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THE DIRICHLET PROBLEM FOR THE COMPLEX HESSIAN OPERATOR IN THE CLASS $\mathcal{N}_m(\Omega, f)$

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Abstract

Let $\Omega \subset \mathbb{C}^n$ be a bounded m-hyperconvex domain, where m is an integer such that $1 \leq m \leq n$. Let μ be a positive Borel measure on Ω. We show that if the complex Hessian equation $H_m(u) = \mu$ admits a (weak) subsolution in Ω, then it admits a (weak) solution with a prescribed least maximal m-subharmonic majorant in Ω.

1. Introduction

Complex Monge-Ampère equations play an important role in pluripotential theory, complex analysis, and have many applications in Kähler geometry. See [5], [7], [8], [16], [13], and the references therein.

Recently complex Hessian equations have attracted the attention of many researchers. See [17], [6], [10], [11], [9], [18], [19], [15], [20], and [22]. They appear in many geometric problems, including J-flow [23] and quaternionic geometry [4]. They are also important from PDE's point of view as they can be considered as non linear PDE's of second order which interpolate between (linear) complex Poisson equations ($m = 1$) and (non linear) complex Monge-Ampère equations ($m = n$).

One of the central problems in this area is the resolution of the Dirichlet problem for complex m-Hessian equations. Given a positive Borel measure μ on Ω and a continuous function ϕ on $\partial\Omega$, find a function u in the domain of definition of the Hessian operator H_m such that

$$H_m(u) := (dd^c u)^m \wedge \beta^{n-m} = \mu \ \text{ in } \ \Omega \quad \text{and} \quad u = \phi \ \text{ in } \ \partial\Omega, \tag{1.1}$$

where $\beta := dd^c|z|^2$. Observe that when u is a $\mathscr{C}^2$ function, then $H_m(u)$ is the m-th elementary symmetric polynomial of the eigenvalues of the complex Hessian $(\partial^2 u/\partial z_j \partial \bar{z}_k)$ of u. In the general case, equation (1.1) should be understood in the sense of currents on Ω.

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A related problem is to characterize the range of the complex Hessian operator. One of the main steps in this study is the so called *subsolution problem*. Namely, assuming the necessary condition that the Dirichlet problem (1.1) admits a subsolution in a given class of admissible functions, does it admit a solution in the same class? This problem was solved by Kołodziej [16] for the complex Monge-Ampère equation in the class of bounded plurisubharmonic (psh) functions and generalized later by Åhag, Cegrell, Czyż, and Hiệp [2] to unbounded psh functions.

Recall that a function $w \in \mathcal{E}_m(\Omega)$ is a subsolution to the Hessian equation $H_m(u) = \mu$ in a given class $\mathcal{U} \subset \mathcal{E}_m(\Omega)$ if $w \in \mathcal{U}$ and satisfies the inequality $H_m(w) \geq \mu$ in the weak sense on Ω.

For the complex Hessian equation the subsolution problem in the class of bounded m-subharmonic functions was solved by Ngoc Cuong Nguyen [20]. Recently, Vu Viet Hung and Nguyen Van Phu [15] solved the subsolution problem for positive Borel measures in the class $\mathcal{F}_m(\Omega)$. See Definition 2.4.

In order to introduce our main result let us first make the following observation. When the measure μ is singular with respect to the Lebesgue measure, a solution with zero boundary values in the classical sense does not always exist, even in the simplest case of the unit disc in $\mathbb{C}$ and μ with finite mass. In this case, the Green potential of the measure μ is the unique solution whose least harmonic majorant is zero.

Our main result solves the subsolution problem for complex Hessian equations and positive Borel measures, with prescribed least maximal m-subharmonic majorants.

Main Theorem. *Let $\Omega \subset \mathbb{C}^n$ be a bounded m-hyperconvex domain and μ be a positive Borel measure on Ω. Assume that there exists a function $\omega \in \mathcal{E}_m(\Omega)$ such that $\mu \leq H_m(\omega)$ in the sense of currents on Ω. Then for every function $f \in \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$, there exists a function $u \in \mathcal{E}_m(\Omega)$ such that $H_m(u) = \mu$ and $\omega + f \leq u \leq f$. In particular, if we require ω to be in $\mathcal{N}_m(\Omega)$, then $u \in \mathcal{N}_m(\Omega, f)$.*

Note that in general a solution does not exist if the Borel measure μ is not dominated by the Hessian measure of a function in the class $\mathcal{E}_m(\Omega)$, even if $f = 0$ and μ has finite mass (see Example 5.5). Note also that Main Theorem is not true for any $f \in \mathcal{E}_m(\Omega)$ as observed in Remark 5.4. The maximality of f is a natural condition which generalizes boundary values in the classical Dirichlet problem and the inequality $\omega + f \leq u \leq f$ is for m-subharmonic functions a natural non-linear substitute for the Riesz decomposition theorem for subharmonic functions as explained in the same remark.

Let us mention that Main Theorem gives a non trivial generalization of the result of Hung and Phu [15], who considered the case of a Borel measure with finite mass and $f = 0$. Indeed, we can find and increasing sequence $(\mu_j)_{j\in\mathbb{N}}$ of Borel measures with finite mass converging to μ and by [15] we have a corresponding sequence of solutions $(u_j)_{j\in\mathbb{N}}$, but we are not able to conclude that it converges to a solution u of $H_m(u) = \mu$.

The proof of Main Theorem is done in several steps. We first establish an appropriate comparison principle in the class $\mathcal{N}_m(\Omega, f)$ which is interesting in its own (see Section 3). Then we solve the Dirichlet problem when the Borel measure μ is non m-polar i.e. μ puts no mass on m-polar sets of Ω, under a technical extra condition that there is a m-subharmonic function which is μ-integrable in Ω. (See Section 4.)

The next step is to use a generalized Radon-Nikodym theorem to write $\mu = \mu_1 + \mu_2$, where μ_1 is absolutely continuous with respect to m-capacity and μ_2 a singular measure supported on a set of m-capacity zero. Then, we approximate μ_i, $i = 1, 2$, by measures μ_i^N for which we know how to solve the Dirichlet problem in $\mathcal{N}_m(\Omega, f)$.

Finally we use the classical Perron envelope to construct a sequence (u_N) in $\mathcal{N}_m(\Omega, f)$ such that $H_m(u_N) = \mu_1^N + \mu_2^N$. To conclude, we show that the sequence (u_N) converges to some $u \in \mathcal{N}_m(\Omega, f)$ that satisfies $H_m(u) = \mu$. (Section 5.1.)

These results, as well as the entire content of this paper, are already in our preprint [12] posted on arXiv in December 2017. We learned recently that Van Thien Nguyen published an article [21] on the same subject without mentioning our preprint. However, the overlap is not so important and our results are more general.

2. Preliminaries

2.1. The class of m-subharmonic functions

In this section we give some basic properties of admissible functions for the complex Hessian equation. Such functions are called m-subharmonic (m-sh). They are subharmonic and non-smooth in general. Define

$$\breve{\Gamma}_m := \left\{\alpha \in \mathbb{C}_{(1,1)} : \alpha \wedge \beta^{n-1} \geq 0, \alpha^2 \wedge \beta^{n-2} \geq 0, \dots, \alpha^m \wedge \beta^{n-m} \geq 0\right\},$$

where $\mathbb{C}_{(1,1)}$ is the space of real $(1, 1)$-forms with constant coefficients in $\mathbb{C}^n$.

A $(1, 1)$-form belonging to $\breve{\Gamma}_m$ is called m-positive.

If T is a current of bidegree $(n - k, n - k)$, with $k \leq m$. Then T is called m-positive if for all m-positive $(1, 1)$-forms $\alpha_1, \dots, \alpha_k$ we have

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \wedge T \geq 0.$$

DEFINITION 2.1. Let $u: \Omega \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function in Ω.

(i) If $u \in \mathscr{C}^2(\Omega)$, then u is m-sh if the form $dd^c u$ belongs pointwise to $\breve{\Gamma}_m$.

(ii) In general, u is called m-sh if the inequality

$$dd^c u \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0, \quad \alpha_1, \ldots, \alpha_{m-1} \in \breve{\Gamma}_m,$$

holds in the weak sense of currents in Ω.

(iii) We denote by $\mathcal{SH}_m(\Omega)$ the set of all m-sh functions in Ω and by $\mathcal{SH}_m^-(\Omega)$ the set of all $u \leq 0$ in $\mathcal{SH}_m(\Omega)$.

(iv) A function $u \in \mathcal{SH}_m(\Omega)$ is said to be maximal if for any $v \in \mathcal{SH}_m(\Omega)$ such that $v \leq u$ outside a compact subset of Ω we have $v \leq u$ in Ω.

