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STRONG MORITA EQUIVALENCE FOR INCLUSIONS OF C^*-ALGEBRAS INDUCED BY TWISTED ACTIONS OF A COUNTABLE DISCRETE GROUP

KAZUNORI KODAKA

Abstract

We consider two twisted actions of a countable discrete group on σ-unital C^*-algebras. Then by taking the reduced crossed products, we get two inclusions of C^*-algebras. We suppose that they are strongly Morita equivalent as inclusions of C^*-algebras. Also, we suppose that one of the inclusions of C^*-algebras is irreducible, that is, the relative commutant of one of the σ-unital C^*-algebra in the multiplier C^*-algebra of the reduced twisted crossed product is trivial. We show that the two actions are then strongly Morita equivalent up to some automorphism of the group.

1. Introduction

In the previous papers [11], [12], we discussed strong Morita equivalence for twisted coactions of a finite dimensional C^*-Hopf algebra on unital C^*-algebras and unital inclusions of unital C^*-algebras. Also, in [13], we clarified the relation between strong Morita equivalence for such twisted coactions and strong Morita equivalence for the associated unital inclusions of unital C^*-algebras. In the present paper, we shall discuss the same subject as above in the case of twisted actions of a countable discrete group on σ-unital C^*-algebras.

Let (α, w_α) and (β, w_β) be twisted actions of a countable discrete group G on σ-unital C^*-algebras A and B, and let $A \times_{\alpha, w_\alpha, r} G$ and $B \rtimes_{\beta, w_\beta, r} G$ denote the twisted reduced crossed products of A and B by (α, w_α) and (β, w_β), respectively. Then we obtain the inclusions of C^*-algebras $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$. In this paper, we first review that if (α, w_α) and (β, w_β) are strongly Morita equivalent, then in the same way as in Combes [6] or Curto, Muphy and Williams [7], the inclusions $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent. However, it is not clear that the converse result holds in general. Our main result, Theorem 5.5, is as follows: We suppose that the inclusion $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ is irreducible, that is, $A' \cap M(A \rtimes_{\alpha, w_\alpha, r} G) = \mathbf{C}1$. If the inclusions $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita

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equivalent, then there is an automorphism ϕ of G such that $(\alpha^{\phi}, w_{\alpha}^{\phi})$ and (β, w_{β}) are strongly Morita equivalent, where $(\alpha^{\phi}, w_{\alpha}^{\phi})$ is the twisted action of G on A defined by $\alpha_t^{\phi}(a) = \alpha_{\phi(t)}(a)$ and $w_{\alpha}^{\phi}(t, s) = w_{\alpha}(\phi(t), \phi(s))$ for any $t, s \in G$, $a \in A$. This result is similar to [13, Theorem 6.2]. We remark that the condition $A' \cap M(A \rtimes_{\alpha, w_{\alpha}, r} G) = \mathbf{C}1$ holds if and only if (α, w_{α}) is a free twisted action of G on A (cf. Corollary 4.7). This freeness of (α, w_{α}) plays an important role in §5.

We let $\mathbf{K}$ denote the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space and $\{e_{ij}\}_{ij \in \mathbf{N}}$ its system of matrix units. By a homomorphism between two C^*-algebras, we will always mean a *-homomorphism. This also applies to isomorphisms and automorphisms. For each C^*-algebra A, we denote by $M(A)$ the multiplier C^*-algebra of A. Let π be an isomorphism of A onto a C^*-algebra B. Then there is a unique strictly continuous isomorphism of $M(A)$ onto $M(B)$ extending π by Jensen and Thomsen [9, Corollary 1.1.15]. We denote it by $\underline{\pi}$. For an algebra A, we denote by id_A the identity map on A; if A is unital, we denote by 1_A the unit of A. If no confusion arises, we denote them by id and 1, respectively. Throughout this paper, we denote by G a countable discrete group with unit element e.

2. Preliminaries

First, we give some definitions. Let A be a C^*-algebra. Let $\mathrm{Aut}(A)$ be the group of all automorphisms of A and $U(M(A))$ the group of all unitary elements in $M(A)$.

DEFINITION 2.1. By a *twisted action* (α, w_{α}) of G on A, we mean a map α from G to $\mathrm{Aut}(A)$ and a map w_{α} from $G \times G$ to $U(M(A))$ satisfying the following conditions:

(1) $\alpha_t \circ \alpha_s = \mathrm{Ad}(w_{\alpha}(t, s)) \circ \alpha_{ts}$,

(2) $w_{\alpha}(t, s) w_{\alpha}(ts, r) = \underline{\alpha_t}(w_{\alpha}(s, r)) w_{\alpha}(t, sr)$,

(3) $w_{\alpha}(t, e) = w_{\alpha}(e, t) = 1_{M(A)}$,

for all $t, s, r \in G$.

By easy computations, for any $t \in G$,

$$\alpha_e = \mathrm{id}, \quad w_{\alpha}(t, t^{-1}) = \underline{\alpha_t}(w_{\alpha}(t^{-1}, t)),$$
$$\alpha_t^{-1} = \alpha_{t^{-1}} \circ \mathrm{Ad}(w_{\alpha}(t, t^{-1})^*) = \mathrm{Ad}(w_{\alpha}(t^{-1}, t)^*) \circ \alpha_{t^{-1}}$$

Let (α, w_{α}) be a twisted action of G on a C^*-algebra A. Then we get a twisted action $(\underline{\alpha}, w_{\alpha})$ of G on $M(A)$ such that $\underline{\alpha_t}$ is the unique strictly continuous automorphism of $M(A)$ extending α_t to $M(A)$ for any $t \in G$.

We regard A and $M(A)$ as C^*-subalgebras of $A \rtimes_{\alpha, w_\alpha, r} G$ and $M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G$ in the usual way, respectively. By considering the universal representation of A on a Hilbert space $\mathscr{H}_u$, we may regard $M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G$ and $A \rtimes_{\alpha, w_\alpha, r} G$ as acting non-degenerately on the same Hilbert space $\ell^2(G, \mathscr{H}_u)$. Then we can see that $M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G \subset M(A \rtimes_{\alpha, w_\alpha, r} G)$ (cf. Pedersen [15, §§3.12 and 7.7] for the case of an untwisted action). Thus we have the following inclusions of C^*-algebras:

$$A \subset A \rtimes_{\alpha, w_\alpha, r} G \subset M(A \rtimes_{\alpha, w_\alpha, r} G)$$
$$M(A) \subset M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G \subset M(A \rtimes_{\alpha, w_\alpha, r} G).$$

We may also form $(\alpha \otimes \mathrm{id}_{\mathbf{K}}, w_\alpha \otimes 1_{M(\mathbf{K})})$, the twisted action of G on $A \otimes \mathbf{K}$. As is well-known, there is an isomorphism of $(A \otimes \mathbf{K}) \rtimes_{\alpha \otimes \mathrm{id}, w_\alpha \otimes 1, r} G$ onto $(A \rtimes_{\alpha, w_\alpha, r} G) \otimes \mathbf{K}$ such that its restriction to $A \otimes \mathbf{K}$ is the identity map on $A \otimes \mathbf{K}$. Thus

$$A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{\alpha \otimes \mathrm{id}, w_\alpha \otimes 1, r} G$$

and

$$A \otimes \mathbf{K} \subset (A \rtimes_{\alpha, w_\alpha, r} G) \otimes \mathbf{K}$$

are isomorphic as inclusions of C^*-algebras. We identify them as inclusions of C^*-algebras in this paper.

It is also well-known that there exists a canonical faithful conditional expectation from $A \rtimes_{\alpha, w_\alpha, r} G$ onto A. The definition of a conditional expectation can be found in Blackadar [2, II. 6.10]. One way to construct E^A is as follows.

Let $E^{M(A)}$ be the faithful canonical conditional expectation from $M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G$ onto $M(A)$ defined in Bédos and Conti [1, §3]. Then we may let E^A be the restriction of $E^{M(A)}$ to $A \rtimes_{\alpha, w_\alpha, r} G$, that is, $E^A = E^{M(A)}|_{A \rtimes_{\alpha, w_\alpha, r} G}$.

For each $t \in G$, let δ_t be the function from G to $M(A)$ defined by

$$\delta_t(s) = \begin{cases} 1_{M(A)}, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

Since $\delta_t \in \ell^1(G, M(A))$, $\delta_t \in M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G$. We regard δ_t as an element in $M(A \rtimes_{\alpha, w_\alpha, r} G)$ by the above inclusion $M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G \subset M(A \rtimes_{\alpha, w_\alpha, r} G)$.

