsf{u}|+|\mathsf{v}|-1).$$

Therefore, by iterating this process one concludes that, after repeating this step $|\mathsf{u}|\cdot|\mathsf{v}|$ times, the identity

$$(\operatorname{ad} B^{\mathsf{u}})(B^{\mathsf{v}}) := B^{\mathsf{u}}B^{\mathsf{v}} - B^{\mathsf{v}}B^{\mathsf{u}} = \sum_{|\mathsf{w}|\leq|\mathsf{u}|+|\mathsf{v}|-1} c_{\mathsf{w}}B^{\mathsf{w}} \tag{2.2}$$

arises, where the sum has, at most, $d\cdot|\mathsf{u}|\cdot|\mathsf{v}|$ summands. With all of this in mind, one can obtain the estimates

$$\|(\operatorname{ad} B^{\mathsf{u}})(B^{\mathsf{v}})(x)\| \leq k|\mathsf{u}||\mathsf{v}|\rho_{|\mathsf{u}|+|\mathsf{v}|-1}(x), \quad x\in\mathscr{D}, \tag{2.3}$$

where k is a non-negative constant which does not depend on u or v, and is related only with the coefficients that come from (2.1). These estimates will be extremely useful in the proof of the main theorem.

The following identity will play a central role in the main theorem of the paper:

LEMMA 2.4 (A commutation relation). *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space, $\mathscr{D}\subseteq\mathscr{X}$ a linear subspace of $\mathscr{X}$ and $\mathscr{L}\subseteq \operatorname{End}(\mathscr{D})$ a real finite-dimensional Lie algebra of linear operators acting on $\mathscr{X}$ having an ordered basis $(B_k)_{1\leq k\leq d}$ consisting of closable operators. Also, let B^{u} and B^{v} be two monomials in the basis elements $(B_k)_{1\leq k\leq d}$. Suppose H_m is an element of order m in the complexification $\mathfrak{U}(\mathscr{L})_{\mathbb{C}}$ of the universal enveloping algebra of $\mathscr{L}$ such that $-H_m$ is a pregenerator of a strongly continuous semigroup $t\mapsto S_t$ satisfying $S_t[\mathscr{X}]\subseteq\mathscr{D}$, for all $t>0$, and*

$$\rho_n(x) \leq \epsilon^{m-n}\|H_m(x)\| + \frac{N}{\epsilon^n}\|x\|, \quad x\in\mathscr{D}, \tag{2.4}$$

for all $0<n\leq m-1$, $0<\epsilon\leq 1$ and a constant $N>0$. Define the element $(\operatorname{ad} B^{\mathsf{u}})(H_m)$ *of $\mathfrak{U}(\mathscr{L})_{\mathbb{C}}$ via an extension of the operator* $\operatorname{ad} B^{\mathsf{u}}$ *just defined in*

(2.2), *by linearity. Then, the identity*

$$\int_0^s B^{\mathsf{v}} S_r[(\operatorname{ad} B^{\mathsf{u}})(H_m)]S_{t-r}(x)\,dr = B^{\mathsf{v}}[(\operatorname{ad} S_s)(B^{\mathsf{u}})]S_{t-s}(x),$$

is valid for all $t > 0$, $0 \le s \le t$ *and* $x \in \mathscr{D}$, *where* $(\operatorname{ad} S_s)(B^{\mathsf{u}})$ *is defined to be the operator* $S_s B^{\mathsf{u}} - B^{\mathsf{u}} S_s$ *on* $\mathscr{D}$; *the identity above is called Duhamel formula; see* [1, p. 356] *and* [16].

PROOF. Fix $t > 0$. The first task will be to establish continuity of the function $r \mapsto B^{\mathsf{v}} S_r B^{\mathsf{u}} S_{t-r}(x)$ on $[0, t]$, for all $x \in \mathscr{D}$. To this purpose, it will first be obtained the differentiability of $r \mapsto B^{\mathsf{w}} S_r(x)$ at $r_0 \in [0, t]$, for all monomials B^{w} of size $(q-1)(m-1) \le |\mathsf{w}| \le q(m-1)$ in the elements of $(B_k)_{1\le k\le d}$ and $x \in \mathscr{D}$, a fact which will be proved by induction on $q \ge 1$. To deal with the case $q = 1$, first note that $H_m S_s = S_s H_m$ on $\mathscr{D}$, for all $s \in [0, \infty)$, which together with (2.4), gives

$$\left\| B^{\mathsf{w}} S_{r_0}\left(\frac{S_{r-r_0} - I}{r - r_0}x + H_m(x)\right)\right\| \le \epsilon^{m-n}\left\| S_{r_0}\left(\frac{S_{r-r_0} - I}{r - r_0}H_m(x) + H_m^2(x)\right)\right\| + \frac{N}{\epsilon^n}\left\| S_{r_0}\left(\frac{S_{r-r_0} - I}{r - r_0}x + H_m(x)\right)\right\|,$$

$x \in \mathscr{D}$, if $r > r_0$, and

$$\left\| B^{\mathsf{w}} S_{r}\left(\frac{I - S_{r_0-r}}{r - r_0}x + H_m(x)\right)\right\| \le \epsilon^{m-n}\left\| S_{r}\left(\frac{I - S_{r_0-r}}{r - r_0}H_m(x) + H_m^2(x)\right)\right\| + \frac{N}{\epsilon^n}\left\| S_{r}\left(\frac{I - S_{r_0-r}}{r - r_0}x + H_m(x)\right)\right\|,$$

$x \in \mathscr{D}$, if $r < r_0$, for all $0 < \epsilon \le 1$. These estimates together with local equicontinuity of $t \mapsto S_t$ yield the desired differentiability.

Now, suppose $r \mapsto B^{\mathsf{w}} S_r(x)$ is differentiable at $r_0 \in [0, t]$, for all $x \in \mathscr{D}$ and all monomials B^{w} in the elements of $(B_k)_{1\le k\le d}$, with $(q-1)(m-1) \le |\mathsf{w}| \le q(m-1)$, for some $q \ge 1$. A fixed monomial B^{w} of size $q(m-1) \le |\mathsf{w}| := n \le (q+1)(m-1)$ may be decomposed as $B^{\mathsf{w}} = B^{\mathsf{w}_0} B^{\mathsf{w}'}$, with $|\mathsf{w}_0| = m-1$ and $|\mathsf{w}'| = n - (m-1)$. Using (2.4), one obtains

$$\begin{aligned}\|B^{\mathsf{w}} S_r(x)\| &= \|B^{\mathsf{w}_0} B^{\mathsf{w}'} S_r(x)\| \\ &\le \epsilon \|H_m B^{\mathsf{w}'} S_r(x)\| + \frac{N}{\epsilon^{m-1}}\|B^{\mathsf{w}'} S_r(x)\|, \quad x \in \mathscr{D},\end{aligned}$$

for all $r \in [0, \infty)$ and $0 < \epsilon \le 1$. On the other hand, by (2.3),

$$\|H_m B^{\mathsf{w}'} S_r(x)\| \le k(n - (m-1))m\rho_n(S_r(x)) + \|B^{\mathsf{w}'} H_m S_r(x)\|.$$