(v) We let $\mathcal{MSH}_m(\Omega)$ denote the family of maximal functions in $\mathcal{SH}_m(\Omega)$.

Błocki [6] observed that up to a point pluripotential theory can be adapted to m-subharmonic functions.

For locally bounded m-sh functions $u_1, \ldots, u_k$, $k \leq m$, the complex Hessian operator is defined inductively by

$$\begin{aligned} H_m(u_1, \ldots, u_k) &:= dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \beta^{n-k} \\ &= dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_k \wedge \beta^{n-k}). \end{aligned}$$

This is a closed positive current. In particular, the complex m-Hessian measure of $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ is defined to be

$$H_m(u) = (dd^c u)^m \wedge \beta^{n-m}.$$

Observe that if $u \in \mathscr{C}^2(\Omega)$ then

$$H_m(u) = \sigma_m(\lambda_1, \ldots, \lambda_n)\beta^n,$$

where $\sigma_m(\lambda_1, \ldots, \lambda_n)$ is the elementary symmetric polynomial of degree m in the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the complex Hessian of u.

Definition 2.2.

(i) A set $P \subset \mathbb{C}^n$ is called m-polar if for any $z \in P$ there exists a neighborhood V of z and a function $u \in \mathcal{SH}_m(V)$ such that $P \cap V \subset \{u = -\infty\}$.

(ii) We will say that a Borel measure on Ω is non m-polar if it vanishes on all m-polar sets in Ω.

Definition 2.3. We say that a function u is strictly m-sh on Ω if it is m-sh on Ω and for every $P \in \Omega$ there exists a constant $C_P > 0$ such that the function $z \mapsto u(z) - C_P|z|^2$ is m-subharmonic in a neighborhood of P.

Definition 2.4. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain.

(i) We say that Ω is m-hyperconvex if there exists a continuous m-sh function $\varphi\colon \Omega \to \mathbb{R}^-$ such that $\{\varphi < c\} \Subset \Omega$, for every $c < 0$.

(ii) We say that Ω is strictly m-pseudoconvex if there exists a smooth strictly m-sh function ρ on some open neighborhood Ω' of $\bar{\Omega}$ such that $\nabla\rho \neq 0$ on $\partial\Omega$ and $\Omega = \{z \in \Omega' : \rho(z) < 0\}$.

(iii) A fundamental exhaustion (Ω_j) of an m-hyperconvex domain Ω is a sequence of strictly m-pseudoconvex domains such that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ for every j, and $\cup_{j=1}^{\infty}\Omega_j = \Omega$.

Note that a fundamental exhaustion (Ω_j) of an m-hyperconvex domain Ω always exists. Indeed, it follows from Åhag, Czyż, and Hed [3] that Ω admits an exhaustion function ρ that is negative, smooth, and strictly m-sh. Then one can choose a sequence (a_j) decreasing to 0 for which $\Omega_j = \{z \in \Omega : \rho(z) < -a_j\}$ is strictly m-pseudoconvex.

2.2. Cegrell's classes and approximation of m-sh functions

The following classes of m-sh functions were introduced and investigated by Lu in [18], [19].

DEFINITION 2.5. Let $\Omega \subset \mathbb{C}^n$ be an m-hyperconvex domain.

(i) We denote by $\mathscr{E}_m^0(\Omega)$ the class of bounded functions $u \in \mathscr{SH}_m^-(\Omega)$ such that $\lim_{z\to\xi} u(z) = 0$, $\forall\xi \in \partial\Omega$ and $\int_\Omega H_m(u) < +\infty$.

(ii) We denote by $\mathscr{F}_m(\Omega)$ the class of functions $u \in \mathscr{SH}_m^-(\Omega)$ such that there exists a sequence (u_j) in $\mathscr{E}_m^0(\Omega)$ decreasing to u in Ω and $\sup_j \int_\Omega H_m(u_j) < +\infty$.

(iii) For every $p > 0$ we denote by $\mathscr{E}_m^p(\Omega)$ the class of functions $\psi \in \mathscr{SH}_m^-(\Omega)$ such that there exists a decreasing sequence (ψ_j) in $\mathscr{E}_m^0(\Omega)$ satisfying $\lim_{j\to+\infty} \psi_j(z) = \psi(z)$ and $\sup_j \int_\Omega (-\psi_j)^p H_m(\psi_j) < +\infty$.

(iv) We denote by $\mathscr{F}_m^p(\Omega)$ the subclass of functions ψ in $\mathscr{E}_m^p(\Omega)$ for which $\sup_j \int_\Omega H_m(\psi_j) < +\infty$.

(v) We denote by $\mathscr{E}_m(\Omega)$ the class of all u in $\mathscr{SH}_m^-(\Omega)$ such that for every $z_0 \in \Omega$ there exists a neighborhood $U \subset \Omega$ of z_0 and a decreasing sequence (u_j) in $\mathscr{E}_m^0(\Omega)$ converging to u on U with $\sup_j \int_\Omega H_m(u_j) < +\infty$.

(vi) Let $E \subset \Omega$ be a Borel subset: The C_m-capacity of E with respect to Ω is defined by

$$C_m(E) := C_m(E, \Omega) = \sup\left\{\int_E H_m(\theta), \theta \in \mathscr{SH}_m(\Omega), -1 \le \theta \le 0\right\}.$$

Let $u^k \in \mathscr{E}_m(\Omega)$, $k = 1, \ldots, m$. It follows from [18, Theorem 3.14] that if $(u_j^k)_j$ are sequences in $\mathscr{E}_m^0(\Omega)$, decreasing to u^k as $j \to +\infty$, then the sequence of measures $H_m(u_j^1, \ldots, u_j^m)$ converges weakly to a positive Radon measure which does not depend on the choice of the sequence $(u_j^k)_j$. One then can define $H_m(u^1, \ldots, u_j^m)$ to be this weak limit. In particular, for $u \in \mathscr{E}_m(\Omega)$, $H_m(u)$ is well defined and it is a Radon measure on Ω.

It follows from the work of Błocki [6] and the local property of the class $\mathscr{E}_m(\Omega)$ that maximal m-sh functions u are precisely the m-sh functions satisfying $H_m(u) = 0$. See also Hung [14]. For $m = 1$, maximal subharmonic functions coincide with harmonic functions. This is why they play an important role in the m-subharmonic potential theory.

Now we introduce an important notion to describe boundary values of m-subharmonic functions.

DEFINITION 2.6. Let Ω be an m-hyperconvex domain, $u \in \mathcal{SH}_m^-(\Omega)$, and define the function $\tilde{u}$ on Ω by $\tilde{u} = (\lim_{j\to+\infty} u^j)^*$, where v^* denotes the upper semicontinuous regularization of a locally upper bounded function v on Ω, the sequence (u^j) in $\mathcal{SH}_m^-(\Omega)$ is defined by

$$u^j = \sup\{\phi \in \mathcal{SH}_m(\Omega) : \phi \le u \text{ on } \Omega \setminus \Omega_j\},$$

and (Ω_j) is a fundamental exhaustion of Ω.

The sequence (u^j) is increasing, which implies that $\sup_j u^j = \lim_{j\to+\infty} u^j$ and that $\tilde{u} \in \mathcal{SH}_m(\Omega)$. Moreover, by definition $u \le \tilde{u}$. It follows from [18, Theorem 1.7.5] that if $u \in \mathscr{E}_m(\Omega)$ then $\tilde{u} \in \mathscr{E}_m(\Omega)$.

PROPOSITION 2.7.