Let $\{u_i\}_{i \in I}$ be an approximate unit of A with $\|u_i\| \leq 1$ for any $i \in I$. We fix the approximate unit $\{u_i\}_{i \in I}$ of A in this paper. Let $x \in M(A \rtimes_{\alpha, w_\alpha, r} G)$. Then $\{E^A(x(u_i \delta_t^*))\}_{i \in I}$ is a Cauchy net in $M(A)$ under the strict topology for any $t \in G$ since $\alpha_t(a) = \delta_t a \delta_t^*$ for any $t \in G$ and $a \in A$. Indeed, for any

$i, j \in I, a \in A$,

$$\begin{aligned}
&\|a(E^A(xu_i\delta_t^*) - E^A(xu_j\delta_t^*))\| \\
&\quad = \|aE^A(x\delta_t^*(\alpha_t(u_i) - \alpha_t(u_j)))\| \\
&\quad = \|E^A(ax\delta_t^*(\alpha_t(u_i) - \alpha_t(u_j)))\| \\
&\quad = \|E^A(ax\delta_t^*)(\alpha_t(u_i) - \alpha_t(u_j))\| \\
&\quad = \|\alpha_t^{-1}(E^A(ax\delta_t^*))(u_i - u_j)\| \longrightarrow 0 \quad (i, j \to \infty).
\end{aligned}$$

Similarly,

$$\|(E^A(xu_i\delta_t^*) - E^A(xu_j\delta_t^*))a\| \to 0 \quad (i, j \to \infty).$$

Let $x_t = \lim_i E^A(xu_i\delta_t^*)$ for every $x \in M(A \rtimes_{\alpha, w_\alpha, r} G)$ and $t \in G$, where the limit is taken under the strict topology in $M(A)$. Then $x_t \in M(A)$ for all $t \in G$.

DEFINITION 2.2. Let $x \in M(A \rtimes_{\alpha, w_\alpha, r} G)$. For any $t \in G$, let x_t be the element in $M(A)$ defined in the above. We call x_t the *Fourier coefficient* of x at $t \in G$, and call $\{x_t\}_{t \in G}$ the *Fourier coefficients* of x.

We note that if $x \in A \rtimes_{\alpha, w_\alpha, r} G$, then $x_t = E^A(x\delta_t^*)$ is the usual Fourier coefficient of x at $t \in G$, and that if $x \in M(A)$, then

$$x_t = \begin{cases} x, & \text{if } t = e, \\ 0, & \text{if } t \neq e. \end{cases}$$

LEMMA 2.3. *Let $x \in A \rtimes_{\alpha, w_\alpha, r} G$. If $x_t = 0$ for all $t \in G$, then $x = 0$.*

PROOF. For any $a \in A$, $t \in G$, we have

$$0 = E^A(x\delta_t^*)\alpha_t(a) = E^A(x\delta_t^*\alpha_t(a)) = E^A(xa\delta_t^*)$$

since $\alpha_t(a) = \delta_t a \delta_t^*$. Since $A \rtimes_{\alpha, w_\alpha, r} G$ is the closed linear span of $\{a\delta_t^* \mid a \in A, t \in G\}$, $E^A(xy) = 0$ for all $y \in A \rtimes_{\alpha, w_\alpha, r} G$. Since E^A is faithful, $x = 0$.

LEMMA 2.4. *Let $x \in M(A \rtimes_{\alpha, w_\alpha, r} G)$. If $x_t = 0$ for all $t \in G$, then $x = 0$.*

PROOF. For any $a \in A$, $t \in G$, we have

$$0 = x_t\alpha_t(a) = \lim_i E^A(x(u_i\delta_t^*))\alpha_t(a) = \lim_i E^A(x(u_i a)\delta_t^*) = E^A(xa\delta_t^*)$$

since $\alpha_t(a) = \delta_t a \delta_t^*$. Since $E^A(xa\delta_t^*)$ is the Fourier coefficient of the element xa at $t \in G$, by Lemma 2.3, $xa = 0$ for any $a \in A$. Thus for any $a \in A$, $t \in G$, $xa\delta_t = 0$. Since $A \rtimes_{\alpha, w_\alpha, r} G$ is the closed linear span of $\{a\delta_t \mid a \in A, t \in G\}$,

$xy = 0$ for any $y \in A \rtimes_{\alpha, w_\alpha, r} G$. We note that by [15, Proposition 3.12.3], x can be regarded as a double centralizer on $A \rtimes_{\alpha, w_\alpha, r} G$ by multiplication. Hence $x = 0$.

DEFINITION 2.5. Let (α, w_α) and (β, w_β) be twisted actions of G on a C^*-algebra A. We say that (α, w_α) and (β, w_β) are *exterior equivalent* if there are unitary elements $\{w_t\}_{t\in G} \subset M(A)$ satisfying the following:

(1) $\beta_t = \mathrm{Ad}(w_t) \circ \alpha_t$,

(2) $w_{ts} = w_\beta(t, s)^* w_t \underline{\alpha_t}(w_s) w_\alpha(t, s)$,

for any $t, s \in G$.

We refer to Packer and Raeburn [14] for more details about this notion.

Let A and B be C^*algebras. Let X be an $A - B$-equivalence bimodule (see Gracia-Bondía, Várilly and Figueroa [8]). Note that X is often called an $A - B$-imprimitivity bimodule (see e.g. Raeburn and Williams [16]). For any $a \in A, b \in B, x \in X$, we denote by $a \cdot x$ the left A-action on X and by $x \cdot b$ the right B-action on X. Let $\widetilde{X}$ be the dual $B - A$-equivalence bimodule of X and let $\widetilde{x}$ denote the element in $\widetilde{X}$ associated to an element $x \in X$. Also, we regard X as a Hilbert $M(A) - M(B)$-bimodule in the sense of Brown, Mingo and Shen [4] as follows: Let $\mathbf{B}_B(X)$ be the C^*-algebra of all adjointable right B-linear operators on X. We recall that an adjointable right B-linear operator on X is bounded. Then $\mathbf{B}_B(X)$ can be identified with $M(A)$. Similarly let ${}_A\mathbf{B}(X)$ be the C^*-algebra of all adjointable left A-linear operators on X. Then ${}_A\mathbf{B}(X)$ can be identified with $M(B)$. Using these identifications, we may regard X as a Hilbert $M(A) - M(B)$-bimodule. Let $\mathrm{Aut}(X)$ be the group of all bijective bounded linear maps on X.

DEFINITION 2.6. Let (α, w_α) and (β, w_β) be twisted actions of G on C^*-algebra A and B, respectively. We say that (α, w_α) and (β, w_β) are *strongly Morita equivalent* if there are an $A - B$-equivalence bimodule X and a map λ from G to $\mathrm{Aut}(X)$ satisfying the following:

(1) $\alpha_t({}_A\langle x, y\rangle) = {}_A\langle \lambda_t(x), \lambda_t(y)\rangle$,

(2) $\beta_t(\langle x, y\rangle_B) = \langle \lambda_t(x), \lambda_t(y)\rangle_B$,

(3) $(\lambda_t \circ \lambda_s)(x) = w_\alpha(t, s) \cdot \lambda_{ts}(x) \cdot w_\beta(t, s)^*$,

for any $t, s \in G$, $x, y \in X$, where we regard X as a Hilbert $M(A) - M(B)$-bimodule in the above way.

We note that if the twisted actions (α, w_α) and (β, w_β) of G on A are exterior equivalent, then they are strongly Morita equivalent. Indeed, let $\{w_t\}_{t\in G}$ be unitary elements in $M(A)$ satisfying Conditions (1), (2) in Definition 2.5. We regard A as the trivial $A-A$-equivalence bimodule in the usual way and we denote it by X_0. For any $t \in G$, let $\lambda_t \in \mathrm{Aut}(X_0)$ be defined by

$$\lambda_t(x) = \alpha_t(x)w_t^*$$

for all $x \in X_0$. Then one can easily check that λ satisfies Conditions (1)–(3) in Definition 2.6.

We recall the definition of strong Morita equivalence for inclusions of C^*-algebras.

Definition 2.7. Two inclusions of C^*-algebras $A \subset C$ and $B \subset D$ with $\overline{AC} = C$ and $\overline{BD} = D$ are said to be *strongly Morita equivalent* if there exists a $C-D$-equivalence bimodule Y having a closed subspace X satisfying the following conditions:

(1) $a \cdot x \in X$ and ${}_C\langle x, y\rangle \in A$ for all $a \in A, x, y \in X$,
(2) $\overline{{}_C\langle X, X\rangle} = A$ and $\overline{{}_C\langle Y, X\rangle} = C$,
(3) $x \cdot b \in X$ and $\langle x, y\rangle_B \in B$ for all $b \in B, x, y \in X$,
(4) $\overline{\langle X, X\rangle_D} = B$ and $\overline{\langle Y, X\rangle_D} = D$.