Hence,

$$\|B^{\mathsf{w}} S_r(x)\| \leq \epsilon[k(n-(m-1))m\rho_n(S_r(x)) \\ + \|B^{\mathsf{w}'} H_m S_r(x)\|] + \frac{N}{\epsilon^{m-1}} \|B^{\mathsf{w}'} S_r(x)\|, \quad x \in \mathscr{D},$$

for all $r \in [0, \infty)$ and $0 < \epsilon \leq 1$. Choosing $1 \geq \epsilon_0 > 0$ such that the inequality $\epsilon_0 k(n-(m-1))m < 1$ holds and taking the maximum over all monomials of size $n-(m-1)$ and n gives

$$\rho_n(S_r(x)) \leq \frac{\epsilon_0 \rho_{n-(m-1)}(H_m S_r(x)) + \frac{N}{\epsilon_0^{m-1}} \rho_{n-(m-1)}(S_r(x))}{1 - \epsilon_0 k(n-(m-1))m},$$

so the induction hypothesis together with a similar argument made in the case $q = 1$ ends the induction proof.

Now, fix $t > 0$, $x \in \mathscr{D}$, $g \in \mathscr{X}'$ and define on $[0, t]$ the functions

$$g_1 \colon s \mapsto \int_0^s g(B^{\mathsf{v}} S_r[(\operatorname{ad} B^{\mathsf{u}})(H_m)] S_{t-r}(x))\, dr$$

and

$$g_2 \colon s \mapsto g(B^{\mathsf{v}}[(\operatorname{ad} S_s)(B^{\mathsf{u}})] S_{t-s}(x)) = g(B^{\mathsf{v}} S_s B^{\mathsf{u}} S_{t-s}(x)) - g(B^{\mathsf{v}} B^{\mathsf{u}} S_t(x)).$$

Note that the operators $S_t|_{\mathscr{D}}$, $t \geq 0$, are continuous with respect to the projective C^∞-topology on $\mathscr{D}$ defined by the operators $\{\overline{B_k}\}_{1 \leq k \leq d}$; to see this, use the induction proof above together with the closability hypothesis on the basis operators of $\mathscr{L}$. Therefore, an application of [9, Theorem A.1, p. 440] together with what was just proved above gives, in particular, that the map $r \mapsto g(B^{\mathsf{v}} S_r(\operatorname{ad} B^{\mathsf{u}})(H_m) S_{t-r}(x))$ is continuous on $[0, t]$, so g_1 is a differentiable function on $[0, t]$ with

$$g_1'(s) = g(B^{\mathsf{v}} S_s[(\operatorname{ad} B^{\mathsf{u}})(H_m)] S_{t-s}(x)).$$

Also, if $\mathscr{D}$ is equipped with the projective C^∞-topology, then it is clear by the induction proof above that $f \colon s \mapsto B^{\mathsf{u}} S_{t-s}(x)$ is a differentiable map from $[0, t]$ to $\mathscr{D}$ and $f'(s) = B^{\mathsf{u}} S_{t-s} H_m(x)$. The same applies for the function $s \mapsto B^{\mathsf{v}} S_s(y)$, for every fixed $y \in \mathscr{D}$. Therefore, by the product rule of [9, Theorem A.1, p. 440], one sees that $H \colon s \mapsto B^{\mathsf{v}} S_s B^{\mathsf{u}} S_{t-s}(x)$ is differentiable on $(0, t)$ and, defining $K \colon s \mapsto B^{\mathsf{v}} S_s$, it follows that

$$\begin{aligned} H'(s) = [K(s)(f(s))]' &= K'(s) f(s) + K(s) f'(s) \\ &= -B^{\mathsf{v}} S_s H_m B^{\mathsf{u}} S_{t-s}(x) + B^{\mathsf{v}} S_s B^{\mathsf{u}} S_{t-s} H_m(x) \\ &= B^{\mathsf{v}} S_s[(\operatorname{ad} B^{\mathsf{u}})(H_m)] S_{t-s}(x), \end{aligned}$$

where the derivative K' is taken with respect to the strong operator topology. This implies $g_1' = g_2'$ on $(0, t)$ and, since $g_1(0) = g_2(0) = 0$, g_1 must be equal to g_2 on $[0, t]$. By a corollary of the Hahn-Banach Theorem, the equality

$$\int_0^s B^{\mathsf{v}} S_r[(\operatorname{ad} B^{\mathsf{u}})(H_m)]S_{t-r}(x)\, dr = B^{\mathsf{v}}[(\operatorname{ad} S_s)(B^{\mathsf{u}})]S_{t-s}(x),$$

must hold for all $x \in \mathscr{D}$ and $0 \leq s \leq t$, as claimed.

For the definition of strongly elliptic operators, we will follow the one given by D. W. Robinson [16, p. 28]:

DEFINITION 2.5 (Strongly elliptic operators). Let $\mathscr{X}$ be a Banach space, G a Lie group with Lie algebra $\mathfrak{g}$, with $(X_k)_{1\leq k\leq d}$ being an ordered basis for $\mathfrak{g}$, and $V\colon G \to \mathscr{L}(\mathscr{X})$ a strongly continuous representation, where

$$\partial V\colon \mathfrak{g} \ni X \mapsto \partial V(X) \in \operatorname{End}(C^\infty(V))$$

is the induced Lie algebra representation. Assume, as in Definition 2.2, without loss of generality, that ∂V is *faithful*. It is said that an operator

$$H_m := \sum_{|\alpha|\leq m} c_\alpha \partial V(X^\alpha)$$

of order m belonging to the complexified enveloping algebra of $\mathfrak{g}$, $(\mathfrak{U}[\partial V[\mathfrak{g}]])_{\mathbb{C}} := \mathfrak{U}[\partial V[\mathfrak{g}]] + i\mathfrak{U}[\partial V[\mathfrak{g}]]$, is *strongly elliptic* if $\operatorname{Re}(-1)^{m/2} \sum_{|\alpha|=m} c_\alpha \xi^\alpha > 0$, for all $\xi \in \mathbb{R}^d\backslash\{0\}$ (for the definition of order of an element in $(\mathfrak{U}[\partial V[\mathfrak{g}]])_{\mathbb{C}}$, see the comments on notations right after Definition 2.3). This definition is basis independent; see [16, p. 29].

The prototypical example of a strongly elliptic operator is the "minus Laplacian" operator $-\sum_{k=1}^d [\partial V(X_k)]^2$.

For a matter of terminological convenience, in the future, the same definition which was just made for strongly elliptic operators will be extended to (complexifications of) universal enveloping algebras of Lie algebras of operators which are *not* necessarily associated with a strongly continuous Lie group representation.