(i) *If $u \in \mathscr{E}_m(\Omega)$, then $\tilde{u} \in \mathscr{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$ and $\tilde{u}$ is the smallest maximal m-subharmonic majorant of u in Ω. Moreover, if u is maximal then $\widetilde{u} = u$.*

(ii) *Let $u, v \in \mathscr{E}_m(\Omega)$ and $\alpha \in \mathbb{R}$, $\alpha \ge 0$. Then $\widetilde{u+v} \ge \widetilde{u}+\widetilde{v}$, and $\widetilde{\alpha u} = \alpha\widetilde{u}$. Moreover, if $u \le v$ then $\widetilde{u} \le \widetilde{v}$.*

PROOF. The statements in (ii) follow directly from the definition. We prove (i). By [19, Theorem 3.1] there exists a sequence (u_k) in $\mathscr{E}_m^0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that $u_k \searrow u$. Fix $j \ge 1$ and put

$$\varphi_k^j = \sup\{\phi \in \mathcal{SH}_m(\Omega) : \phi \le u_k \text{ on } \Omega \setminus \Omega_j\}.$$

Then $H_m(\varphi_k^j) = 0$ on Ω_j for all $k \ge 1$, $(\varphi_k^j)_k$ is decreasing and $u^j \le \varphi_k^j$ for all $k \ge 1$, where

$$u^j = \sup\{\phi \in \mathcal{SH}_m(\Omega) : \phi \le u \text{ on } \Omega \setminus \Omega_j\}.$$

Set $\varphi^j := \lim_{k\to+\infty} \varphi_k^j \in \mathcal{SH}_m(\Omega)$. Since $\varphi^j(z) \le u(z)$ for all $z \in \Omega \setminus \Omega_j$, we have $\varphi^j \le u^j$ on Ω. Hence $\varphi^j = u^j$ on Ω. It follows from the continuity of the complex Hessian operator along decreasing sequences that $H_m(u^j) = 0$ on Ω_j. Using [15, Theorem 3.10], we conclude that $\tilde{u}$ is maximal.

For the second statement let $v \in \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$ such that $v \ge u$ on Ω. If h is a candidate defining u^j then $h \le v$ outside the compact set $\overline{\Omega}_j \Subset \Omega$. Since v is maximal we deduce that $h \le v$ on Ω. We thus have $u^j \le v$ on Ω, which implies $\tilde{u} \le v$.

Remark 2.8. The definition of the function $\tilde{u}$ is independent of the choice of the fundamental exhaustion of Ω.

Indeed, let (Ω_j) and $(\Omega_k^{'})$ be two fundamental exhaustions of Ω and take v^j and u^k as in Definition 2.6. Then we have that $u \le \tilde{v}$. This implies, by (i) in Proposition 2.7, that $\tilde{u} \le \tilde{v}$. Then by symmetry we get $\tilde{u} = \tilde{v}$.

Definition 2.9. Let $\mathcal{N}_m(\Omega) := \{u \in \mathcal{E}_m(\Omega) : \widetilde{u} = 0\}$.

It follows from Proposition 2.7 that $\mathcal{N}_m(\Omega)$ is a convex cone and that it is precisely the set of functions in $\mathcal{E}_m(\Omega)$ with smallest maximal m-sh majorant identically zero.

Remark 2.10. $\mathcal{F}_m(\Omega) \subset \mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega)$.

Indeed, by definition $\mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega)$. For the first inclusion take $u \in \mathcal{F}_m(\Omega)$. By definition of $\tilde{u}$ we have $u \le \tilde{u}$, hence $\tilde{u} \in \mathcal{F}_m(\Omega)$. Since $\tilde{u}$ is maximal, it follows from [14, Lemma 1] that $\tilde{u} = 0$. This proves that $u \in \mathcal{N}_m(\Omega)$.

Definition 2.11. Let $\mathcal{K}_m(\Omega)$ be one of the function classes $\mathcal{E}_m^0(\Omega)$, $\mathcal{F}_m(\Omega)$, $\mathcal{F}_m^p(\Omega)$, or $\mathcal{N}_m(\Omega)$ and $f \in \mathcal{E}_m(\Omega)$.

(i) We denote by $\mathcal{K}_m(\Omega, f)$ the class of all $u \in \mathcal{SH}_m(\Omega)$ satisfying $f \ge u \ge \varphi + f$ for some $\varphi \in \mathcal{K}_m(\Omega)$.

(ii) We denote by $\mathcal{K}_m^a(\Omega, f)$ the class of all $u \in \mathcal{SH}_m(\Omega)$ satisfying $f \ge u \ge \varphi + f$ for some $\varphi \in \mathcal{K}_m(\Omega)$ such that $H_m(\varphi)$ is non m-polar.

Proposition 2.12. *Assume that $f \in \mathcal{E}_m(\Omega)$ and $u \in \mathcal{SH}_m^-(\Omega)$ such that $u \le f$. Then there exists a decreasing sequence (u_j) in $\mathcal{E}_m^0(\Omega, f)$ that converges pointwise to u on Ω, as $j \to +\infty$. Moreover, if $f \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then (u_j) can be chosen such that $u_j \in \mathcal{E}_m^0(\Omega, f) \cap \mathcal{C}(\overline{\Omega})$.*

Proof. By [19, Theorem 3.1] there exists a sequence (φ_j) in $\mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $\varphi_j \searrow u$. Set

$$u_j = \sup\{\psi \in \mathcal{SH}_m(\Omega) : \psi \le \min(\varphi_j, f)\}.$$

Observe that $\varphi_j + f$ is a candidate defining u_j, hence $\varphi_j + f \leq u_j \leq f$. This reveals that $u_j \in \mathcal{E}_m^0(\Omega, f)$. We also have that $u_j \searrow u$.

Now, if f is continuous on $\bar{\Omega}$ then, using $\varphi_j + f \leq u_j \leq f$ we have

$$\lim_{z\to\xi}(u_j)_*(z) = \lim_{z\to\xi}(u_j)^*(z) = f(\xi), \quad \text{for all } \xi \in \partial\Omega.$$

It thus follows from [6, Proposition 3.2] that u_j is continuous on $\bar{\Omega}$.

PROPOSITION 2.13. *Let $f \in \mathcal{E}_m(\Omega)$ and $\varphi \in \mathcal{SH}_m^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. If (u_j) is a decreasing sequence in $\mathcal{N}_m(\Omega, f)$ which converges pointwise to $u \in \mathcal{N}_m(\Omega, f)$ as $j \to +\infty$, then* $\lim_{j\to+\infty} \int_\Omega (-\varphi) H_m(u_j) = \int_\Omega (-\varphi) H_m(u)$.

PROOF. Assume first that $\int_\Omega (-\varphi) H_m(u) < +\infty$. By Lemma 2.14 below, the sequence $\left(\int_\Omega (-\varphi) H_m(u_j)\right)_j$ is increasing and bounded from above by $\int_\Omega (-\varphi) H_m(u)$. Since a function in $\mathcal{E}_m(\Omega)$ is locally in $\mathcal{F}_m(\Omega)$, it follows from [15, Theorem 3.5] that the sequence $((-\varphi) H_m(u_j))_j$ converges weakly to $(-\varphi) H_m(u)$ and the desired limit of the total masses is valid.

If $\int_\Omega (-\varphi) H_m(u) = +\infty$, using again [15, Theorem 3.5], we get

$$\lim_{j\to+\infty} \int_\Omega (-\varphi) H_m(u_j) = +\infty.$$

LEMMA 2.14. *Let $u, v \in \mathcal{N}_m(\Omega, f)$, $u \leq v$, and T be a closed m-positive current of the form $T = dd^c\omega_2 \wedge \ldots \wedge dd^c\omega_m \wedge \beta^{n-m}$, where $\omega_j \in \mathcal{E}_m(\Omega)$, $2 \leq j \leq m$. Then for all $\varphi \in \mathcal{SH}_m^-(\Omega)$ the following inequality holds*

$$\int_\Omega (-\varphi)\, dd^c v \wedge T \leq \int_\Omega (-\varphi)\, dd^c u \wedge T. \tag{2.1}$$

PROOF. If $\int_\Omega (-\varphi)\, dd^c u \wedge T = +\infty$ then the conclusion holds trivially. Assume that $\int_\Omega (-\varphi)\, dd^c u \wedge T < +\infty$. It follows from [19, Theorem 3.1] that there exists a sequence (φ_k) in $\mathcal{E}_m^0(\Omega) \cap C(\overline{\Omega})$ such that $\varphi_k \searrow \varphi$, as $k \to +\infty$. Since $u \in \mathcal{N}_m(\Omega, f)$, there exists $\psi \in \mathcal{N}_m(\Omega)$ such that $f \geq u \geq f + \psi$. Let (Ω_j) be a fundamental exhaustion of Ω and set $v_j = \max(u, \psi^j + v)$, where ψ^j is given by Definition 2.6. Then $v_j \in \mathcal{E}_m(\Omega)$, $v_j = u$ on $\Omega \setminus \Omega_j$, $u \leq v_j$, and (v_j) is an increasing sequence that converges pointwise to v quasi-everywhere on Ω, as $j \to +\infty$. For $r \geq j$, let $\tilde{u} \in \mathcal{F}_m(\Omega)$ such that $\tilde{u} \geq u$ and $\tilde{u} = u$ on Ω_r. Put $\tilde{v}_j = \max\{\tilde{u}, v_j\}$. Then $\tilde{v}_j \in \mathcal{F}_m(\Omega)$, $\tilde{v}_j = v_j$ on Ω_r and $\tilde{v}_j = \tilde{u}$ on $\Omega \setminus \Omega_j \supset \Omega \setminus \Omega_r$. Put $\tilde{T} = dd^c\tilde{\omega}_2 \wedge \cdots \wedge dd^c\tilde{\omega}_m \wedge \beta^{n-m}$, where $\tilde{\omega}_\ell \in \mathcal{F}_m(\Omega)$