For each twisted action (α, w_α) of G on a C^*-algebra A, we get an inclusion of C^*-algebras $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ with $\overline{A(A \rtimes_{\alpha, w_\alpha, r} G)} = A \rtimes_{\alpha, w_\alpha, r} G$.

Let (α, w_α) and (β, w_β) be twisted actions of G on A and B, respectively. We suppose that they are strongly Morita equivalent so that there are an $A-B$-equivalence bimodule X and a map λ from G to $\mathrm{Aut}(X)$ satisfying Conditions (1)–(3) in Definition 2.6. We show that the inclusions of C^*-algebras $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent in the same way as in Combes [6] or Curto, Muhly and Williams [7]. Let L_X be the linking C^*-algebra for X defined by

$$L_X = \begin{bmatrix} A & X \\ \widetilde{X} & B \end{bmatrix} = \left\{ \begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix} \,\middle|\, a \in A, b \in B, x, y \in X \right\}.$$

For more details about linking C^*-algebras, we refer to [16, §3.2]. Let $t, s \in G$. Let γ_t be the map on L_X defined by

$$\gamma_t\left(\begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix}\right) = \begin{bmatrix} \alpha_t(a) & \lambda_t(x) \\ \widetilde{\lambda_t(y)} & \beta_t(b) \end{bmatrix}$$

for all $\left[\begin{smallmatrix} a & x \\ \widetilde{y} & b \end{smallmatrix}\right] \in L_X$. Also, let $w_\gamma(t, s)$ be the unitary element in $M(L_X)$ defined by

$$w_\gamma(t, s) = \begin{bmatrix} w_\alpha(t, s) & 0 \\ 0 & w_\beta(t, s) \end{bmatrix}$$

for all $t, s \in G$. Then by routine computations, one verifies that (γ, w_γ) is a twisted action of G on L_X. Let $L_X \rtimes_{\gamma, w_\gamma, r} G$ be the twisted reduced crossed product of L_X by (γ, w_γ). Let $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then p and q are projections in $M(L_X) \subset M(L_X \rtimes_{\gamma, w_\gamma, r} G)$ and they are full in $M(L_X \rtimes_{\gamma, w_\gamma, r} G)$. Let $Y = p(L_X \rtimes_{\gamma, w_\gamma, r} G)q$ and $X = pL_X q$. Then clearly X is a closed subspace of Y and Y is an $A \rtimes_{\alpha, w_\alpha, r} G - B \rtimes_{\beta, w_\beta, r} G$-equivalence bimodule by easy computations, where $A \rtimes_{\alpha, w_\alpha, r} G$ and $B \rtimes_{\beta, w_\beta, r} G$ are identified with $p(L_X \rtimes_{\gamma, w_\gamma, r} G)p$ and $q(L_X \rtimes_{\gamma, w_\gamma, r} G)q$, respectively. Furthermore, identifying A and B with $pL_X p$ and $qL_X q$, respectively, one readily obtains that the inclusions $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent with respect to Y and X. Hence we obtain the following proposition.

PROPOSITION 2.8. *Let (α, w_α) and (β, w_β) be twisted actions of a countable discrete group G on C^*-algebras A and B, respectively. If (α, w_α) and (β, w_β) are strongly Morita equivalent up to some automorphism of G, then the inclusions of C^*algebras $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent.*

PROOF. Let ϕ be an automorphism of G. Let $(\alpha^\phi, w_\alpha^\phi)$ be the twisted action of G on A defined by

$$\alpha_t^\phi(a) = \alpha_{\phi(t)}(a) \qquad w_\alpha^\phi(t, s) = w_\alpha(\phi(t), \phi(s))$$

for any $a \in A$, $t, s \in G$. Then there is an isomorphism π of $A \rtimes_{\alpha, w_\alpha, r} G$ onto $A \rtimes_{\alpha^\phi, w_\alpha^\phi, r} G$ with $\pi|_A = \mathrm{id}$ on A. Thus we may assume that (α, w_α) and (β, w_β) are strongly Morita equivalent. Hence by the above discussion, we get that $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent.

3. Relative commutants of inclusions of σ-unital C^*-algebras

Let A be a C^*-algebra and p a projection in $M(A \otimes \mathbf{K})$. We note that by [9, E 1.1.8], there is an isomorphism of $M(p(A \otimes \mathbf{K})p)$ onto $pM(A \otimes \mathbf{K})p$ such that its restriction to $p(A \otimes \mathbf{K})p$ is the identity map on $p(A \otimes \mathbf{K})p$. We identify $M(p(A \otimes \mathbf{K})p)$ with $pM(A \otimes \mathbf{K})p$ by the above isomorphism. We begin this section with the following lemma:

LEMMA 3.1. *Let $A \subset C$ be an inclusion of C^*-algebras with $\overline{AC} = C$. Then the following conditions are equivalent:*

(1) $A' \cap M(C) = \mathbf{C}1$,

(2) $(A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}) = \mathbf{C}1$.

PROOF. (1) $\Rightarrow$ (2): Let $x \in (A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K})$. Then for any $a \in A$, $k \in \mathbf{K}$,

$$x(a \otimes k) = (a \otimes k)x.$$

Since A is dense in $M(A)$ under the strict topology and $\overline{AC} = C$, $x(1 \otimes k) = (1 \otimes k)x$ for any $k \in \mathbf{K}$, where 1 denotes $1_{M(C)}$. We note that

$$\begin{aligned}(1 \otimes e_{11})M(C \otimes \mathbf{K})(1 \otimes e_{11}) &= M((1 \otimes e_{11})(C \otimes \mathbf{K})(1 \otimes e_{11})) \\ &= M(C \otimes e_{11}) = M(C) \otimes e_{11}.\end{aligned}$$

For any $a \in A$,

$$(1 \otimes e_{11})x(1 \otimes e_{11})(a \otimes e_{11}) = (a \otimes e_{11})(1 \otimes e_{11})x(1 \otimes e_{11}).$$

Since $A' \cap M(C) = \mathbf{C}1$, there is a $c \in \mathbf{C}$ such that $(1 \otimes e_{11})x(1 \otimes e_{11}) = c(1 \otimes e_{11})$. Thus $x(1 \otimes e_{11}) = c(1 \otimes e_{11})$. For any $n \in \mathbf{N}$,

$$\begin{aligned}x(1 \otimes e_{nn}) &= x(1 \otimes e_{n1})(1 \otimes e_{11})(1 \otimes e_{1n}) \\ &= (1 \otimes e_{n1})x(1 \otimes e_{11})(1 \otimes e_{1n}) \\ &= (1 \otimes e_{n1})c(1 \otimes e_{11})(1 \otimes e_{1n}) \\ &= c(1 \otimes e_{nn}).\end{aligned}$$

Hence $x\big(1 \otimes \sum_{i=1}^n e_{ii}\big) = c\big(1 \otimes \sum_{i=1}^n e_{ii}\big)$ for any $n \in \mathbf{N}$. Since $1 \otimes \sum_{i=1}^n e_{ii}$ is strictly convergent to $1_{M(C \otimes \mathbf{K})}$ as $n \to \infty$, $x = c1$. Thus $(A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}) = \mathbf{C}1$.

(2) $\Rightarrow$ (1): Let $x \in A' \cap M(C)$. Then $x \otimes 1_{M(\mathbf{K})} \in M(C \otimes \mathbf{K})$. For any $a \in A$, $k \in \mathbf{K}$,

$$(a \otimes k)(x \otimes 1_{M(\mathbf{K})}) = ax \otimes k = xa \otimes k = (x \otimes 1_{M(\mathbf{K})})(a \otimes k).$$

Thus $x \otimes 1_{M(\mathbf{K})} \in (A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}) = \mathbf{C}1_{M(C \otimes \mathbf{K})}$. Therefore, there is a $c \in \mathbf{C}$ such that $x \otimes 1_{M(\mathbf{K})} = c1_{M(C \otimes \mathbf{K})}$. Hence, $x = c1_{M(C \otimes \mathbf{K})}$. Thus $A' \cap M(C) = \mathbf{C}1$.