In order to avoid unnecessary confusions, it should be noted that references [1] and [16, see p. 30] use a slightly different convention from the one employed in this manuscript to define infinitesimal generators: if an operator H_m is strongly elliptic then, according to their definitions, $\overline{H_m}$ will be the generator of a one-parameter semigroup. However, with the definition of infinitesimal generators used in the present paper, if H_m is strongly elliptic, then $-\overline{H_m}$ will be the generator of a one-parameter group. Therefore, for example, $-\sum_{k=1}^{d}[\partial V(X_k)]^2$ is a strongly elliptic operator and $\overline{\sum_{k=1}^{d}[\partial V(X_k)]^2}$ generates a one-parameter semigroup, with the definitions employed here.

Finally, we can move on to the

3. Main theorems

THEOREM 3.1 (Exponentiation: general case). *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space, $\mathfrak{g}$ a real finite-dimensional Lie algebra with an ordered basis $(X_k)_{1\le k\le d}$, $(\mathscr{D}, \mathfrak{g}, \eta)$ a representation of $\mathfrak{g}$ by closed linear operators on $\mathscr{X}$, where $B_k := \eta(X_k)$, $1 \le k \le d$, and H_m an element of order $m \ge 2$ of $\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])_{\mathbb{C}}$, where $\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])_{\mathbb{C}} := \mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}]) + i\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])$ is the complexification of $\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])$. Suppose that the following hypotheses are valid:*

(1) *for each $X \in \mathfrak{g}$, there exist $\sigma_x \ge 0$ and $M_x > 0$ such that*

$$\|(\mu I - \eta(X))^n y\| \ge M_x^{-1}(|\mu| - \sigma_x)^n \|y\|, \tag{3.1}$$

for all $n \in \mathbb{N}$, $|\mu| > \sigma_x$ and $y \in \mathscr{D}$;

(2) *$-H_m$ is an infinitesimal pregenerator of a strongly continuous semigroup $t \mapsto S_t$ satisfying $S_t[\mathscr{X}] \subseteq \mathscr{D}$, for each $t > 0$ and, for each $n \in \mathbb{N}$ satisfying $0 < n \le m-1$, there exists $C_n > 0$ for which the estimates*

$$\rho_n(S_t(y)) \le C_n t^{-\frac{n}{m}} \|y\| \tag{3.2}$$

are verified for all $t \in (0, 1]$ and all $y \in \mathscr{X}$, where $B_0 := I$ and

$$\rho_n(x) := \max\big\{\|B_{i_1} \dots B_{i_k} \dots B_{i_n} x\| : 1 \le k \le n, 0 \le i_k \le d\big\},$$

for all $x \in \mathscr{D}$.

Then, $(\mathscr{D}, \mathfrak{g}, \eta)$ exponentiates.

The idea is to divide the proof into a few auxiliary lemmas. After proving Lemma 3.5, a proof of Theorem 3.1 will be given.

LEMMA 3.2. *Assume the hypotheses of Theorem 3.1, with the exception of* (1). *Then, the estimates*

$$\rho_n(y) \le \epsilon^{m-n}\|H_m(y)\| + \frac{E_n}{\epsilon^n}\|y\|, \quad y \in \mathscr{D}, \tag{3.3}$$

are valid for all $0 < \epsilon \le 1$, $0 < n \le m-1$ and some $E_n > 0$.

PROOF. The semigroup $t \mapsto S_t$ is strongly continuous, so there exist constants $M \ge 1$ and $w \ge 0$ such that

$$\|S_t(y)\| \le M \exp(wt)\|y\|,$$

for all $y \in \mathscr{X}$ and $t \geq 0$. Now, if $0 < \delta < 1$ satisfies $0 \leq w\delta < 1$, the resolvent $\mathrm{R}(1, -\delta\overline{H_m}) = (I + \delta\overline{H_m})^{-1}$ is well-defined and

$$\mathrm{R}(1, -\delta\overline{H_m}) = \int_0^\infty \exp(-t) S_{\delta t}(y)\, dt, \quad y \in \mathscr{X},$$

with

$$\int_0^\infty \exp(-t) \|S_{\delta t}(y)\|\, dt < \infty.$$

Fix a monomial B^{u} of size $0 < n \leq m-1$ in the operators $\{B_k\}_{1 \leq k \leq d}$ and $t > 0$. If $\delta t \leq 1$, then $\|B^{\mathrm{u}} S_{\delta t}(y)\| \leq C_n (\delta t)^{-\frac{n}{m}} \|y\|$, $y \in \mathscr{X}$, by (3.2); if $\delta t > 1$, then

$$\|B^{\mathrm{u}} S_{\delta t}(y)\| = \|B^{\mathrm{u}} S_{1+(\delta t-1)}(y)\| \leq C_n M \exp(w(\delta t - 1)) \|y\|, \quad y \in \mathscr{X}.$$

These inequalities and

$$\frac{1}{\exp(t)} \leq \sqrt[m]{\frac{1}{\exp(t)}} \leq t^{-\frac{k}{m}} \sqrt[m]{k!},$$

with $k = m - n$ and $k = n$, respectively, allow one to obtain, for every $0 < \delta < 1$ satisfying $0 \leq w\delta < \frac{1}{2}$, the estimates

$$\begin{aligned}
&\int_0^\infty \exp(-t) \|B^{\mathrm{u}} S_{\delta t}(y)\|\, dt \\
&\quad \leq C_n \|y\| \left(\int_0^{1/\delta} \frac{\exp(-t)}{(\delta t)^{n/m}}\, dt + M \exp(-w) \int_{1/\delta}^\infty \exp((w\delta - 1)t)\, dt \right) \\
&\quad = C_n \|y\| \Bigg(\delta^{-\frac{n}{m}} \left[\exp(-t) \frac{m}{m-n} t^{\frac{m-n}{m}} \right] \Bigg|_{t=0}^{t=1/\delta} \\
&\qquad + \delta^{-\frac{n}{m}} \frac{m}{m-n} \int_0^{1/\delta} \exp(-t) t^{\frac{m-n}{m}}\, dt + M \exp(-w) \frac{\exp(w - 1/\delta)}{1 - w\delta} \Bigg) \\
&\quad \leq C_n \|y\| \Bigg(\delta^{-\frac{n}{m}} \delta^{\frac{m-n}{m}} \sqrt[m]{(m-n)!} \frac{m}{m-n} \delta^{\frac{n-m}{m}} \\
&\qquad + \delta^{-\frac{n}{m}} \frac{m}{m-n} \int_0^{+\infty} \exp(-t) t^{\frac{m-n}{m}}\, dt + 2M \delta^{\frac{n}{m}} \sqrt[m]{n!} \Bigg) \\
&\quad \leq C_n \Bigg(\sqrt[m]{(m-n)!} \frac{m}{m-n} \\
&\qquad + \frac{m}{m-n} \int_0^\infty \exp(-t) t^{\frac{m-n}{m}}\, dt + 2M \sqrt[m]{n!} \Bigg) \delta^{-\frac{n}{m}} \|y\|,
\end{aligned}$$

for all $y \in \mathscr{X}$ (in the last inequality it was used that $\delta^{-\frac{n}{m}} > \delta^{\frac{n}{m}}$). Therefore, the integral $\int_0^\infty \exp(-t)B^{\mathsf{u}}S_{\delta t}(y)\,dt$ defines an element in $\mathscr{X}$, for all $y \in \mathscr{X}$. By closedness of each operator B_k it follows that $(I + \delta\overline{H_m})^{-1}(y) \in \operatorname{Dom} B^{\mathsf{u}}$ and

$$B^{\mathsf{u}}(I + \delta\overline{H_m})^{-1}(y) = \int_0^\infty \exp(-t)B^{\mathsf{u}}S_{\delta t}(y)\,dt.$$