such that $\tilde{\omega}_\ell = \omega_\ell$ on Ω_r. Integrating by parts as in [19, Theorem 3.16] we get

$$\begin{aligned}
\int_{\Omega_r} (-\varphi_k)\, dd^c u \wedge T - \int_{\Omega_r} (-\varphi_k)\, dd^c v_j \wedge T \\
&= \int_{\Omega_r} (-\varphi_k)\, dd^c \tilde{u} \wedge \tilde{T} - \int_{\Omega_r} (-\varphi_k)\, dd^c \tilde{v}_j \wedge \tilde{T} \\
&= \int_{\Omega} (-\varphi_k)\, dd^c \tilde{u} \wedge \tilde{T} - \int_{\Omega} (-\varphi_k)\, dd^c \tilde{v}_j \wedge \tilde{T} \\
&= \int_{\Omega_r} (\tilde{v}_j - \tilde{u})\, dd^c \varphi_k \wedge \tilde{T} \\
&\geq 0.
\end{aligned}$$

By letting $r \to +\infty$, we get

$$\int_\Omega (-\varphi_k)\, dd^c u \wedge T \geq \int_\Omega (-\varphi_k)\, dd^c v_j \wedge T.$$

Since (v_j) converges to v quasi everywhere, it follows that (v_j) converges to v in C_m-capacity in Ω. Since φ_k is bounded, it follows from [15, Theorem 3.10] that $(-\varphi_k)\, dd^c v_j \wedge T$ converges weakly to $(-\varphi_k)\, dd^c v \wedge T$ as $j \to +\infty$. Thus

$$\lim_{j\to+\infty} \int_\Omega (-\varphi_k)\, dd^c v_j \wedge T \geq \int_\Omega (-\varphi_k)\, dd^c v \wedge T.$$

Then (2.1) holds for φ_k. Since $\int_\Omega (-\varphi)\, dd^c u \wedge T < +\infty$, letting $k \to +\infty$ and using monotone convergence theorem one obtains the desired inequality.

THEOREM 2.15. *Assume that μ is a positive measure on Ω. Then there exists $\phi \in \mathscr{E}_m^0(\Omega)$, $0 \leq \ell \in L^1_{\mathrm{loc}}(H_m(\phi))$, and a positive measure ν carried by an m-polar set, such that $\mu = \ell H_m(\phi) + \nu$.*

This result is known as Cegrell's decomposition theorem. PROOF. This can be deduced from Cegrell's decomposition theorem in [7] and the same arguments as in [19, Theorem 5.3].

LEMMA 2.16. *If $u \in \mathscr{E}_m(\Omega)$ then $\mathbf{1}_{\{u>-\infty\}} H_m(u)$ vanishes on m-polar sets.*

PROOF. Set $v = e^u$. A direct computation shows that $H_m(v) \geq e^{mu} H_m(u)$. Let $E \subset \Omega$ be an m-polar set and let $E_k := E \cap \{u > -k\}$. Then, since v is bounded,

$$0 = \int_{E_k} H_m(v) \geq \int_{E_k} e^{-km} H_m(u).$$

It follows that $\int_{E_k} H_m(u) = 0$ for all k. Letting $k \to +\infty$ we arrive at the conclusion.

3. A comparison principle for the class $\mathcal{N}_m(\Omega, f)$

The goal of this section is to establish several versions of the comparison principle for the class $\mathcal{N}_m(\Omega, f)$ that will be used later.

LEMMA 3.1. *Let $u, v \in \mathcal{E}_m(\Omega)$ be such that* $\liminf_{z\to\xi}[u(z) - v(z)] \geq 0$, *for all $\xi \in \partial\Omega$. Let $T = dd^c\omega_1 \wedge \cdots \wedge dd^c\omega_m \wedge \beta^{n-m}$, where $\omega_j \in \mathcal{SH}_m(\Omega)$, $-1 \leq \omega_j \leq 0$, $j = 1, \ldots, m$. Then*

$$\frac{1}{m!}\int_{\{u<v\}} (v-u)^m T + \int_{\{u<v\}} (-\omega_1) H_m(v) \leq \int_{\{u<v\}\cup\{u=v=-\infty\}} (-\omega_1) H_m(u).$$

Note that the condition $\liminf_{z\to\xi}[u(z) - v(z)] \geq 0$, for all $\xi \in \partial\Omega$ is equivalent to $\{u < v - \varepsilon\} \Subset \Omega$, for all $\varepsilon > 0$.

PROOF. Fix $\varepsilon > 0$ and an open set Ω' such that $\{u < v - \varepsilon\} \Subset \Omega' \Subset \Omega$. Let $\tilde{u}$ be a function in $\mathcal{F}_m(\Omega)$ such that $\tilde{u} \geq u$ and $\tilde{u} = u$ on Ω'. Set

$$v_\varepsilon := \max(\tilde{u}, v - 2\varepsilon).$$

Then $v_\varepsilon \in \mathcal{F}_m(\Omega)$, $v_\varepsilon \geq \tilde{u}$, and $v_\varepsilon = \tilde{u}$ on $\Omega \setminus \Omega'$. It follows from [10, Theorem 6] that

$$\frac{1}{m!}\int_\Omega (v_\varepsilon - \tilde{u})^m T + \int_\Omega (-\omega_1) H_m(v_\varepsilon) \leq \int_\Omega (-\omega_1) H_m(\tilde{u}).$$

By the maximum principle, see [15, Theorem 3.6], we also have

$$\mathbf{1}_{\{\tilde{u}<v-2\varepsilon\}} H_m(v_\varepsilon) = \mathbf{1}_{\{\tilde{u}<v-2\varepsilon\}} H_m(v),$$

and

$$\mathbf{1}_{\{\tilde{u}>v-2\varepsilon\}} H_m(v_\varepsilon) = \mathbf{1}_{\{\tilde{u}>v-2\varepsilon\}} H_m(\tilde{u}).$$

Note also that $\int_\Omega (-\omega_1) H_m(\tilde{u}) < +\infty$ because $\tilde{u} \in \mathcal{F}_m(\Omega)$ and ω_1 is bounded. We thus have

$$\frac{1}{m!}\int_{\{\tilde{u}<v-2\varepsilon\}} (v-2\varepsilon-\tilde{u})^m T + \int_{\{\tilde{u}<v-2\varepsilon\}} (-\omega_1) H_m(v) \leq \int_{\{\tilde{u}\leq v-2\varepsilon\}} (-\omega_1) H_m(\tilde{u}).$$

Since $\{\tilde{u} < v - 2\varepsilon\} \subset \Omega'$ and $u = \tilde{u}$ on Ω', it follows that

$$\frac{1}{m!}\int_{\{u<v-2\varepsilon\}} (v-2\varepsilon-u)^m T + \int_{\{u<v-2\varepsilon\}} (-\omega_1) H_m(v) \leq \int_{\{\tilde{u}\leq v-2\varepsilon\}} (-\omega_1) H_m(\tilde{u}).$$

Observe also that $\{\tilde{u} \le v - 2\varepsilon\} \subset \{\tilde{u} < v - \varepsilon\} \cup \{\tilde{u} = v = -\infty\}$. We thus have

$$\begin{aligned}\frac{1}{m!}\int_{\{u<v-2\varepsilon\}} &(v - 2\varepsilon - u)^m T + \int_{\{u<v-2\varepsilon\}} (-\omega_1) H_m(v)\\ &\le \int_{\{\tilde{u}<v-\varepsilon\}} (-\omega_1) H_m(\tilde{u}) + \int_{\{\tilde{u}=v=-\infty\}} (-\omega_1) H_m(\tilde{u})\\ &= \int_{\{u<v-\varepsilon\}} (-\omega_1) H_m(u) + \int_{\{\tilde{u}=v=-\infty\}} (-\omega_1) H_m(\tilde{u}).\end{aligned}$$

To obtain the last line we note that $\{\tilde{u} < v - \varepsilon\} \subset \Omega'$ and $u = \tilde{u}$ on Ω'.