Let A and B be σ-unital C^*-algebras and $A \subset C$ and $B \subset D$ inclusions of C^*-algebras with $\overline{AC} = C$ and $\overline{BD} = D$. Then C and D are also σ-unital. We suppose that $A \subset C$ and $B \subset D$ are strongly Morita equivalent. Then in the same way as in the proof of [10, Proposition 3.5] or Brown, Green and Rieffel [3, Proposition 3.1], there is an isomorphism θ of $D \otimes \mathbf{K}$ onto $C \otimes \mathbf{K}$ such that $\theta|_{B \otimes \mathbf{K}}$ is an isomorphism of $B \otimes \mathbf{K}$ onto $A \otimes \mathbf{K}$. Let $p = \underline{\theta}(1_{M(B)} \otimes e_{11})$. Then p is a projection in $M(A \otimes \mathbf{K}) \subset M(C \otimes \mathbf{K})$ and p is full in $A \otimes \mathbf{K}$ and $C \otimes \mathbf{K}$, that is,

$$\overline{(A \otimes \mathbf{K})p(A \otimes \mathbf{K})} = A \otimes \mathbf{K}$$

and

$$\overline{(C \otimes \mathbf{K})p(C \otimes \mathbf{K})} = C \otimes \mathbf{K}.$$

By the definitions of θ and p, the inclusion of C^*-algebras

$$(1_{M(B)} \otimes e_{11})(B \otimes \mathbf{K})(1_{M(B)} \otimes e_{11}) \subset (1_{M(B)} \otimes e_{11})(D \otimes \mathbf{K})(1_{M(B)} \otimes e_{11})$$

is isomorphic to the inclusion of C^*-algebras

$$p(A \otimes \mathbf{K})p \subset p(C \otimes \mathbf{K})p$$

as inclusions of C^*-algebras. We will show that if $B' \cap M(D) = \mathbf{C}1$, then $A' \cap M(C) = \mathbf{C}1$.

LEMMA 3.2. *With the above notation and assumptions,*

$$(A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}) \cong ((A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}))p.$$

PROOF. Let π be the map from $(A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K})$ to $((A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}))p$ defined by $\pi(x) = xp$ for any $x \in (A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K})$. Let $x \in (A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K})$. Then for any $a \in A \otimes \mathbf{K}$, $xa = ax$. Hence $xa = ax$ for any $a \in M(A \otimes \mathbf{K})$. This implies that π is a homomorphism of $(A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K})$ to $((A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}))p$. It is clear that π is surjective. We show that π is injective. We suppose that $\pi(x) = 0$. Then $xp = 0$. For any $y, z \in A \otimes \mathbf{K}$, $yxpz = 0$. Since $x \in (A \otimes \mathbf{K})'$, $xypz = 0$. Since $\overline{(A \otimes \mathbf{K})p(A \otimes \mathbf{K})} = A \otimes \mathbf{K}$, $xa = 0$ for any $a \in A \otimes \mathbf{K}$. Let c be any element in $C \otimes \mathbf{K}$ and let $\{u_i\}$ be an approximate unit of $A \otimes \mathbf{K}$. Then $u_i c$ is norm convergent to c since

$$\overline{(A \otimes \mathbf{K})(C \otimes \mathbf{K})} = C \otimes \mathbf{K}.$$

Thus $xc = \lim_i xu_i c = 0$ since $xu_i = 0$ for any i. Since x can be regarded as a double centralizer on A by multiplication by [15, Proposition 3.12.3], $x = 0$. Since π is injective, we obtain the conclusion.

LEMMA 3.3. *With the above notation and assumptions,*

$$((A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}))p \subset (p(A \otimes \mathbf{K})p)' \cap pM(C \otimes \mathbf{K})p.$$

PROOF. Let $x \in (A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K})$. We show that

$$xp \in (p(A \otimes \mathbf{K})p)' \cap pM(C \otimes \mathbf{K})p.$$

We note that $xa = ax$ for any $a \in A \otimes \mathbf{K}$. Hence $xp = pxp$ since $p \in M(A \otimes \mathbf{K})$. Thus $xp \in pM(C \otimes \mathbf{K})p$. Also, for any $a \in A \otimes \mathbf{K}$, $xp(pap) = xpap = (pap)xp$. Hence $xp \in (p(A \otimes \mathbf{K})p)'$. The claim clearly follows.

LEMMA 3.4. *With the above notation and assumptions, if $B' \cap M(D) = \mathbf{C}1$, then $A' \cap M(C) = \mathbf{C}1$.*

PROOF. By the discussion before Lemma 3.2,

$$(1_{M(B)} \otimes e_{11})(B \otimes \mathbf{K})(1_{M(B)} \otimes e_{11}) \subset (1_{M(B)} \otimes e_{11})(D \otimes \mathbf{K})(1_{M(B)} \otimes e_{11})$$

is isomorphic to $(p(A \otimes \mathbf{K})p \subset p(C \otimes \mathbf{K})p$ as inclusions of C^*-algebras. Also,

$$(1_{M(B)} \otimes e_{11})(B \otimes \mathbf{K})(1_{M(B)} \otimes e_{11}) \subset (1_{M(B)} \otimes e_{11})(D \otimes \mathbf{K})(1_{M(B)} \otimes e_{11})$$

is isomorphic to $B \subset D$ as inclusions of C^*-algebras. Hence since $B' \cap D = \mathbf{C}1$, $(p(A \otimes \mathbf{K})p)' \cap p(C \otimes \mathbf{K})p = \mathbf{C}p$. Thus by Lemma 3.3 $((A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}))p = \mathbf{C}p$. It follows that $(A \otimes \mathbf{K})' \cap M(C \otimes \mathbf{K}) = \mathbf{C}1$ by Lemma 3.2. Therefore, by Lemma 3.1, $A' \cap M(C) = \mathbf{C}1$.

4. Free twisted actions of a countable discrete group

We recall the following definitions (see [5], [17] and references therein). Let α be an automorphism of a C^*-algebra A.

DEFINITION 4.1. We say that α is *inner* if there is a unitary element $w \in M(A)$ such that $\alpha = \mathrm{Ad}(w)$. We say that α is *outer* if α is not inner.

DEFINITION 4.2. We say that α is *free* if α satisfies the following condition: If $x \in M(A)$ satisfies that $xa = \alpha(a)x$ for all $a \in A$, then $x = 0$.

Let (α, w_α) be a twisted action of G on a C^*-algebra A.

DEFINITION 4.3. We say that (α, w_α) is *outer* if α_t is outer for every $t \in G \setminus \{e\}$.

DEFINITION 4.4. We say that (α, w_α) is *free* if the automorphism α_t is free for every $t \in G \setminus \{e\}$.

LEMMA 4.5. *Let (α, w_α) be a twisted action of G on a C^*-algebra A. If (α, w_α) is free, then (α, w_α) is outer. Furthermore, if $A' \cap M(A) = \mathbf{C}1$, then the converse holds.*

PROOF. We suppose first that (α, w_α) is free and that (α, w_α) is not outer. Then there are an element $t \in G \setminus \{e\}$ and a unitary element $w \in M(A)$ such that $\alpha_t = \mathrm{Ad}(w)$. Thus for any $a \in A$, $\alpha_t(a) = waw^*$. Hence $wa = \alpha_t(a)w$ for all $a \in A$. Since (α, w_α) is free, $w = 0$. This gives a contradiction. Therefore, (α, w_α) is outer. Next, we suppose that $A' \cap M(A) = \mathbf{C}1$ and that (α, w_α) is outer. Assume that (α, w_α) is not free. Then there are an element $t \in G \setminus \{e\}$

and a non-zero element $x \in M(A)$ such that $xa = \alpha_t(a)x$ for all $a \in A$. Hence $ax^* = x^*\alpha_t(a)$ for any $a \in A$. Thus $x^*xa = x^*\alpha_t(a)x = ax^*x$ for any $a \in A$. Hence $x^*x \in \mathbf{C}1$. Also, $\alpha_t(a)xx^* = xax^* = xx^*\alpha_t(a)$ for any $a \in A$. Thus $xx^* \in \mathbf{C}1$. A computation gives $(x^*x - xx^*)^*(x^*x - xx^*) = 0$, i.e., x is normal, so we get that there exists some positive $c \in \mathbf{R}$ such that $x^*x = xx^* = c1$. Let $w = (1/\sqrt{c})x$. Then w is a unitary element in $M(A)$ such that $\alpha_t(a) = waw^*$ for all $a \in A$. This gives a contradiction. Therefore, (α, w_α) is free.

Let (α, w_α) be a twisted action of G on a unital C^*algebra A. Let E^A be the faithful canonical conditional expectation from $A \rtimes_{\alpha, w_\alpha, r} G$ onto A defined in §2. In the same way as Zarikian [17, Theorem 3.1.2], we obtain the following proposition.

PROPOSITION 4.6. *With the above assumption, the following conditions are equivalent:*

(1) *The conditional expectation E^A is unique,*

(2) $A' \cap (A \rtimes_{\alpha, w_\alpha, r} G) = A' \cap A$,

(3) *the twisted action (α, w_α) is free.*

PROOF. (1) $\Rightarrow$ (2): This is immediate by [17, Lemma 3.1.1].