Also, the estimates above show that there exists $E'_n > 0$ independent of δ such that

$$\|B^{\mathsf{u}}(I + \delta\overline{H_m})^{-1}(y)\| \leq E'_n\delta^{-\frac{n}{m}}\|y\|, \quad y \in \mathscr{X}.$$

Hence, if y is of the form $y = (I + \delta\overline{H_m})(y') = (I + \delta H_m)(y')$, $y' \in \mathscr{D}$, one obtains

$$\begin{aligned}\|B^{\mathsf{u}}(y')\| &= \|B^{\mathsf{u}}(I + \delta\overline{H_m})^{-1}[(I + \delta\overline{H_m})(y')]\| \\ &\leq E'_n\delta^{-\frac{n}{m}}\|(I + \delta\overline{H_m})(y')\| \\ &\leq E'_n\delta^{\frac{m-n}{m}}\|H_m(y')\| + E'_n\delta^{-\frac{n}{m}}\|y'\|.\end{aligned}$$

Now, define $\epsilon_n(\delta) := (E'_n)^{\frac{1}{m-n}}\delta^{\frac{1}{m}}$. If $\epsilon_n(\delta_0) > 1$, for some δ_0, then (3.3) is proved since, by continuity, the range of the function ϵ_n must contain the interval (0, 1], when δ runs through (0, 1) and the values satisfying $0 \leq w\delta < \frac{1}{2}$. Then, defining $E_n := (E'_n)^{\frac{m}{m-n}}$, one obtains

$$E_n\epsilon_n(\delta)^{-n} = (E'_n)^{\frac{m}{m-n}}\epsilon_n(\delta)^{-n} = E'_n\delta^{-\frac{n}{m}},$$

so

$$\rho_n(y') \leq \epsilon^{m-n}\|H_m(y')\| + \frac{E_n}{\epsilon^n}\|y'\|, \quad y' \in \mathscr{D}, 0 < \epsilon \leq 1,$$

as desired. However, if $\epsilon_n(\delta) < 1$, for all δ satisfying $0 < \delta < 1$ and $0 \leq w\delta < \frac{1}{2}$, then multiply the function ϵ_n by a real constant a greater than 1 so that the range of $\epsilon'_n := a\epsilon_n$ contains 1. This reduces the problem to the previous case and finishes the proof of Lemma 3.2.

LEMMA 3.3. *Assume the hypotheses of Theorem 3.1, with the exception of* (1). *Then, given* $q \in \mathbb{N}$, $q \geq 1$, *there exists a strictly positive constant* K_q *such that*

$$\rho_n(S_t(y)) \leq K_q \sup_{0 \leq j \leq q} \|H_m^j(y)\|,$$

for all $y \in \mathscr{D}$, $(q-1)(m-1) < n \leq q(m-1)$ *and* $t \in (0, 1]$.

PROOF. The procedure is by induction on q. By what was proved in Lemma 3.2, for each fixed $0 < n \leq m-1$ the estimates

$$\rho_n(y) \leq \epsilon^{m-n}\|H_m(y)\| + \frac{E_n}{\epsilon^n}\|y\|, \quad y \in \mathscr{D},$$

are valid for some $E_n > 0$ and all $0 < \epsilon \le 1$. This shows the existence of a $K_1 > 0$ satisfying

$$\rho_n(S_t(y)) \le K_1 \sup_{0\le j\le 1} \|H_m^j(y)\|, \quad y \in \mathscr{D}, 0 < n \le m-1, t \in (0,1],$$

since $H_m S_t = S_t H_m$ on Dom H_m and $\|S_t\| \le Me^{wt}$, for some $M \ge 1$, $w \ge 0$ and all $t \ge 0$. Now, suppose that, for some fixed $q \in \mathbb{N}$, $q \ge 1$, the inequality

$$\rho_n(S_t(y)) \le K_q \sup_{0\le j\le q} \|H_m^j(y)\|$$

is true, for all $y \in \mathscr{D}$, $(q-1)(m-1) < n \le q(m-1)$, $t \in (0,1]$. Fix $y \in \mathscr{D}$. A monomial B^{u} of size $q(m-1) < n \le (q+1)(m-1)$ in the operators $(B_k)_{1\le k\le d}$ may be decomposed as $B^{\mathsf{u}} = B^{\mathsf{u}_0}B^{\mathsf{u}'}$, with $|\mathsf{u}_0| = m-1$ and $|\mathsf{u}'| = n-(m-1)$. Using (3.3), one obtains

$$\begin{aligned}\|B^{\mathsf{u}}S_t(y)\| &= \|B^{\mathsf{u}_0}B^{\mathsf{u}'}S_t(y)\| \\ &\le \epsilon^{m-(m-1)}\|H_m B^{\mathsf{u}'}S_t(y)\| + \frac{E_{m-1}}{\epsilon^{m-1}}\|B^{\mathsf{u}'}S_t(y)\| \\ &= \epsilon\|H_m B^{\mathsf{u}'}S_t(y)\| + \frac{E_{m-1}}{\epsilon^{m-1}}\|B^{\mathsf{u}'}S_t(y)\|, \quad 0 < t \le 1, 0 < \epsilon \le 1.\end{aligned}$$

On the other hand, by the induction hypothesis,

$$\begin{aligned}\|H_m B^{\mathsf{u}'}S_t(y)\| &\le \|[\mathrm{ad}(H_m)(B^{\mathsf{u}'})]S_t(y)\| + \|B^{\mathsf{u}'}H_m S_t(y)\| \\ &\le k\big(n-(m-1)\big)m\rho_n(S_t(y)) + K_q \sup_{0\le j\le q}\|H_m^j H_m(y)\|\end{aligned}$$

for $0 < t \le 1$, k being a constant depending only on d, H_m and on the numbers c_{ij} which are defined by $[B_i|_{\mathscr{D}}, B_j|_{\mathscr{D}}] = \sum_{k=1}^d c_{ij}^{(k)} B_k|_{\mathscr{D}}$, for all $1 \le i \ne j \le d$. Hence, using the induction hypothesis again, one obtains

$$\begin{aligned}\|B^{\mathsf{u}}S_t(y)\| \le{}& \epsilon\big[k(n-(m-1))m\rho_n(S_t(y)) + K_q \sup_{0\le j\le q+1}\|H_m^j(y)\|\big] \\ &+ \frac{E_{m-1}}{\epsilon^{m-1}}K_q \sup_{0\le j\le q}\|H_m^j(y)\|, \quad 0 < t \le 1, \quad 0 < \epsilon \le 1.\end{aligned}$$

Choosing an $1 \ge \epsilon_0 > 0$ such that $\epsilon_0 k(n-(m-1))m < 1$ and taking the maximum over all monomials B^{u} of size n gives

$$\begin{aligned}\rho_n(S_t(y)) &\le \frac{\epsilon_0 K_q \sup_{0\le j\le q+1}\|H_m^j(y)\| + \frac{E_{m-1}}{\epsilon_0^{m-1}}K_q \sup_{0\le j\le q}\|H_m^j(y)\|}{1-\epsilon_0 k(n-(m-1))m} \\ &\le \frac{K_q\big(\epsilon_0 + \frac{E_{m-1}}{\epsilon_0^{m-1}}\big)}{1-\epsilon_0 k(n-(m-1))m}\sup_{0\le j\le q+1}\|H_m^j(y)\|, \quad 0 < t \le 1.\end{aligned}$$

This concludes the induction proof.