On the other hand, it follows from [15, Proposition 5.2] that

$$\mathbf{1}_{\{\tilde{u}=-\infty\}} H_m(\tilde{u}) \le \mathbf{1}_{\{u=-\infty\}} H_m(u).$$

We therefore have

$$\begin{aligned}\frac{1}{m!}\int_{\{u<v-2\varepsilon\}} (v - 2\varepsilon - u)^m T &+ \int_{\{u<v-2\varepsilon\}} (-\omega_1) H_m(v)\\ &\le \int_{\{u<v-\varepsilon\}} (-\omega_1) H_m(u) + \int_{\{u=v=-\infty\}} (-\omega_1) H_m(u).\end{aligned}$$

Letting $\varepsilon \searrow 0$ we arrive at the conclusion.

THEOREM 3.2. *Let $f, v \in \mathscr{E}_m(\Omega)$ and $u \in \mathcal{N}_m(\Omega, f)$ and assume that $v \le f$ on Ω. Let $T = dd^c\omega_1 \wedge \ldots \wedge dd^c\omega_m \wedge \beta^{n-m}$, where $\omega_j \in \mathcal{SH}_m(\Omega)$, $-1 \le \omega_j \le 0$, $j = 1, \ldots, m$. Then*

$$\frac{1}{m!}\int_{\{u<v\}} (v - u)^m T + \int_{\{u<v\}} (-\omega_1) H_m(v) \le \int_{\{u<v\}\cup\{u=v=-\infty\}} (-\omega_1) H_m(u).$$

PROOF. Since $u \in \mathcal{N}_m(\Omega, f)$, by definition there exists $\phi \in \mathcal{N}_m(\Omega)$ such that $\phi + f \le u \le f$. Let (Ω_j) be a fundamental exhaustion of Ω and define ϕ^j as in Definition 2.6. Then $\phi^j \nearrow 0$ quasi everywhere. Since $v \le f$, we have

$$u \ge \phi + f = \phi^j + f \ge \phi^j + v, \quad \text{on } \Omega \setminus \Omega_j.$$

Fix $\varepsilon > 0$. By elementary reason we have $H_m(v + \phi^j) \ge H_m(v)$. Applying Lemma 3.1 to u and $v + \phi^j - \varepsilon$, we get

$$\begin{aligned}\frac{1}{m!}\int_{\{u<v+\phi^j-\varepsilon\}} &(v + \phi^j - \varepsilon - u)^m T\\ &+ \int_{\{u<v+\phi^j-\varepsilon\}} (-\omega_1) H_m(v) \le \int_{\{u\le v+\phi^j-\varepsilon\}} (-\omega_1) H_m(u). \quad (3.1)\end{aligned}$$

On the other hand, $(\mathbf{1}_{\{u<v+\phi^j-\varepsilon\}}(v+\phi^j-\varepsilon-u)^m)_{j=1}^{\infty}$ and $(\mathbf{1}_{\{u<v+\phi^j-\varepsilon\}})_{j=1}^{\infty}$ are increasing sequences that converge to $\mathbf{1}_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^m$ and $\mathbf{1}_{\{u<v-\varepsilon\}}$, respectively, as $j \to +\infty$ quasi everywhere on Ω. Note also that T puts no mass on m-polar sets. By Lemma 2.16, $\mathbf{1}_{\{v>-\infty\}}H_m(v)$ also vanishes on m-polar sets. Therefore, $(\mathbf{1}_{\{u<v-\varepsilon+\phi^j\}})_{j=1}^{\infty}$ converges to $\mathbf{1}_{\{u<v-\varepsilon\}}$ almost everywhere with respect to $\mathbf{1}_{\{v>-\infty\}}H_m(v)$, and $(\mathbf{1}_{\{u<v-\varepsilon+\phi^j\}}(v-\varepsilon+\phi^j-u)^m)_{j=1}^{\infty}$ converges to $\mathbf{1}_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^m$ almost everywhere with respect to T. Now using the monotone convergence theorem in (3.1) we obtain

$$\frac{1}{m!}\int_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^m T+\int_{\{u<v-\varepsilon\}}(-\omega_1)H_m(v)\le\int_{\{u\le v-\varepsilon\}}(-\omega_1)H_m(u).$$

The desired inequality is obtained by letting $\varepsilon \to 0^+$.

Corollary 3.3 (Weak Comparison Principle). *Let $u, v, f \in \mathcal{E}_m(\Omega)$ be such that $H_m(u)$ is a non m-polar measure on Ω and $H_m(u) \le H_m(v)$. Then $u \ge v$ on Ω if one of the following conditions is satisfied:*

(1) $\liminf_{z\to\zeta}[u(z)-v(z)] \ge 0$ *for every* $\zeta \in \partial\Omega$,

(2) $u \in \mathcal{N}_m(\Omega, f)$ *and* $v \le f$.

Proof. Suppose that $\liminf_{z\to\zeta}[u(z)-v(z)] \ge 0$, for all $\zeta \in \partial\Omega$. Fix $\omega \in \mathcal{SH}_m(\Omega)$ such that $-1 \le \omega \le 0$. Fix $\varepsilon > 0$. Then $\{u+\varepsilon < v\} \Subset \Omega$, hence $\int_{\{u+\varepsilon<v\}} H_m(v) < +\infty$. By Lemma 3.1 we get

$$\begin{aligned}\frac{\varepsilon^m}{m!}\int_{\{u+2\varepsilon<v\}}H_m(\omega) &\le \frac{1}{m!}\int_{\{u+\varepsilon<v\}}(v-\varepsilon-u)^m H_m(\omega)\\ &\le \int_{\{u+\varepsilon<v\}}(-\omega)[H_m(u)-H_m(v)] \le 0.\end{aligned}$$

Hence $C_m(\{u+2\varepsilon<v\}) \le 0$. Thus, $u+2\varepsilon \ge v$. Let ε tends to 0, then we obtain that $u \ge v$ on Ω.

Suppose now that $u \in \mathcal{N}_m(\Omega, f)$ and $v \le f$. Then there exists $\varphi \in \mathcal{N}_m(\Omega)$ such that $f+\varphi \le u \le f$. Let φ^j be the sequence associated to φ and a fundamental exhaustion (Ω_j) as in Definition 2.6 and let $\varepsilon > 0$. The same argument as above applied to $u+2\varepsilon$ and $v+\varphi^j$ yields $u+2\varepsilon \ge v+\varphi^j$ in Ω. Letting $j \to +\infty$ and then $\varepsilon \to 0^+$, we get the desired inequality.

4. The Dirichlet problem in $\mathcal{N}_m(\Omega, f)$ for $f \in \mathcal{MSH}_m(\Omega)$

4.1. The Perron envelope

We shall first study the following Dirichlet problem with continuous boundary data:

$$\begin{cases} u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega), \\ H_m(u) = \mu, \\ \limsup\limits_{z\to\xi} u(z) = f(\xi), \quad \forall \xi \in \partial\Omega. \end{cases} \tag{4.1}$$

Here Ω is a bounded m-pseudoconvex domain, $f \in \mathcal{MSH}_m(\Omega) \cap \mathscr{C}(\partial\Omega)$ and μ is a positive measure on Ω.

A function $v \in \mathcal{SH}_m \cap L^\infty_{\mathrm{loc}}(\Omega)$ is said to be a *subsolution* of the Dirichlet problem (4.1) if $H_m(v) \geq \mu$ and $\limsup_{z\to\xi} v(z) \leq f(\xi)$ for every $\xi \in \partial\Omega$. We let $\mathcal{B}_m(\mu, f)$ denote the set of all subsolutions. The function

$$U_m(\mu, f) = \sup\{v : v \in \mathcal{B}_m(\mu, f)\}$$

is called the *Perron envelope* for the Dirichlet problem (4.1).

Lemma 4.1. *If $\mathcal{B}_m(\mu, f)$ is non-empty, then the Perron envelope for the Dirichlet problem* (4.1) *is a subsolution, i.e., $U_m(\mu, f) \in \mathcal{B}(\mu, f)$.*

Proof. It follows from Choquet's lemma that there exists a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $\mathcal{B}_m(\mu, f)$ such that $(\sup_{j\in\mathbb{N}} \varphi_j)^* = U^*_m(\mu, f)$. Thus $U^*_m(\mu, f) \in \mathcal{SH}_m(\Omega)$. Since f is maximal, $\varphi_j \leq f$ on Ω. We thus have that

$$\limsup_{z\to\xi} U^*_m(\mu, f)(z) \leq f(\xi),$$

for all $\xi \in \partial\Omega$. On the other hand, by [18, Theorem 1.3.16],

$$H_m(\max_{1\leq j\leq k}(\varphi_j)) \geq \mu.$$

Since $\max_{1\leq j\leq k}(\varphi_j) \nearrow U^*_m(\mu, f)$ almost everywhere as $k \to +\infty$, it follows from [18, Theorem 1.3.10] that $H_m(U^*_m(\mu, f)) \geq \mu$. We deduce that $U^*_m(\mu, f) \in \mathcal{B}_m(\mu, f)$, hence $U^*_m(\mu, f) \leq U_m(\mu, f)$. We finally have $U_m(\mu, f) = U^*_m(\mu, f) \in \mathcal{B}_m(\mu, f)$ as desired.