(2) $\Rightarrow$ (3): Let $t \in G \setminus \{e\}$ and $x \in A$. We suppose that $xa = \alpha_t(a)x$ for all $a \in A$. Then since $\alpha_t(a) = \delta_t a \delta_t^*$, $\delta_t^* xa = a\delta_t^* x$ for all $a \in A$. Hence $\delta_t^* x \in A' \cap (A \rtimes_{\alpha, w_\alpha, r} G)$. By the definition of E^A, $E^A(\delta_t^* x) = 0$. On the other hand, since $A' \cap (A \rtimes_{\alpha, w_\alpha, r} G) = A' \cap A$, $E^A(\delta_t^* x) = \delta_t^* x$. Hence $x = 0$.

(3) $\Rightarrow$ (1): We suppose that (α, w_α) is free. Let F be a conditional expectation from $A \rtimes_{\alpha, w_\alpha, r} G$ onto A. Let $t \in G \setminus \{e\}$. For any $a \in A$,

$$F(\delta_t)a = F(\delta_t a) = F(\alpha_t(a)\delta_t) = \alpha_t(a)F(\delta_t)$$

since $\alpha_t(a) = \delta_t a \delta_t^*$. Since (α, w_α) is free, $F(\delta_t) = 0$. Hence $F(a\delta_t) = aF(\delta_t) = 0 = E^A(a\delta_t)$ for all $a \in A$, $t \in G \setminus \{e\}$, and it follows that $F = E^A$.

COROLLARY 4.7. *Let (α, w_α) be a twisted action of G on a C^*-algebra A. Then the following conditions are equivalent:*

(1) $A' \cap M(A \rtimes_{\alpha, w_\alpha, r} G) = \mathbf{C}1$,

(2) $A' \cap M(A) = \mathbf{C}1$ *and (α, w_α) is free,*

(3) $A' \cap M(A) = \mathbf{C}1$ *and (α, w_α) is outer.*

PROOF. (1) $\Rightarrow$ (2): It is clear that $A' \cap M(A) = \mathbf{C}1$. We show that (α, w_α) is free. Since $M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G \subset M(A \rtimes_{\alpha, w_\alpha, r} G)$,

$$M(A)' \cap (M(A) \rtimes_{\underline{\alpha}, w_\alpha, r} G) \subset A' \cap M(A \rtimes_{\alpha, w_\alpha, r} G) = \mathbf{C}1.$$

Hence by Proposition 4.6, $(\underline{\alpha}, w_\alpha)$ is free. Let $t \in G \setminus \{e\}$ and $x \in M(A)$. We suppose that $xa = \alpha_t(a)x$ for all $a \in A$. Then since $\underline{\alpha_t}$ is strictly continuous, $xa = \underline{\alpha_t}(a)x$ for all $a \in M(A)$. Hence $x = 0$ since $(\underline{\alpha}, w_\alpha)$ is free. Thus (α, w_α) is free.

(2) $\Rightarrow$ (1): Let $x \in A' \cap M(A \rtimes_{\alpha, w_\alpha, r} G)$. For any $t \in G$ let x_t be the Fourier coefficient of x at $t \in G \setminus \{e\}$ defined in §2. Then for any $a \in A$,

$$\begin{aligned} x_t a &= \lim_i E^A(xu_i\delta_t^*)a = \lim_i E^A(xu_i\delta_t^* a) = \lim_i E^A(xu_i\alpha_t^{-1}(a)\delta_t^*) \\ &= E^A(x\alpha_t^{-1}(a)\delta_t^*) = E^A(\alpha_t^{-1}(a)x\delta_t^*) = \lim_i E^A(\alpha_t^{-1}(a)xu_i\delta_t^*) \\ &= \alpha_t^{-1}(a)\lim_i E^A(xu_i\delta_t^*) = \alpha_t^{-1}(a)x_t. \end{aligned}$$

Since (α, w_α) is free $x_t = 0$ for every $t \in G \setminus \{e\}$. Thus, setting $y = x - x_e \in M(A \rtimes_{\alpha, w_\alpha, r} G)$, we get that $y_t = 0$ for all $t \in G$. Lemma 2.4 gives that $y = 0$. Hence, $x = x_e \in A' \cap M(A) = \mathbf{C}1$. This shows that (1) holds.

The equivalence (2) $\Leftrightarrow$ (3) follows from Lemma 4.5.

Lemma 4.8. *Let (α, w_α) be a free twisted action of G on a C^*-algebra A. Then $(\alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ is a free twisted action of G on $A \otimes \mathbf{K}$.*

Proof. Let $t \in G \setminus \{e\}$. Let x be an element in $M(A \otimes \mathbf{K})$ satisfying that

$$xy = (\alpha_t \otimes \mathrm{id})(y)x$$

for all $y \in A \otimes \mathbf{K}$. This implies that

$$xy = (\underline{\alpha_t} \otimes \mathrm{id})(y)x$$

for all $y \in M(A) \otimes \mathbf{K}$. Hence we get that

$$x(1 \otimes e_{ii}) = (1 \otimes e_{ii})x$$

for all $i \in \mathbf{N}$. Also, we have that

$$x(a \otimes e_{ii}) = (\alpha_t(a) \otimes e_{ii})x$$

for any $a \in A$, $i \in \mathbf{N}$. Hence

$$(1 \otimes e_{ii})x(1 \otimes e_{ii})(a \otimes e_{ii}) = (\alpha_t(a) \otimes e_{ii})(1 \otimes e_{ii})x(1 \otimes e_{ii})$$

for all $a \in A$, $i \in \mathbf{N}$. Since (α, w_α) is a free twisted action on A and we can identify A with $(1 \otimes e_{ii})(A \otimes \mathbf{K})(1 \otimes e_{ii})$,

$$(1 \otimes e_{ii})x(1 \otimes e_{ii}) = 0$$

for all $i \in \mathbf{N}$. Thus, since $x(1 \otimes e_{ii}) = 0$ for any $i \in \mathbf{N}$, $x = 0$.

5. Strong Morita equivalence for inclusions of C^*-algebras

Let (α, w_α) and (β, w_β) be twisted actions of G on σ-unital C^*-algebras A and B and let $A \rtimes_{\alpha, w_\alpha, r} G$ and $B \rtimes_{\beta, w_\beta, r} G$ denote the twisted reduced crossed products of A and B by (α, w_α) and (β, w_β), respectively. Then we obtain the inclusions of C^*-algebras $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$.

We suppose that $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent. We regard $\mathbf{K}$ as the trivial $\mathbf{K} - \mathbf{K}$-equivalence bimodule. Then the inclusions of C^*algebras $A \otimes \mathbf{K} \subset (A \rtimes_{\alpha, w_\alpha, r} G) \otimes \mathbf{K}$ and $B \otimes \mathbf{K} \subset (B \rtimes_{\beta, w_\beta, r} G) \otimes \mathbf{K}$ are also strongly Morita equivalent. Hence in the same way as in the proof of [10, Proposition 3.5] or Brown, Green and Rieffel [3, Proposition 3.1], there is an isomorphism θ of $(A \rtimes_{\alpha, w_\alpha, r} G) \otimes \mathbf{K}$ onto $(B \rtimes_{\beta, w_\beta, r} G) \otimes \mathbf{K}$ such that $\theta|_{A \otimes \mathbf{K}}$ is an isomorphism of $A \otimes \mathbf{K}$ onto $B \otimes \mathbf{K}$. Also, as explained in §2, the inclusion of C^*-algebras

$$A \otimes \mathbf{K} \subset (A \rtimes_{\alpha, w_\alpha, r} G) \otimes \mathbf{K}$$

is isomorphic to

$$A \otimes \mathbf{K} \subset (A \otimes \mathbf{K}) \rtimes_{\alpha \otimes \mathrm{id}, w_\alpha \otimes 1, r} G$$

as inclusions of C^*-algebras, and

$$B \otimes \mathbf{K} \subset (B \rtimes_{\beta, w_\beta, r} G) \otimes \mathbf{K}$$

is isomorphic to

$$B \otimes \mathbf{K} \subset (B \otimes \mathbf{K}) \rtimes_{\beta \otimes \mathrm{id}, w_\beta \otimes 1, r} G$$

as inclusions of C^*-algebras. Thus θ can be regarded as an isomorphism of $(A \otimes \mathbf{K}) \rtimes_{\alpha \otimes \mathrm{id}, w_\alpha \otimes 1, r} G$ onto $(B \otimes \mathbf{K}) \rtimes_{\beta \otimes \mathrm{id}, w_\beta \otimes 1, r} G$ such that $\theta|_{A \otimes \mathbf{K}}$ is an isomorphism of $A \otimes \mathbf{K}$ onto $B \otimes \mathbf{K}$. Set $\theta_r = \theta|_{A \otimes \mathbf{K}}$, $\gamma_t = \theta_r \circ (\alpha_t \otimes \mathrm{id}) \circ \theta_r^{-1}$ and $w_\gamma(t, s) = \underline{\theta_r}(w_\alpha(t, s) \otimes 1)$ for all $t, s \in G$.