Actually, it is the following corollary which will be invoked, later:

COROLLARY 3.4. *Assume the hypotheses of Theorem 3.1, with the exception of* (1). *Then, given* $q \in \mathbb{N}$, $q \geq 1$, *and* $y \in \mathscr{D}$, *there exists* $K_{y,q} > 0$ *satisfying* $\rho_n(S_t(y)) \leq K_{y,q}$, *for all* $t \in (0, 1]$ *and* $(q-1)(m-1) < n \leq q(m-1)$.

LEMMA 3.5. *Assume the hypotheses of Theorem 3.1, with the exception of* (1). *Then, there exist* $K, L \geq 1$ *such that*

$$\rho_n(S_t(y)) \leq K L^n n! t^{-\frac{n}{m}} \|y\|,$$

for all $n \in \mathbb{N}$, $n \geq 1$, $y \in \mathscr{X}$ *and* $t \in (0, 1]$.

PROOF. The idea is, again, to proceed by induction. The strategy will be to prove that there exist $K, L \geq 1$ such that, given $q \in \mathbb{N}$, $q \geq 1$ and $(q-1)(m-1) < n \leq q(m-1)$, there exists a strictly positive constant C_n defined by $C_n := K L^n n!$ satisfying

$$\rho_n(S_t(y)) \leq C_n t^{-\frac{n}{m}} \|y\|, \quad y \in \mathscr{X}, t \in (0, 1].$$

The case $q = 1$ follows from (3.2), with

$$K := D_1 := \max\big\{1, \max\{C_j : 0 < j \leq m-1\}\big\}$$

and $L := 1$, so it is already known to be true. Fix $q \geq 1$ and suppose that there exists, for every $p \leq q$ and $\ell \in \mathbb{N}$ satisfying $(p-1)(m-1) < \ell \leq p(m-1)$, a strictly positive constant C_ℓ (not necessarily of the form $K L^\ell \ell!$, $K, L \geq 1$) such that

$$\rho_\ell(S_t(y)) \leq C_\ell t^{-\frac{\ell}{m}} \|y\|, \quad y \in \mathscr{X}, t \in (0, 1].$$

Fix $n \in \mathbb{N}$ satisfying $q(m-1) < n \leq (q+1)(m-1)$ (in particular, $n > m-1$), $0 < t \leq 1$ and $y \in \mathscr{D}$. As in the proof of Lemma 3.3, take a monomial B^{u} of size $q(m-1) < n \leq (q+1)(m-1)$ in the operators $(B_k)_{1 \leq k \leq d}$ and decompose it as $B^{\mathsf{u}} = B^{\mathsf{u}_0} B^{\mathsf{u}'}$, with $|\mathsf{u}_0| = m-1$ and $|\mathsf{u}'| = n-(m-1)$. Then, by Lemma 2.4, above,

$$\begin{aligned} B^{\mathsf{u}} S_t(y) &= B^{\mathsf{u}_0} S_s B^{\mathsf{u}'} S_{t-s}(y) + B^{\mathsf{u}_0}[(\operatorname{ad} B^{\mathsf{u}'})(S_s)] S_{t-s}(y) \\ &= B^{\mathsf{u}_0} S_s B^{\mathsf{u}'} S_{t-s}(y) - \int_0^s B^{\mathsf{u}_0} S_r [(\operatorname{ad} B^{\mathsf{u}'})(H_m)] S_{t-r}(y)\, dr, \end{aligned} \tag{3.4}$$

for all $0 < s < t$. By (2.3), one has the inequality

$$\|[(\operatorname{ad} B^{\mathsf{u}'})(H_m)] S_{t-r}(y)\| \leq k(n-(m-1)) m \rho_n(S_{t-r}(y))$$

where k, again, depends only on d, H_m and on the numbers c_{ij} defined by $[B_i|_{\mathscr{D}}, B_j|_{\mathscr{D}}] = \sum_{j=1}^{d} c_{ij}^{(k)} B_k|_{\mathscr{D}}$, for all $1 \leq i \neq j \leq d$. Applying this inequality together with the induction hypothesis, one obtains from (3.4) the estimates

$$\begin{aligned}\|B^{\mathsf{u}} S_t(y)\| &\leq C_{m-1} s^{-\frac{m-1}{m}} \|B^{\mathsf{u}'} S_{t-s}(y)\| \\ &\qquad + \int_0^s C_{m-1} r^{-\frac{m-1}{m}} \|[\operatorname{ad}(B^{\mathsf{u}'})(H_m)] S_{t-r}(y)\| \, dr \\ &\leq C_{m-1} s^{-\frac{m-1}{m}} C_{n-(m-1)} (t-s)^{-\frac{n-(m-1)}{m}} \|y\| \\ &\qquad + C_{m-1} k(n-(m-1)) m \int_0^s r^{-\frac{m-1}{m}} \rho_n(S_{t-r}(y)) \, dr.\end{aligned}$$

Making the changes of variables $s = \lambda t$, where $\lambda \in (0, 1)$, and $r = ut$, inside the integral, such inequality becomes

$$\begin{aligned}\|B^{\mathsf{u}} S_t(y)\| &\leq C_{m-1} C_{n-(m-1)} t^{-\frac{n}{m}} \lambda^{-\frac{m-1}{m}} (1-\lambda)^{-\frac{n-(m-1)}{m}} \|y\| \\ &\qquad + C_{m-1} k(n-(m-1)) m t^{\frac{1}{m}} \int_0^{\lambda} u^{-\frac{m-1}{m}} \rho_n(S_{t(1-u)}(y)) \, du.\end{aligned}$$