4.2. The Dirichlet problem with smooth boundary data

Our goal in this section is to prove an analogue of a result of Cegrell [7] for the complex Hessian equation.

Theorem 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded strictly m-pseudoconvex domain $f \in \mathcal{MSH}_m(\Omega) \cap \mathscr{C}^\infty(\partial\Omega)$. Assume that μ is a positive measure with* *finite mass in Ω. Then for $p \geq 1$ there exists a unique $u \in \mathcal{F}_m^p(\Omega, f)$ with $H_m(u) = \mu$ if and only if ther exists $A > 0$ such that*

$$\int (-\varphi)^p d\mu \leq A\left(\int (-\varphi)^p H_m(\varphi)\right)^{\frac{p}{p+m}}, \quad \forall \varphi \in \mathcal{E}_m^0(\Omega). \tag{4.2}$$

Note that it follows from the work of Li [17] that $U_m(dV, f), U_m(dV, -f) \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$. For simplicity denote by $\bar{f} := U_m(0, -f)$. Using [19, Theorem 3.22] we then get

$$\int_\Omega H_m(f + \bar{f}) \leq \int_\Omega H_m(U_m(dV, f) + U_m(dV, -f)) < +\infty.$$

This implies that $f + \bar{f} \in \mathcal{E}_m^0(\Omega)$.

Proof of Theorem 4.2. Assume that $\mu = H_m(u)$ with $u \in \mathcal{F}_m^p(\Omega, f)$. Then $0 \geq u + \bar{f} \geq \varphi + f + \bar{f}$ for some $\varphi \in \mathcal{F}_m^p(\Omega)$. Since $\varphi + f + \bar{f} \in \mathcal{F}_m^p(\Omega)$, then by [19, Theorem 3.9], $u + \bar{f} \in \mathcal{F}_m^p(\Omega)$. Therefore, by [19, Proposition 5.4 and Theorem 6.2], the measure $\nu := H_m(u + \bar{f})$ satisfies the inequality (4.2) with μ replaced by ν. Since $\mu = H_m(u) \leq \nu$, so does μ.

To prove the "only if" statement we first observe that the inequality (4.2) implies that μ puts no mass on m-polar sets. Indeed, let $E \Subset \Omega$ be a compact set and let

$$u_E := \sup\{\psi \in \mathcal{SH}_m^-(\Omega) : \psi \leq -1 \text{ in } E\}$$

be its relative m-extremal function in Ω. Let (E_j) be a decreasing sequence of regular compact sets in Ω such that $E = \cap_j E_j$. We can choose E_j to be the $(1/j)$-neighborhood of E for $j \geq 1$. Then it follows from [18, Proposition 1.4.9] that $u_{E_j}^* = u_{E_j}$ is continuous in Ω. Moreover, by [18, Proposition 1.4.2] $u_{E_j} \nearrow u_E^*$ quasi everywhere in Ω and $H_m(u_{E_j}) \longrightarrow H_m(u_E^*)$ weakly in Ω as $j \to +\infty$. Therefore, since $u_{E_j} = -1$ in E_j, it follows from inequality (4.2) applied to u_{E_j} that for any $j \in \mathbb{N}$,

$$\begin{aligned}\int_E d\mu \leq \int_{E_j} d\mu \leq \int_\Omega (-u_{E_j})^p \, d\mu &\leq A\left(\int_\Omega (-u_{E_j})^p H_m(u_{E_j})\right)^{p/p+m} \\ &= A(\mathrm{Cap}_m(E_j))^{p/p+m}.\end{aligned}$$

Observe that the last equality follows from [18, Proposition 1.4.11 and Proposition 1.4.12]. On the other hand, since $\mathrm{Cap}_m(E_j) \to \mathrm{Cap}_m(E)$ as $j \to +\infty$, then $\mu(E) \leq A(\mathrm{Cap}_m(E))^{\frac{p}{p+m}}$. This inequality is true for any compact subset in Ω. Hence by outer regularity it follows that $\mu(E) \leq A(\mathrm{Cap}_m^*(E))^{\frac{p}{p+m}}$ for any Borel set in Ω, which proves our claim that μ puts no mass on m-polar sets. Applying the decomposition theorem ([19, Theorem 5.3]) we get

$$\mu = gH_m(\phi), \quad \phi \in \mathcal{E}_m^0(\Omega), \quad g \in L^1_{\mathrm{loc}}(\Omega, H_m(\phi)).$$

Let $\mu_j = \min(g, j)H_m(\phi)$. Then $\mu_j \le H_m(j^{1/m}\phi) \le H_m(j^{1/m}\phi + f)$ and it follows from [20, Theorem 2.2] that there exists $u_j \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that $H_m(u_j) = \mu_j$ and $u_j = f$ on $\partial\Omega$. Since f is maximal then $u_j \le f$ on Ω. This, coupled with the weak comparison principle applied to u_j and $j^{1/m}\phi + f$ implies that $j^{1/m}\phi + f \le u_j$ (since $\lim_{z\longrightarrow\xi}(u_j - j^{1/m}\phi - f) = 0$ for all $\xi \in \partial\Omega$), which yields that $u_j \in \mathcal{N}_m(\Omega, f)$. On the other hand, it follows from [18, Theorem 1.8.17] that $\mu = H_m(v)$ with $v \in \mathcal{E}_m^p(\Omega)$. However, since $\int_\Omega H_m(v) < +\infty$, this implies that $v \in \mathcal{F}_m^p(\Omega)$. Furthermore, we have $H_m(v + f) \ge \mu \ge \mu_j = H_m(u_j)$. Thus using again the weak comparison principle, we obtain $v + f \le u_j \le f$ for all $j \ge 0$. To finish the proof it is enough to observe that by construction (u_j) decreases to some m-sh function u. Letting $j \longrightarrow +\infty$ we infer that $u \in \mathcal{F}_m^p(\Omega, f)$ and $H_m(u) = \mu$.

4.3. The Dirichlet problem for non m-polar measures

In this section we solve the Dirichlet problem for non m-polar measures. We first establish several intermediate results.

PROPOSITION 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded m-pseudoconvex domain and let $f \in \mathcal{MSH}_m(\Omega) \cap \mathcal{C}(\partial\Omega)$. If μ is a positive measure on Ω such that the envelope $U_m(\mu, f) \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ and $\lim_{z\to\xi} U_m(\mu, f)(z) = f(\xi)$, $\forall\xi \in \partial\Omega$, then for every positive measure ν dominated by μ, the envelope $U_m(\nu, f)$ solves $H_m(U_m(\nu, f)) = \nu$ and satisfies the inequality $f \ge U_m(\nu, f) \ge U_m(\mu, f)$.*

PROOF. We follow the ideas in [7]. Suppose that Ω is smoothly bounded and strictly m-pseudoconvex. Assume further that ν is compactly supported and $f \in \mathcal{MSH}_m(\Omega) \cap \mathcal{C}^\infty(\partial\Omega)$. Since $\nu \le \mu$, then $\mathcal{B}_m(\mu, f) \subset \mathcal{B}_m(\nu, f)$, which yields that $\mathcal{B}_m(\nu, f) \ne \varnothing$ and that $U_m(\mu, f) \le U_m(\nu, f)$. Since $U_m(\nu, f) + \bar{f} \in \mathcal{E}_m^0(\Omega) \cap \mathcal{B}_m(\nu, 0)$, then $\mathcal{B}_m(\nu, 0) \ne \varnothing$, so Lemma 4.1 yields that $U_m(\nu, 0) \in \mathcal{E}_m^0(\Omega) \cap \mathcal{B}_m(\nu, 0)$, therefore, by [19, Lemma 5.1] one can find $\psi \in \mathcal{E}_m^0(\Omega) \subset \mathcal{F}_m^p(\Omega)$ such that $H_m(\psi) = \nu$. Hence ν satisfies (4.2) for every $p \ge 1$. By Theorem 4.2 there is a unique $\vartheta \in \mathcal{F}_m^p(\Omega, f)$ such that $H_m(\vartheta) = \nu$ and $\lim_{z\to\xi} \vartheta(z) = f(\xi)$, $\forall\xi \in \partial\Omega$. In fact the definition of $U_m(\nu, f)$ implies that $\vartheta = U_m(\nu, f)$.