Lemma 5.1. *With the above notation and assumptions,* $(\alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ *and* (γ, w_γ) *are strongly Morita equivalent.*

Proof. Let θ_r be the isomorphism of $A \otimes \mathbf{K}$ onto $B \otimes \mathbf{K}$ defined as above. Let $X_{\theta_r^{-1}}$ be the $A \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodule obtained in the following way: Set $X_{\theta_r^{-1}} = A \otimes \mathbf{K}$ as a vector space over $\mathbf{C}$. For any $a \in A \otimes \mathbf{K}$, $b \in B \otimes \mathbf{K}$, $x, y \in X_{\theta_r^{-1}}$, define

$$a \cdot x = ax, \quad x \cdot b = x\theta_r^{-1}(b),$$
$${}_{A \otimes \mathbf{K}}\langle x, y\rangle = xy^*, \quad \langle x, y\rangle_{B \otimes \mathbf{K}} = \theta_r(x^* y).$$

By easy computations, $X_{\theta_r^{-1}}$ is an $A \otimes \mathbf{K} - B \otimes \mathbf{K}$-equivalence bimodule. For any $t \in G$, let $\lambda_t \in \mathrm{Aut}(X_{\theta_r^{-1}})$ defined by

$$\lambda_t(x) = (\alpha_t \otimes \mathrm{id})(x)$$

for all $x \in X_{\theta_r^{-1}}$. Then

$$\begin{aligned} {}_{A\otimes\mathbf{K}}\langle \lambda_t(x), \lambda_t(y)\rangle &= (\alpha_t \otimes \mathrm{id})(xy^*) = (\alpha_t \otimes \mathrm{id})({}_{A\otimes\mathbf{K}}\langle x, y\rangle), \\ \langle \lambda_t(x), \lambda_t(y)\rangle_{B\otimes\mathbf{K}} &= \theta_r((\alpha_t \otimes \mathrm{id})(x^*y)) = \gamma_t(\theta_r(x^*y)) = \gamma_t(\langle x, y\rangle_{B\otimes\mathbf{K}}) \end{aligned}$$

for all $x, y \in X_{\theta_r^{-1}}$, $t \in G$. Therefore, we obtain the conclusion.

Since the systems $(A \otimes \mathbf{K}, G, \alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ and $(B \otimes \mathbf{K}, G, \gamma, w_\gamma)$ are conjugate, there is an isomorphism π_θ of $(A \otimes \mathbf{K}) \rtimes_{\alpha\otimes\mathrm{id}, w_\alpha\otimes 1, r} G$ onto $(B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G$ such that $\pi_\theta|_{A\otimes\mathbf{K}} = \theta_r$. Then we have the following commutative diagram:

$$\begin{tikzcd} (B \otimes \mathbf{K}) \rtimes_{\beta\otimes\mathrm{id}, w_\beta\otimes 1, r} G & (A \otimes \mathbf{K}) \rtimes_{\alpha\otimes\mathrm{id}, w_\alpha\otimes 1, r} G \arrow[l, "\theta"'] \arrow[r, "\pi_\theta"] & (B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G \\ B \otimes \mathbf{K} \arrow[u] & A \otimes \mathbf{K} \arrow[l, "\theta_r"'] \arrow[r, "\theta_r"] \arrow[u] & B \otimes \mathbf{K}. \arrow[u] \end{tikzcd}$$

Let ρ be the isomorphism of $(B \otimes \mathbf{K}) \rtimes_{\beta\otimes\mathrm{id}, w_\beta\otimes 1, r} G$ onto $(B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G$ defined by $\rho = \pi_\theta \circ \theta^{-1}$. Then $\rho|_{B\otimes\mathbf{K}} = \mathrm{id}$ on $B \otimes \mathbf{K}$. For any $t \in G$, let $u_t^{\beta\otimes\mathrm{id}}$ be the canonical unitary element in $M((B \otimes \mathbf{K}) \rtimes_{\beta\otimes\mathrm{id}, w_\beta\otimes 1, r} G)$ implementing $\beta_t \otimes \mathrm{id}$. Set $v_t = \underline{\rho}(u_t^{\beta\otimes\mathrm{id}})$ for each $t \in G$. Then for any $b \in B \otimes \mathbf{K}$,

$$v_t b v_t^* = \underline{\rho}(u_t^{\beta\otimes\mathrm{id}})\rho(b)\underline{\rho}(u_t^{\beta\otimes\mathrm{id}\,*}) = \rho((\beta_t \otimes \mathrm{id})(b)) = (\beta_t \otimes \mathrm{id})(b).$$

Hence $v_t b = (\beta_t \otimes \mathrm{id})(b)v_t$ for all $b \in B \otimes \mathbf{K}$, $t \in G$.

Let $E^{B\otimes\mathbf{K}}$ be the canonical conditional expectation from $(B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G$ onto $B \otimes \mathbf{K}$ and let $\{U_i\}_{i\in I}$ be an approximate unit of $B \otimes \mathbf{K}$. Let $\{b_s^t\}_{s\in G}$ be the Fourier coefficients of v_t in $M((B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G)$ with respect to $\{U_i\}_{i\in I}$.

LEMMA 5.2. *With the above notation and assumptions, for any $t, s \in G$, $b \in B \otimes \mathbf{K}$, we have*

$$b_s^t b = ((\beta_t \otimes \mathrm{id}) \circ \gamma_s^{-1})(b) b_s^t. \qquad (*)$$

PROOF. Let u_s^γ be the canonical unitary element in $M((B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G)$ implementing γ_s for any $s \in G$. Let $t \in G$ and $b \in B \otimes \mathbf{K}$. Then since $\|bU_i - U_i b\| \to 0\,(i \to \infty)$, the Fourier coefficient of $v_t b$ at $s \in G$ is given by

$$\lim_i E^{B\otimes\mathbf{K}}(v_t b U_i u_s^{\gamma *}) = \lim_i E^{B\otimes\mathbf{K}}(v_t U_i b u_s^{\gamma *}) = \lim_i E^{B\otimes\mathbf{K}}(v_t U_i u_s^{\gamma *}\gamma_s(b))$$
$$= \lim_i E^{B\otimes\mathbf{K}}(v_t U_i u_s^{\gamma *})\gamma_s(b) = b_s^t \gamma_s(b).$$

Moreover the Fourier coefficient of $(\beta_t \otimes \mathrm{id})(b)v_t$ at $s \in G$ is given by

$$\lim_i E^{B\otimes\mathbf{K}}((\beta_t \otimes \mathrm{id})(b)v_t U_i u_s^{\gamma *}) = (\beta_t \otimes \mathrm{id})(b) \lim_i E^{B\otimes\mathbf{K}}(v_t U_i u_s^{\gamma *})$$
$$= (\beta_t \otimes \mathrm{id})(b) b_s^t.$$

Since $v_t b = (\beta_t \otimes \mathrm{id})(b)v_t$, we get that

$$b_s^t \gamma_s(b) = (\beta_t \otimes \mathrm{id})(b) b_s^t$$

for all $t, s \in G$, $b \in B \otimes \mathbf{K}$. Since b is an arbitrary element in $B \otimes \mathbf{K}$, replacing b by $\gamma_s^{-1}(b)$, we obtain that

$$b_s^t b = (\beta_t \otimes \mathrm{id})(\gamma_s^{-1}(b)) b_s^t$$

for all $t, s \in G$, $b \in B \otimes \mathbf{K}$, as desired.

Assume now that the actions $(\alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ and $(\beta \otimes \mathrm{id}, w_\beta \otimes 1)$ are free on $A \otimes \mathbf{K}$ and $B \otimes \mathbf{K}$, respectively and that $(B \otimes \mathbf{K})' \cap M(B \otimes \mathbf{K}) = \mathbf{C}1_{M(B\otimes\mathbf{K})}$. Let t be any element in G. Since $v_t \neq 0$, there is an element $s_0 \in G$ such that $b_{s_0}^t \neq 0$. Since $s_0 \in G$ is depending on $t \in G$, we denote it by $\phi(t)$. Hence $b_{\phi(t)}^t \neq 0$, and by Lemma 5.2 for any $b \in B \otimes \mathbf{K}$, we have

$$b_{\phi(t)}^t b = ((\beta_t \otimes \mathrm{id}) \circ \gamma_{\phi(t)}^{-1})(b) b_{\phi(t)}^t.$$