Putting $\lambda = n^{-m}$, one has that $\lambda^{-\frac{m-1}{m}} = n^{m-1}$ and

$$\begin{aligned}(1-\lambda)^{-\frac{n-(m-1)}{m}} &= (1-\lambda)^{-\frac{n}{m}} (1-\lambda)^{\frac{m-1}{m}} \\ &\leq (1-\lambda)^{-\frac{n}{m}} \leq c_m,\end{aligned}$$

where $1 < c_m := \sup_{\ell \geq m} \{(1-\ell^{-m})^{-\frac{\ell}{m}}\} < \infty$. Hence, by taking the maximum over all the monomials of size n, the above inequality becomes

$$\begin{aligned}\rho_n(S_t(y)) &\leq C_{m-1} C_{n-(m-1)} t^{-\frac{n}{m}} n^{m-1} c_m \|y\| \\ &\qquad + C_{m-1} k(n-(m-1)) m t^{\frac{1}{m}} \int_0^{n^{-m}} u^{-\frac{m-1}{m}} \rho_n(S_{t(1-u)}(y)) \, du.\end{aligned}$$

But, since the $0 < t \leq 1$ fixed was arbitrary, one may repeat this procedure and iterate $j-1$ times the last inequality to obtain

$$\rho_n(S_t(y)) \leq C_{m-1} C_{n-(m-1)} t^{-\frac{n}{m}} n^{m-1} c_m \|y\| \sum_{i=0}^{j-1} a_i + R_j, \tag{3.5}$$

where $a_0 := 1$,

$$\begin{aligned}
a_i &:= \big(C_{m-1}k(n-(m-1))mt^{\frac{1}{m}}\big)^i \int_0^{n^{-m}} u_1^{-\frac{m-1}{m}}(1-u_1)^{-\frac{n}{m}}\,du_1 \\
&\quad \cdots \int_0^{n^{-m}} u_i^{-\frac{m-1}{m}}(1-u_i)^{-\frac{n}{m}}\,du_i \\
&\le (C_{m-1}knmt^{\frac{1}{m}})^i \big[\sup_{\ell\ge m}(1-\ell^{-m})^{-\frac{\ell}{m}}\big]^i \left(\int_0^{n^{-m}} u^{-\frac{m-1}{m}}\,du\right)^i \\
&= (C_{m-1}knmc_m t^{\frac{1}{m}})^i (mn^{-1})^i = (C_{m-1}km^2c_m t^{\frac{1}{m}})^i,
\end{aligned}$$

if $i \ge 1$, and

$$\begin{aligned}
R_j &:= \big(C_{m-1}k(n-(m-1))mt^{\frac{1}{m}}\big)^j \\
&\quad \times \int_0^{n^{-m}} u_1^{-\frac{m-1}{m}} \cdots \int_0^{n^{-m}} u_j^{-\frac{m-1}{m}} \rho_n(S_{t(1-u_1)\ldots(1-u_j)}(y))\,du_1\ldots du_j,
\end{aligned}$$

for all $j \ge 1$. Therefore, by Corollary 3.4,

$$R_j \le (C_{m-1}km^2t^{\frac{1}{m}})^j K_{y,q+1}.$$

If $t_m > 0$ is chosen so that $C_{m-1}km^2c_m(t_m)^{\frac{1}{m}} = \frac{1}{2}$ (equivalently, $t_m = (2C_{m-1}km^2c_m)^{-m}$), then for all $0 < t \le t_m$,

$$\begin{aligned}
\sum_{j=0}^{\infty} a_j &\le \sum_{j=0}^{\infty} (C_{m-1}km^2c_m t^{\frac{1}{m}})^j \\
&\le \sum_{j=0}^{\infty} (C_{m-1}km^2c_m (t_m)^{\frac{1}{m}})^j = \sum_{j=0}^{\infty} \frac{1}{2^j} = 2.
\end{aligned}$$

Also, the limit $\lim_{j\to\infty} R_j$ exists and equals zero, for all $0 < t \le t_m$, since

$$\begin{aligned}
0 &\le \lim_{j\to\infty} R_j \le K_{y,q+1}\big[\lim_{j\to\infty}(C_{m-1}km^2c_m t^{\frac{1}{m}})^j\big] \\
&\le K_{y,q+1}\big[\lim_{j\to\infty}(C_{m-1}km^2c_m (t_m)^{\frac{1}{m}})^j\big] = K_{y,q+1}\left[\lim_{j\to\infty}\frac{1}{2^j}\right] = 0,
\end{aligned}$$

where in the second inequality it was used that $c_m > 1$. Hence, taking the limit $j \to \infty$ in (3.5), it follows that

$$\rho_n(S_t(y)) \le 2C_{m-1}C_{n-(m-1)}n^{m-1}c_m t^{-\frac{n}{m}}\|y\|,$$

if $t \in (0, t_m]$ and $y \in \mathscr{D}$. Solving the recurrence relation gives

$$\rho_n(S_t(y)) \leq D_1[2D_1c_m]^q n^{q(m-1)} t^{-\frac{n}{m}} \|y\|, \quad t \in (0, t_m], y \in \mathscr{D},$$

where we recall that $D_1 := \max\big\{1, \max\{C_j : 0 < j \leq m-1\}\big\}$; more precisely, an induction proof on $q \geq 1$, on the formula

$$\rho_n(S_t(y)) \leq D_1[2D_1c_m]^{q-1} n^{(q-1)(m-1)} t^{-\frac{n}{m}} \|y\|,$$
$$(q-1)(m-1) < n \leq q(m-1)$$

subjected to the restrictions $y \in \mathscr{D}$ and $t \in (0, t_m]$, gives the desired inequality: the base case follows easily from (3.2) and from $D_1 \geq 1$; the inductive step follows from a repetition of the argument done so far, in the present lemma, with C_ℓ substituted by $D_1[2D_1c_m]^{p-1}\ell^{(p-1)(m-1)}$.

Consequently,

$$\rho_n(S_t(y)) \leq D_1[2eD_1c_m]^n n! t^{-\frac{n}{m}} \|y\|, \quad t \in (0, t_m], y \in \mathscr{D},$$

where it was used the estimate $n^{q(m-1)} \leq e^n[q(m-1)]!$. Therefore, defining $K_0 := D_1$ and $L_0 := 2eD_1c_m$, the constants C_ℓ, with $p(m-1) < \ell \leq (p+1)(m-1)$ and $0 < p \leq q$, may be chosen as $C_\ell = K_0 L_0^\ell \ell!$, if $t \in (0, t_m]$ and $y \in \mathscr{D}$.