Now if ν does not necessarily have a compact support, consider $\nu_j = \mathbf{1}_{K_j}\nu$ where $(K_j)_j$ is an increasing sequence of compact subsets of Ω such that $\cup_j K_j = \Omega$. Then (ν_j) is an increasing sequence of compactly supported measures that converges weakly to ν. By the first part, we get $H_m(U_m(\nu_j, f)) = \nu_j$ and $U_m(\mu, f) \leq U_m(\nu_j, f) \leq f$. Since $(U_m(\nu_j, f))_j$ is decreasing, then by the continuity of the complex Hessian operator along decreasing sequences $H_m(U_m(\nu, f)) = \nu$ and $f \geq U_m(\nu, f) \geq U_m(\mu, f)$.

For the general case, when Ω is only m-pseudoconvex and $f \in \mathcal{MSH}_m(\Omega) \cap \mathscr{C}(\partial\Omega)$, choose an increasing sequence $(\Omega_k)_{k=1}^{\infty}$ of smoothly bounded strictly m-pseudoconvex domains of Ω such that supp $\nu \Subset \Omega_1$. Since each $f_k := f_{|\partial\Omega_k} = U_m(0, f)_{|\partial\Omega_k}$ is upper semicontinuous, there exists $f_{k,j} \in \mathscr{C}^\infty(\partial\Omega_k)$ with $f_{k,j} \searrow f_k$, as $j \to +\infty$. By the first part of the proof, there exists a unique $u_{k,j} \in \mathcal{SH}_m(\Omega_k) \cap L^\infty(\Omega_k)$ with $H_m(u_{k,j}) = \nu$, $u_{k,j} = U_m(\nu, f_{k,j})$, and $\lim_{z\to\xi} u_{k,j}(z) = f_k(\xi)$, $\forall \xi \in \partial\Omega_k$. Therefore, $U_m(\nu, f)_{|\Omega_k} \in \mathscr{B}_m(\nu, f_{k,j})$, so $U_m(\nu, f)_{|\Omega_k} \leq u_{k,j}$. Since $u_{k,j} \searrow u_k$, as $j \to +\infty$, it follows from the continuity of the Hessian operator along decreasing sequences that $H_m(u_k) = \nu$ and

$$U_m(\mu, f) \leq U_m(\nu, f) \leq u_k \leq U_m(0, f_k) = U_m(0, f)_{|\Omega_k} \quad \text{on } \Omega_k,$$

where the last equality follows from the weak comparison principle. On the other hand, observe that the inequality

$$u_{k+1|\partial\Omega_k} \leq U_m(0, f_{k+1})_{|\partial\Omega_k} = U_m(0, f)_{|\partial\Omega_k} = f_k,$$

implies that the sequence $(u_k)_{k=1}^{\infty}$ is decreasing. Let u denote its limit. In order to finish the proof it remains to show that $u = U_m(\nu, f)$. Since $H_m(u_k) = \nu$ and

$$\liminf_{z\to\xi} u(z) = f(\xi) = \liminf_{z\to\xi} U_m(\nu, f)(z), \quad \xi \in \partial\Omega$$

and since $H_m(u) = H_m(U_m(\nu, f))$, then applying the weak comparison principle yields that $u = U_m(\nu, f)$.

The following theorem was proved for the Monge-Ampère operator by Åhag. (See [1, Theorem 3.4].)

THEOREM 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded m-hyperconvex domain and μ is a positive non m-polar measure on Ω with finite total mass. Then for every $f \in \mathcal{MSH}_m(\Omega) \cap \mathscr{C}(\overline{\Omega})$ there exists a unique $u \in \mathscr{F}_m^a(\Omega, f)$ with $H_m(u) = \mu$.*

PROOF. The case $f = 0$ follows from [19, Theorem 1.2]. For the general case, we proceed as in [1, Theorem 3.4]. Since μ is non m-polar and has a finite total mass, it follows from Theorem 2.15 that there exist functions $\phi \in \mathscr{E}_m^0(\Omega)$ and $0 \leq \ell \in L^1_{\text{loc}}(H_m(\phi))$ such that $\mu = \ell H_m(\phi)$. For each $j \in \mathbb{N}$, set $\mu_j := \min\{\ell, j\} H_m(\phi)$ and observe that $\mu_j \leq H_m(j^{1/m}\phi)$. So, by [20, Theorem 2.2] there exists a unique $\psi_j \in \mathscr{E}_m^0(\Omega)$ such that $H_m(\psi_j) = \mu_j$. This construction implies that $(\psi_j)_j$ is decreasing, $\psi_j + f \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$,

$\lim_{z\to\xi}(\psi_j+f) = f(\xi)$ for every $\xi \in \partial\Omega$, and $U_m(H_m(\psi_j+f), f) = \psi_j+f$. The above equality implies that $H_m(\psi_j+f) \geq \mu_j$. Therefore, by Theorem 4.3 the function $U_j := U_m(\mu_j, f)$ satisfies $U_j \in \mathscr{E}_m^0(\Omega, f)$, $H_m(U_j) = \mu_J$ and $f \geq U_j \geq \psi_j + f$ for each j. Moreover the sequence $(U_j)_j$ is decreasing. Since $\mu(\Omega) < +\infty$ by assumption, it follows that

$$\sup_j \int_\Omega H_m(\psi_j) = \sup_j \int_\Omega H_m(U_j) \leq \sup_j \mu_j(\Omega) \leq \mu(\Omega) < +\infty.$$

Therefore $\psi := \lim_{j\to+\infty} \psi_j \in \mathscr{F}_m^a(\Omega)$ and $u := \lim_{j\to+\infty} U_j$ satisfy $u \in \mathscr{E}_m(\Omega)$, $H_m(u) = \mu$ and $f \geq u \geq \psi + f$ on Ω. Uniqueness follows from the weak comparison principle.

REMARK 4.5. We define $U_m(\mu, f)$ to be the solution obtained by Theorem 4.4. When $\mu = H_m(v)$ with $v \in \mathscr{E}_m^a(\Omega)$, then we define $U_m(\mu, 0) := \lim_j U_m(\mathbf{1}_{\Omega_j}\mu, 0)$.

THEOREM 4.6. *Suppose $\mu \leq H_m(v)$ where $v \in \mathscr{N}_m^a(\Omega)$. Then for every bounded maximal m-subharmonic function f there exists a unique $\varphi \in \mathscr{N}_m^a(\Omega, f)$ with $\mu = H_m(\varphi)$.*

PROOF. We first claim that if $\phi \in \mathscr{E}_m^0(\Omega)$ with $\operatorname{supp}(H_m(\phi)) \Subset \Omega$, then we can find $u \in \mathscr{E}_m^0(\Omega, f)$ such that $H_m(u) = H_m(\phi)$. Indeed, let $(f_k)_{k\in\mathbb{N}}$ and $(\phi_k)_{k\in\mathbb{N}}$ be sequences of functions in $\mathscr{E}_m^0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that (f_k) decreases to f and (ϕ_k) decreases to ϕ on Ω. Let $(\Omega_j)_{j\in\mathbb{N}}$ be a fundamental exhaustion of Ω satisfying $\operatorname{supp}(H_m(\phi)) \Subset \Omega_0$. For each $k \in \mathbb{N}$, define the sequence $j \longmapsto f_k^j = (f_k)^j$ associated to the sequence (Ω_j) according to Definition 2.6 with $u = f_k$. Fix $k, j \in \mathbb{N}$ and observe that $\phi_k + f_k^j \in \mathscr{C}(\partial\Omega_j)$. Then it follows from [11, Theorem 2.10] that there exists $g_{j,k} \in \mathscr{MSH}_m(\Omega_j) \cap \mathscr{C}(\overline{\Omega}_j)$ such that $g_{j,k_{|\partial\Omega_j}} = \phi_k + f_k$. Using Theorem 4.4 one can find a unique $u_{j,k} \in \mathscr{F}_m^a(\Omega_j, g_{j,k})$ such that $H_m(u_{j,k}) = \mu$ on Ω_j. Now define

$$\begin{cases} \Psi_{j,k} = \max(u_{j,k}, \phi_k + f_k^j), & \text{on } \overline{\Omega}_j, \\ \Psi_{j,k} = \phi_k + f_k^j, & \text{on } \Omega \setminus \overline{\Omega}_j. \end{cases}$$

Then $\Psi_{j,k} \in \mathscr{SH}_m^-(\Omega)$ and $f_k^j \geq \Psi_{j,k} \geq f_k^j + \phi_k$ on Ω. Fix $j \in \mathbb{N}$. Then for any $k \in \mathbb{N}$, we have $u_{j,k+1} \leq u_{j,k}$ on Ω_j, which implies that $\Psi_{j,k+1} \leq \Psi_{j,k}$ on Ω. Hence $f^j \geq \lim_{k\to+\infty} \Psi_{j,k} \geq f^j + \phi$ on Ω. Set $\psi_j := \lim_{k\to+\infty} \Psi_{j,k}$. Since $f \in \mathscr{MSH}_m(\Omega)$ it follows that $f^j = f$ and $f \geq \psi_j \geq f + \phi$ on Ω.