Since $(B \otimes \mathbf{K})' \cap M(B \otimes \mathbf{K}) = \mathbf{C}1$, by the proof of Lemma 4.5, there is a unitary element $w_t \in M(B \otimes \mathbf{K})$ such that

$$((\beta_t \otimes \mathrm{id}) \circ \gamma_{\phi(t)}^{-1})(b) = w_t b w_t^*$$

for all $b \in B \otimes \mathbf{K}$. That is,

$$(\beta_t \otimes \mathrm{id})(b) = w_t \gamma_{\phi(t)}(b) w_t^*$$

for all $b \in B \otimes \mathbf{K}$. Thus for any $s \in G$, $b \in B \otimes \mathbf{K}$,

$$(\beta_t \otimes \mathrm{id})(\gamma_{s^{-1}}(b)) = w_t \gamma_{\phi(t)}(\gamma_{s^{-1}}(b)) w_t^*.$$

By Equation $(*)$ in Lemma 5.2 and the above equation,

$$\begin{aligned}
b_s^t b &= (\beta_t \otimes \mathrm{id})(\gamma_s^{-1}(b))b_s^t \\
&= ((\beta_t \otimes \mathrm{id}) \circ \mathrm{Ad}(w_\gamma(s^{-1}, s)^* \circ \gamma_{s^{-1}})(b)b_s^t \\
&= [\mathrm{Ad}((\underline{\beta_t} \otimes \mathrm{id})(w_\gamma(s^{-1}, s)^*)) \circ (\beta_t \otimes \mathrm{id}) \circ \gamma_{s^{-1}}](b)b_s^t \\
&= [\mathrm{Ad}((\underline{\beta_t} \otimes \mathrm{id})(w_\gamma(s^{-1}, s)^*)) \circ \mathrm{Ad}(w_t) \circ \gamma_{\phi(t)} \circ \gamma_{s^{-1}}](b)b_s^t \\
&= [\mathrm{Ad}((\underline{\beta_t} \otimes \mathrm{id})(w_\gamma(s^{-1}, s)^*)w_t) \circ \mathrm{Ad}(w_\gamma(\phi(t), s^{-1})) \circ \gamma_{\phi(t)s^{-1}}](b)b_s^t \\
&= [\mathrm{Ad}((\underline{\beta_t} \otimes \mathrm{id})(w_\gamma(s^{-1}, s)^*)w_t w_\gamma(\phi(t), s^{-1})) \circ \gamma_{\phi(t)s^{-1}}](b)b_s^t.
\end{aligned}$$

Hence

$$\begin{aligned}
&w_\gamma(\phi(t), s^{-1})^* w_t^* (\underline{\beta_t} \otimes \mathrm{id})(w_\gamma(s^{-1}, s))b_s^t b \\
&\qquad\qquad = \gamma_{\phi(t)s^{-1}}(b) w_\gamma(\phi(t), s^{-1})^* w_t^* (\underline{\beta_t} \otimes \mathrm{id})(w_\gamma(s^{-1}, s))b_s^t
\end{aligned}$$

for any $s \in G, b \in B \otimes \mathbf{K}$. Since $(\alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ is free, so is (γ, w_γ). Hence the automorphism $\gamma_{\phi(t)s^{-1}}$ is free if $s \neq \phi(t)$. Thus $b_s^t = 0$ for any $s \in G$ with $s \neq \phi(t)$. Therefore, for any $t \in G$, there is a unique $\phi(t) \in G$ such that

$$b_{\phi(t)}^t \neq 0, \quad \underline{\rho}(u_t^{\beta \otimes \mathrm{id}}) = v_t = b_{\phi(t)}^t u_{\phi(t)}^\gamma.$$

By the above discussions, we can regard ϕ as a map on G. We show that ϕ is an automorphism of G.

LEMMA 5.3. *With above notation and assumptions, ϕ is an automorphism of G.*

PROOF. Let $t, s \in G$. Then

$$v_t v_s = \underline{\rho}(u_t^{\beta \otimes \mathrm{id}} u_s^{\beta \otimes \mathrm{id}}) = \underline{\rho}((w_\beta(t, s) \otimes 1)u_{ts}^{\beta \otimes \mathrm{id}}) = \underline{\rho}(w_\beta(t, s) \otimes 1)v_{ts}.$$

Since $\rho(b) = b$ for any $b \in B \otimes \mathbf{K}$, $\underline{\rho}(w_\beta(t, s) \otimes 1) = w_\beta(t, s) \otimes 1$. Hence $v_t v_s = (w_\beta(t, s) \otimes 1)v_{ts}$. Then the Fourier coefficient of $v_t v_s$ at $r \in G$ can be computed as follows: Since $v_t = b_{\phi(t)}^t u_{\phi(t)}^\gamma$ and $v_s = b_{\phi(s)}^s u_{\phi(s)}^\gamma$, we get that

$$\begin{aligned}
&\lim_i E^{B \otimes \mathbf{K}}(v_t v_s U_i u_r^{\gamma *}) \\
&\qquad = \lim_i E^{B \otimes \mathbf{K}}(b_{\phi(t)}^t u_{\phi(t)}^\gamma b_{\phi(s)}^s u_{\phi(s)}^\gamma U_i u_r^{\gamma *}) \\
&\qquad = \lim_i E^{B \otimes \mathbf{K}}(b_{\phi(t)}^t \underline{\gamma}_{\phi(t)}(b_{\phi(s)}^s) u_{\phi(t)}^\gamma u_{\phi(s)}^\gamma U_i u_r^{\gamma *}) \\
&\qquad = \lim_i E^{B \otimes \mathbf{K}}(b_{\phi(t)}^t \underline{\gamma}_{\phi(t)}(b_{\phi(s)}^s) w_\gamma(\phi(t), \phi(s)) u_{\phi(t)\phi(s)}^\gamma U_i u_r^{\gamma *})
\end{aligned}$$

$$= \begin{cases} b^t_{\phi(t)}\underline{\gamma}_{\phi(t)}(b^s_{\phi(s)})w_\gamma(\phi(t),\phi(s)), & \text{if } r=\phi(t)\phi(s), \\ 0, & \text{if } r\neq\phi(t)\phi(s). \end{cases}$$

On the other hand, the Fourier coefficient of $(w_\beta(t,s)\otimes 1)v_{ts}$ at $r\in G$ can be computed as follows: Since $v_{ts}=b^{ts}_{\phi(ts)}u^\gamma_{\phi(ts)}$, we get that

$$\begin{aligned} &\lim_i E^{B\otimes\mathbf{K}}((w_\beta(t,s)\otimes 1)v_{ts}U_iu^{\gamma*}_r) \\ &\quad=\lim_i E^{B\otimes\mathbf{K}}((w_\beta(t,s)\otimes 1)b^{ts}_{\phi(ts)}u^\gamma_{\phi(ts)}U_iu^{\gamma*}_r) \\ &\quad= \begin{cases} (w_\beta(t,s)\otimes 1)b^{ts}_{\phi(ts)}, & \text{if } r=\phi(ts), \\ 0, & \text{if } r\neq\phi(ts). \end{cases} \end{aligned}$$

As $v_tv_s=(w_\beta(t,s)\otimes 1)v_{ts}$, we obtain that

$$\begin{gathered} u^\gamma_{\phi(t)\phi(s)}=u^\gamma_{\phi(ts)}, \\ b^t_{\phi(t)}\underline{\gamma}_{\phi(t)}(b^s_{\phi(s)})w_\gamma(\phi(t),\phi(s))=(w_\beta(t,s)\otimes 1)b^{ts}_{\phi(ts)} \qquad (**) \end{gathered}$$

for all $t,s\in G$. Hence $\phi(t)\phi(s)=\phi(ts)$ for all $t,s\in G$, i.e., ϕ is a homomorphism of G to G. We show that ϕ is injective. Let $t\in G\setminus\{e\}$ and suppose that $\phi(t)=e$. By the definition of ϕ, $\underline{\rho}(u^{\beta\otimes\mathrm{id}}_t)=v_t=b^t_eu^\gamma_e=b^t_e$. By Equation $(*)$ in Lemma 5.2,

$$b^t_eb=(\beta_t\otimes\mathrm{id})(b)b^t_e$$

for all $b\in B\otimes\mathbf{K}$. Since $\beta_t\otimes\mathrm{id}$ is a free automorphism of $B\otimes\mathbf{K}$, $b^t_e=0$. Hence $v_t=0$. This is a contradiction. Thus $\phi(t)\neq e$, that is, ϕ is injective. Next we show that ϕ is surjective. Since ρ is an isomorphism of $(B\otimes\mathbf{K})\rtimes_{\beta\otimes\mathrm{id},w_\beta\otimes 1,r}G$ onto $(B\otimes\mathbf{K})\rtimes_{\gamma,w_\gamma,r}G$, $(B\otimes\mathbf{K})\rtimes_{\gamma,w_\gamma,r}G$ is the closed linear span of the set

$$\{bv_t\mid b\in B\otimes\mathbf{K}, t\in G\}=\{bb^t_{\phi(t)}u^\gamma_{\phi(t)}\mid b\in B\otimes\mathbf{K}, t\in G\}$$