The last task is to extend the inequality just obtained to all $y \in \mathscr{X}$ and to $t \in (t_m, 1]$. Since $S_t[\mathscr{X}] \subseteq \mathscr{D}$, for all $t > 0$, one has that, for fixed $t_m \geq t > 0$ and $y \in \mathscr{X}$,

$$\begin{aligned}\rho_n(S_t(y)) &= \inf_{0<\epsilon<t} \rho_n(S_{t-\epsilon}S_\epsilon(y)) \leq K_0L_0^n n!\Big[\inf_{0<\epsilon<t}[(t-\epsilon)^{-\frac{n}{m}}\|S_\epsilon(y)\|]\Big] \\ &\leq K_0L_0^n n!\Big[\inf_{0<\epsilon<t}[(t-\epsilon)^{-\frac{n}{m}} Me^{w\epsilon}]\Big]\|y\| = (MK_0)L_0^n n! t^{-\frac{n}{m}}\|y\|.\end{aligned}$$

If $t_m \geq 1$, the induction proof is complete. So suppose $0 < t_m < 1$. Then, if $1 \geq t > t_m$ and $y \in \mathscr{X}$,

$$\begin{aligned}\rho_n(S_t(y)) = \rho_n(S_{t_m}S_{t-t_m}(y)) &\leq K_0L_0^n n!(t_m)^{-\frac{n}{m}}\|S_{t-t_m}(y)\| \\ &\leq K_0L_0^n n!(t_m)^{-\frac{n}{m}} Me^{w(t-t_m)}\|y\| \\ &\leq (Me^w K_0)(t_m^{-\frac{1}{m}} L_0)^n n! t^{-\frac{n}{m}}\|y\|,\end{aligned}$$

where in the last inequality it was used that $t^{-\frac{n}{m}} \geq 1$ and that $t - t_m < 1$. Hence, with the definitions $K := Me^w K_0$ and $L := t_m^{-\frac{1}{m}} L_0$ the desired inequality

$$\rho_n(S_t(y)) \leq KL^n n! t^{-\frac{n}{m}}\|y\|,$$

for all $y \in \mathscr{X}$ and $t \in (0, 1]$ is finally obtained, concluding the induction step.

Therefore, with the definitions $K := Me^w D_1$ and $L := 2eMD_1 c_m t_m^{-\frac{1}{m}}$, it is true that

$$\rho_n(S_t(y)) \leq KL^n n! t^{-\frac{n}{m}} \|y\|,$$

for all $n \in \mathbb{N}$, $n \geq 1$, $x \in \mathscr{X}$ and $t \in (0, 1]$, completing the proof.

Finally, we are able to prove Theorem 3.1:

Proof of Theorem 3.1. Define $\mathscr{S}_0 := \bigcup_{0<t\leq 1} S_t[\mathscr{X}]$ and

$$C^\omega(\eta) := \left\{x \in \mathscr{D} : \sum_{n=0}^{+\infty} \frac{\rho_n(x)}{n!} s^n < \infty, \text{ for some } s > 0\right\}.$$

Since S is strongly continuous, $\mathscr{S}_0$ is dense in $\mathscr{X}$. Besides, the estimates from Lemma 3.5 show that $\mathscr{S}_0 \subseteq C^\omega(\eta)$, in view of the following argument: fix a number $0 < t \leq 1$ and $y \in \mathscr{X}$; if one chooses $r > 0$ so that

$$r < \frac{1}{Lt^{-\frac{1}{m}}},$$

then the inequality

$$\sum_{n=0}^{\infty} \frac{\rho_n(S_t(y))}{n!} r^n \leq K\|y\| \sum_{n=0}^{\infty} L^n t^{-\frac{n}{m}} r^n < \infty$$

proves the inclusion. Therefore, $C^\omega(\eta)$ is dense in $\mathscr{X}$. Now, because the inclusion

$$C^\omega(\eta) \subseteq C^\omega(B_k) := \left\{x \in \mathscr{D} : \sum_{n=0}^{\infty} \frac{\|B_k^n(x)\|}{n!} s^n < \infty, \text{ for some } s > 0\right\}$$

is valid for every $1 \leq k \leq d$, it follows from [17, Theorem 1] that every $B_k|_{C^\omega(\eta)}$ is an infinitesimal pregenerator of a strongly continuous one parameter group. Hence, since $C^\omega(\eta)$ is also left invariant by the operators in $\{\eta(X) : X \in \mathfrak{g}\}$, it follows from [8, Theorem 3.1] that the finite-dimensional real Lie algebra

$$\{\eta(X)|_{C^\omega(\eta)} : X \in \mathfrak{g}\} \subseteq \mathrm{End}(C^\omega(\eta))$$

is exponentiable. Hence, the equality $\overline{\eta(X)|_{C^\omega(\eta)}} = \eta(X) = \overline{\eta(X)|_{\mathscr{D}}}$ follows from the inclusion $\overline{\eta(X)|_{C^\omega(\eta)}} \subset \overline{\eta(X)|_{\mathscr{D}}} = \eta(X)$ together with the following version of [2, Theorem 3.1.15(3), p. 177]: *Let S be a densely defined closable* *linear operator on the Banach space $\mathscr{Y}$. If $\overline{S}$ generates a strongly continuous semigroup $t \mapsto S_t$ satisfying*

$$\|S_t(y)\| \leq M \exp(wt)\|y\|, \quad y \in \mathscr{Y}, t > 0,$$

for constants $M \geq 1$ and $w \geq 0$, and T is an extension of $\overline{S}$ such that $\mu_0 I - T$ is an injective operator, for some $\mu_0 > w$, then $\overline{S} = T$.

Therefore, $(\mathscr{D}, \mathfrak{g}, \eta)$ is also exponentiable, as claimed.

The next corollary is an immediate consequence of the proof of Theorem 3.1:

COROLLARY 3.6. *If the hypotheses of Theorem 3.1 are assumed, then*

$$C^{\omega}(\eta) := \left\{ x \in \mathscr{D} : \sum_{n=0}^{+\infty} \frac{\rho_n(x)}{n!} s^n < \infty, \text{ for some } s > 0 \right\}$$

is a dense core domain for $\eta[\mathfrak{g}]$ and the finite-dimensional real operator Lie algebras

$$\{\eta(X)|_{C^{\omega}(\eta)} : X \in \mathfrak{g}\} \subseteq \mathrm{End}(C^{\omega}(\eta)) \quad \text{and} \quad \{\eta(X)|_{\mathscr{D}} : X \in \mathfrak{g}\} \subseteq \mathrm{End}(\mathscr{D})$$

both exponentiate to a strongly continuous representation $V : G \to \mathscr{L}(\mathscr{X})$ of the same simply connected Lie group G. In particular, these operator Lie algebras have the same dimension.

REMARK 3.7. Note that the estimates (3.1) were just invoked at the very end, after the proof of Lemma 3.5. In order to obtain an exponentiation result for the isometric case one just needs to impose $\sigma_x = 0$ and $M_x = 1$, for all $X \in \mathfrak{g}$, on the estimates (3.1); in this case, one usually says that the operators $\eta(X)|_{\mathscr{D}}$, $X \in \mathfrak{g}$, are *conservative*. Then, invoking [17, Theorem 1], [8, Theorem 3.1] and [2, Theorem 3.1.15(3), p. 177], as above, one obtains the following corollary:

THEOREM 3.8 (Exponentiation: isometric case). *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space, $\mathfrak{g}$ a real finite-dimensional Lie algebra with an ordered basis $(X_k)_{1 \leq k \leq d}$, $(\mathscr{D}, \mathfrak{g}, \eta)$ a representation of $\mathfrak{g}$ by closed linear operators on $\mathscr{X}$, where $B_k := \eta(X_k)$, $1 \leq k \leq d$, and H_m an element of order $m \geq 2$ of $\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])_{\mathbb{C}}$, where $\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])_{\mathbb{C}} := \mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}]) + i\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])$ is the complexification of $\mathfrak{U}(\eta_{\mathscr{D}}[\mathfrak{g}])$. Suppose that the following hypotheses are valid:*