Also,

$$\{bb^t_{\phi(t)}u^\gamma_{\phi(t)}\mid b\in B\otimes\mathbf{K}, t\in G\}=\{bu^\gamma_{\phi(t)}\mid b\in B\otimes\mathbf{K}, t\in G\}$$

since $b^t_{\phi(t)}$ is a unitary element in $M(B\otimes\mathbf{K})$. Assume that ϕ is not surjective. Then there is some element $t_0\in G$ such that $t_0\notin\phi(G)$. For any $b\in B\otimes\mathbf{K}$ and $t\in G$, the Fourier coefficient of $bu^\gamma_{\phi(t)}$ at $t_0\in G$ is can be computed as follows:

$$\lim_i E^{B\otimes\mathbf{K}}(bu^\gamma_{\phi(t)}U_iu^{\gamma\,*}_{t_0})=\lim_i E^{B\otimes\mathbf{K}}(b\gamma_{\phi(t)}(U_i)u^\gamma_{\phi(t)}u^{\gamma\,*}_{t_0})=0.$$

Thus, for any $x \in (B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G$, the Fourier coefficient of x at $t_0 \in G$ is equal to 0. On the other hand, for an element $b \in B \otimes \mathbf{K}$ with $b \neq 0$, $bu_{t_0}^\gamma \in (B \otimes \mathbf{K}) \rtimes_{\gamma, w_\gamma, r} G$ and the Fourier coefficient of $bu_{t_0}^\gamma$ at $t_0 \in G$ is equal to b. This is a contradiction. Hence ϕ is surjective. Therefore, we have shown that ϕ is an automorphism of G.

LEMMA 5.4. *With the above notation and assumptions,* $(\beta \otimes \mathrm{id}, w_\beta \otimes 1)$ *and* $(\gamma^\phi, w_\gamma^\phi)$ *are exterior equivalent, where* $(\gamma^\phi, w_\gamma^\phi)$ *is the twisted action of* G *on* $B \otimes \mathbf{K}$ *defined by* $\gamma_t^\phi(b) = \gamma_{\phi(t)}(b)$ *and* $w_\gamma^\phi(t, s) = w_\gamma(\phi(t), \phi(s))$ *for all* $t, s \in G$, $b \in B \otimes \mathbf{K}$.

PROOF. By the discussion before Lemma 5.3, $v_t = b_{\phi(t)}^t u_{\phi(t)}^\gamma$ for any $t \in G$. Since v_t and $u_{\phi(t)}^\gamma$ are unitary elements in $M(B \otimes \mathbf{K})$, $b_{\phi(t)}^t$ is a unitary element in $M(B \otimes \mathbf{K})$ for any $t \in G$. Also, for any $b \in B \otimes \mathbf{K}$, $t \in G$

$$\begin{aligned}(\beta_t \otimes \mathrm{id})(b) = v_t b v_t^* = b_{\phi(t)}^t u_{\phi(t)}^\gamma b u_{\phi(t)}^{\gamma *} b_{\phi(t)}^{t *} &= b_{\phi(t)}^t \gamma_{\phi(t)}(b) b_{\phi(t)}^{t*} \\ &= (\mathrm{Ad}(b_{\phi(t)}^t) \circ \gamma_{\phi(t)})(b)\end{aligned}$$

Moreover, Equation $(**)$ in the proof of Lemma 5.3 says that

$$b_{\phi(t)}^t \underline{\gamma}_{\phi(t)}(b_{\phi(s)}^s) w_\gamma(\phi(t), \phi(s)) = (w_\beta(t, s) \otimes 1) b_{\phi(ts)}^{ts}$$

for all $t, s \in G$. Therefore, (β, w_β) and $(\gamma^\phi, w_\gamma^\phi)$ are exterior equivalent.

We can now state the main theorem, which shows that the converse of Proposition 2.8 holds under the irreduciblity condition described in Corollary 4.7.

THEOREM 5.5. *Let* (α, w_α) *and* (β, w_β) *be twisted actions of a countable discrete group* G *on* σ*-unital* C^**-algebras* A *and* B, *and let* $A \rtimes_{\alpha, w_\alpha, r} G$ *and* $B \rtimes_{\beta, w_\beta, r} G$ *denote the twisted reduced crossed products of* A *and* B *by* (α, w_α) *and* (β, w_β), *respectively. We suppose that* $A' \cap M(A \rtimes_{\alpha, w_\alpha, r} G) = \mathbf{C}1$. *If the inclusions of* C^**-algebras* $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ *and* $B \subset B \rtimes_{\beta, w_\beta, r} G$ *are strongly Morita equivalent, then there is an automorphism* ϕ *of* G *such that* $(\alpha^\phi, w_\alpha^\phi)$ *and* (β, w_β) *are strongly Morita equivalent, where* $(\alpha^\phi, w_\alpha^\phi)$ *is the twisted action of* G *on* A *defined by* $\alpha_t^\phi(a) = \alpha_{\phi(t)}(a)$ *and* $w_\alpha^\phi(t, s) = w_\alpha(\phi(t), \phi(s))$ *for all* $t, s \in G$, $a \in A$.

PROOF. Since $A' \cap M(A \rtimes_{\alpha, w_\alpha, r} G) = \mathbf{C}1$, by Lemma 3.4, $B' \cap M(B \rtimes_{\beta, w_\beta, r} G) = \mathbf{C}1$. Also, by Corollary 4.7, $A' \cap M(A) = \mathbf{C}1$, $B' \cap M(B) = \mathbf{C}1$ and (α, w_α), (β, w_β) are free. Thus by Lemma 3.1, $(B \otimes \mathbf{K})' \cap M(B \otimes \mathbf{K}) = \mathbf{C}1_{M(B \otimes \mathbf{K})}$. Furthermore, by Lemma 4.8, $(\alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ and $(\beta \otimes \mathrm{id}, w_\beta \otimes 1)$ are free. On the other hand, since $A \subset A \rtimes_{\alpha, w_\alpha, r} G$ and $B \subset B \rtimes_{\beta, w_\beta, r} G$ are strongly Morita equivalent, by the discussion before Lemma 5.2, there is an isomorphism θ of $(A \otimes \mathbf{K}) \rtimes_{\alpha\otimes\mathrm{id}, w_\alpha\otimes 1, r} G$ onto $(B \otimes \mathbf{K}) \rtimes_{\beta\otimes\mathrm{id}, w_\beta\otimes 1, r} G$ such that $\theta|_{A\otimes\mathbf{K}}$ is an isomorphism of $A \otimes \mathbf{K}$ onto $B \otimes \mathbf{K}$. Set $\theta_r = \theta|_{A\otimes\mathbf{K}}$ and $\gamma_t = \theta_r \circ (\alpha_t \otimes \mathrm{id}) \circ \theta_r^{-1}$, $w_\gamma(t, s) = \underline{\theta_r}(w_\alpha(t, s) \otimes 1)$ for all $t, s \in G$. Then by Lemmas 5.3 and 5.4, there is an automorphism ϕ of G such that $(\beta\otimes\mathrm{id}, w_\beta\otimes 1)$ and $(\gamma^\phi, w_\gamma^\phi)$ are exterior equivalent, hence strongly Morita equivalent, where $(\gamma^\phi, w_\gamma^\phi)$ is the twisted action of G on $B \otimes \mathbf{K}$ defined by $\gamma_t^\phi(b) = \gamma_{\phi(t)}(b)$ and $w_\gamma^\phi(t, s) = w_\gamma(\phi(t), \phi(s))$ for all $t, s \in G$ and $b \in B \otimes \mathbf{K}$. We note that (γ, w_γ) and $(\alpha \otimes \mathrm{id}, w_\alpha \otimes 1)$ are strongly Morita equivalent by Lemma 5.1. It follows that $(\gamma^\phi, w_\gamma^\phi)$ and $(\alpha^\phi \otimes \mathrm{id}, w_\alpha^\phi \otimes 1)$ are strongly Morita equivalent. Hence we can conclude that $(\alpha^\phi\otimes\mathrm{id}, w_\alpha^\phi\otimes 1)$ and $(\beta\otimes\mathrm{id}, w_\beta\otimes 1)$ are strongly Morita equivalent. As $(\alpha^\phi, w_\alpha^\phi)$ and (β, w_β) are strongly Morita equivalent to $(\alpha^\phi \otimes \mathrm{id}, w_\alpha^\phi \otimes 1)$ and $(\beta \otimes \mathrm{id}, w_\beta \otimes 1)$, respectively, we get that $(\alpha^\phi, w_\alpha^\phi)$ and (β, w_β) are strongly Morita equivalent, as asserted.

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DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
RYUKYU UNIVERSITY
NISHIHARA-CHO, OKINAWA, 903-0213
JAPAN
E-mail: kodaka@math.u-ryukyu.ac.jp