(1) *$\eta(X)$ is a conservative operator, for every $X \in \mathfrak{g}$;*

(2) *$-H_m$ is an infinitesimal pregenerator of a strongly continuous semigroup $t \mapsto S_t$ satisfying $S_t[\mathscr{X}] \subseteq \mathscr{D}$, for each $t > 0$ and, for each*

$n \in \mathbb{N}$ satisfying $0 < n \leq m - 1$, there exists $C_n > 0$ for which the estimates

$$\rho_n(S_t(y)) \leq C_n t^{-\frac{n}{m}} \|y\|$$

are verified for all $t \in (0, 1]$ and all $y \in \mathscr{X}$, where $B_0 := I$ and

$$\rho_n(x) := \max\{\|B_{i_1} \ldots B_{i_k} \ldots B_{i_n} x\| : 1 \leq k \leq n, 0 \leq i_k \leq d\},$$

for all $x \in \mathscr{D}$.

Then, $(\mathscr{D}, \mathfrak{g}, \eta)$ exponentiates to an isometric representation.

Note that, if H_m is the usual minus Laplacian operator $-\sum_{k=1}^{d} \eta_{\mathscr{D}}(X_k)^2$, then conservativity of the basis operators $\{\eta_{\mathscr{D}}(X_k)\}_{1 \leq k \leq d}$ imply dissipativity of $\Delta := \sum_{k=1}^{d} \eta_{\mathscr{D}}(X_k)^2$; in other words, Δ satisfies $\|(\mu I - \Delta)y\| \geq \mu\|y\|$, for all $\mu > 0$ and $y \in \mathscr{D}$: first, fix $y \in \mathscr{D}$ and note that each $\eta_{\mathscr{D}}(X_k)^2$ is a dissipative operator, since

$$\left\|\left(I - \lambda^2 \eta_{\mathscr{D}}(X_k)^2\right)y\right\| = \left\|\left(I - \lambda\eta_{\mathscr{D}}(X_k)\right)\left(I + \lambda\eta_{\mathscr{D}}(X_k)\right)y\right\| \geq \|y\|$$

for all $\lambda > 0$, by conservativity of the operators $\eta_{\mathscr{D}}(X_k)$, $1 \leq k \leq d$. By a characterization of dissipative operators on Banach spaces, which may be found on pp. 174 and 175 of [2], if $f \in \mathscr{X}'$ is a tangent functional at y (in other words, a bounded linear functional such that $|f(y)| = \|f\|\|y\|$, where $\|f\|$ denotes the operator norm of f), then $\operatorname{Re} f(\eta_{\mathscr{D}}(X_k)^2(y)) \leq 0$, where $1 \leq k \leq d$. By linearity, $\operatorname{Re} f(\Delta(y)) \leq 0$, proving that Δ is a dissipative operator.

Consequently, assuming the hypotheses of Theorem 3.8, together with $H_m := -\sum_{k=1}^{d} \eta_{\mathscr{D}}(X_k)^2$, $\overline{\Delta}$ will be the generator of a contractive semigroup: under the hypothesis that $\overline{\Delta}$ generates a strongly continuous semigroup it follows from the Feller-Miyadera-Phillips Theorem [5, Theorem 3.8, p. 77], in particular, that $\operatorname{Ran}(\mu I - \overline{\Delta}) = \mathscr{X}$, for all $\mu > 0$ larger than some $w \geq 0$. Therefore, by [5, Proposition 3.14 (ii) and (iv), p. 82], it follows that $\overline{\operatorname{Ran}(\mu I - \Delta)} = \operatorname{Ran}(\mu I - \overline{\Delta}) = \mathscr{X}$, for all $\mu > 0$. Finally, with the aid of the Lumer-Phillips Theorem [5, Theorem 3.15, p. 83], one concludes that $\overline{\Delta}$ actually generates a contractive semigroup.

Combining Theorem 1.1 with Theorem 3.1 gives the following characterization:

Theorem 3.9 (Exponentiation on Banach spaces). *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space and $(\mathscr{X}_\infty, \mathfrak{g}, \eta)$ a representation by closed operators, where $\mathscr{B} = (X_k)_{1 \leq k \leq d}$ is an ordered basis of the finite-dimensional real Lie algebra $\mathfrak{g}$,*

$\eta(X_k) := B_k$ and

$$\mathscr{X}_\infty := \bigcap_{n=1}^{\infty}\left(\bigcap\{\mathrm{Dom}[B_{i_1}\dots B_{i_k}\dots B_{i_n}] : 1 \le k \le n,\ 1 \le i_k \le d\}\right);$$

note that, by the definition of a representation by closed operators, we are assuming $\mathscr{X}_\infty$ is a dense subspace of $\mathscr{X}$ which is also a core domain for $\eta[\mathfrak{g}]$. Suppose that

$$H_m := \sum_{\alpha;|\alpha|\le m} c_\alpha \eta_{\mathscr{X}_\infty}(X_1)^{\alpha_1}\dots\eta_{\mathscr{X}_\infty}(X_k)^{\alpha_k}\dots\eta_{\mathscr{X}_\infty}(X_d)^{\alpha_d}$$

is an element of $\mathfrak{U}(\eta_{\mathscr{X}_\infty}[\mathfrak{g}])_{\mathbb{C}}$ of order $m \ge 2$ which is strongly elliptic. Then, the following are equivalent:

(1) *$(\mathscr{X}_\infty, \mathfrak{g}, \eta)$ exponentiates to a strongly continuous representation $V: G \to \mathscr{L}(\mathscr{X})$ of a simply connected Lie group G, having $\mathfrak{g}$ as its Lie algebra, such that $dV(X) = \overline{\eta(X)|_{\mathscr{X}_\infty}}$ is the generator of a strongly continuous one-parameter group, for all $X \in \mathfrak{g}$.*

(2) *The operators $\{\eta(X)\}_{X\in\mathfrak{g}}$ satisfy* (3.1)*, and $-H_m$ is an infinitesimal pregenerator of a strongly continuous one-parameter semigroup $S: [0,\infty) \to \mathscr{L}(\mathscr{X})$ such that $S_t[\mathscr{X}] \subseteq \mathscr{X}_\infty$, for all $t > 0$ and furthermore, for each $n \in \mathbb{N}$ satisfying $0 < n \le m-1$, there exists $C_n > 0$ for which*

$$\rho_n(S_t(y)) \le C_n t^{-\frac{n}{m}} \|y\|,$$

where $t \in (0, 1]$ and $y \in \mathscr{X}$.

If the operators $\{\eta(X)\}_{X\in\mathfrak{g}}$, above, satisfy the estimates (3.1) with $\sigma_x = 0$ and $M_x = 1$, for all $X \in \mathfrak{g}$, one obtains an analogous characterization for the isometric case.

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