Now for each $j \in \mathbb{N}$, set $u_j := \lim_{k\to+\infty} u_{j,k}$ on Ω_j and proceed to prove that

$$u_j = \psi_j \text{ on } \Omega_j \text{ and } \psi_{j+1} \geq \psi_j \text{ on } \Omega. \tag{4.3}$$

To prove the first statement of (4.3), fix $j \in \mathbb{N}$ and observe that $\psi_j \geq u_j$ on Ω_j, since $u_{j,k} \leq \Psi_{j,k}$ on Ω_j for any $k \in \mathbb{N}$. Moreover we have

$$\begin{cases} \limsup\limits_{z \to \xi}(\phi + f)(z) \leq u_{j,k}(\xi) = \phi_k(\xi) + f_k(\xi), & \text{for } \xi \in \partial\Omega_j, \\ H_m(\phi + f) \geq H_m(u_{j,k}) = H_m(\phi), & \text{on } \Omega_j. \end{cases}$$

Hence, $u_j \geq \phi + f$ on Ω_j and then $\psi_j = \max\{u_j, f + \phi\} \leq u_j$ on Ω_j. In particular we have $H_m(\psi_j) = H_m(\phi)$ in Ω_j. This proves the first statement of (4.3).

To prove the second statement of (4.3), observe that we have for any $j \in \mathbb{N}$,

$$\begin{cases} \psi_{j+1} = \psi_j = \phi + f, & \text{on } \Omega \setminus \Omega_{j+1}, \\ \psi_{j+1} = u_{j+1}, & \text{on } \Omega_{j+1}. \end{cases}$$

Observe that $\psi_{j+1} = u_{j+1} \geq \phi + f = \psi_j$ on $\Omega_{j+1} \cap (\Omega \setminus \Omega_j)$, and then $\psi_{j+1} \geq \psi_j$ on $\partial\Omega_{j+1}$. Since $H_m(\psi_{j+1}) = H_m(\phi) \leq H_m(\psi_j)$ on Ω_{j+1}, it follows from the weak comparison principle Corollary 3.3 that $\psi_{j+1} \geq \psi_j$ on Ω_{j+1}. Hence $\psi_{j+1} \geq \psi_j$ on Ω.

Now set $u = (\lim_{j \to +\infty} \psi_j)^*$. Then $H_m(u) = H_m(\phi)$ and $f \geq u \geq f + \phi$, which proves the required properties.

Finally since μ is non m-polar, then applying Theorem 2.15 we get

$$\mu = \ell H_m(g), \quad g \in \mathcal{E}_m^0(\Omega), \quad 0 \leq \ell \in L^1_{\text{loc}}(H_m(g)).$$

For each k, use [19, Theorem 5.1] to find a function $\psi_k \in \mathcal{E}_m^0(\Omega)$ such that $H_m(\psi_k) = \mu_k := \mathbf{1}_{\Omega_k} \inf(\ell, k) H_m(g)$. By the claim above we can find $u_k \in \mathcal{E}_m^0(\Omega, f)$ such that $H_m(u_k) = \mu_k$, where (u_k) is a decreasing sequence with $f \geq u_k \geq \psi_k + f$. Since $\psi_k \geq v$ (by Corollary 3.3), then $f \geq u_k \geq v + f$. Letting $k \to +\infty$ yields $u := \lim_k u_k \in \mathcal{N}_m(\Omega, f)$ and $H_m(u) = \mu$. This establishes the existence part of the theorem. The uniqueness follows from the weak comparison principle.

We finally prove the main result of this section.

THEOREM 4.7. *Let μ be a non m-polar positive measure on Ω. If there exists a function $\varphi \in \mathcal{E}_m(\Omega)$, $\varphi \not\equiv 0$ such that $\int_\Omega (-\varphi) d\mu < +\infty$, then for every function $f \in \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$, there exists a unique $u \in \mathcal{N}_m(\Omega, f)$ such that*

$$H_m(u) = \mu.$$

PROOF. The uniqueness is due to Corollary 3.3. For the existence part we split the proof into two steps.

Step 1. We first assume that $f \equiv 0$ and prove that there exists $u \in \mathcal{N}_m^a(\Omega)$ such that $H_m(u) = \mu$.

Let (Ω_j) be an exhaustion sequence for Ω and let $\mu_j = \mathbf{1}_{\Omega_j}\mu$, $j \geq 1$. It follows from [19, Theorem 1.2] that there exists $u_j \in \mathcal{F}_m^a(\Omega)$ such that $H_m(u_j) = \mu_j$. Then by the comparison principle (see Corollary 3.3) (u_j) is decreasing.

Fix a function $h \in \mathcal{E}_m^0(\Omega)$ such that $0 > h \geq \max(u_1, \varphi)$. Then $h \geq u_j$ for all j, and $H_m(h)$ is not identically zero. Integrating by parts as in [19, Theorem 3.16] we get for any $j \in \mathbb{N}$,

$$\int_\Omega (-u_j) H_m(h) \leq \int_\Omega (-h) H_m(u_j) \leq \int_\Omega (-\varphi)\, d\mu < +\infty.$$

Set $u := \lim_{j\to+\infty} u_j$. Then taking the limit in the previous inequality, we get $\int_\Omega (-u) H_m(h) \leq \int_\Omega (-\varphi) d\mu < +\infty$, which implies that $u \not\equiv -\infty$, hence $u \in \mathcal{SH}_m(\Omega)$.

Now we prove that $u \in \mathcal{E}_m(\Omega)$. Fix an open relatively compact subset $W \Subset \Omega$, and consider for each $j \in \mathbb{N}$ the following function:

$$v_j := \sup\{v \in \mathcal{SH}_m^-(\Omega) : v \leq u_j \text{ on } W\}.$$

Then $v_j \searrow u$ on W and $H_m(v_j)$ is supported on $\overline{W}$. Since $u_j \leq v_j$, then by Lemma 2.14 we get

$$\int_\Omega (-\varphi) H_m(v_j) \leq \int_\Omega (-\varphi) H_m(u_j) \leq \int_\Omega (-\varphi)\, d\mu < +\infty.$$

Set $t := \sup_{\overline{W}} \varphi < 0$. The above inequality yields

$$\int_\Omega H_m(v_j) = \int_{\overline{W}} H_m(v_j) \leq |t|^{-1} \int_\Omega (-\varphi) H_m(v_j) \leq |t|^{-1} \int_\Omega (-\varphi)\, d\mu < +\infty.$$

Therefore, $u \in \mathcal{E}_m(\Omega)$.

It remains to prove that $u \in \mathcal{N}_m^a(\Omega)$. For each fixed $j \in \mathbb{N}$, set $\nu_j = (1 - \mathbf{1}_{\Omega_j})\mu$ and observe that

$$\int_\Omega (-\varphi) d\nu_j \leq \int_\Omega (-\varphi)\, d\mu < +\infty.$$

We can thus apply the arguments above to find $w_j \in \mathcal{E}_m^a(\Omega)$ such that $H_m(w_j) = \nu_j$. For each $k > j$ we have that $u_k \in \mathcal{F}_m^a(\Omega)$ and $H_m(u_k) \leq H_m(u_j + w_j)$. It follows from Corollary 3.3 that $u_k \geq u_j + w_j$. Letting $k \to +\infty$ we obtain $u \geq u_j + w_j$.

By Remark 2.10 we have $\widetilde{u}_j = 0$. Therefore, it follows from (ii) Proposition 2.7 that

$$\begin{aligned} w_j \le \widetilde{w}_j &= \widetilde{w}_j + \widetilde{u}_j \\ &\le (\widetilde{w_j + u_j}) \\ &\le \widetilde{u}. \end{aligned}$$

Fix $h \in \mathcal{E}_m^0(\Omega)$ such that $h \not\equiv 0$ and $h \ge \max\{\varphi, -1\}$. For each $k > j$ let $w_{j,k}$ be the unique function in $\mathcal{F}_m^a(\Omega)$ such that $H_m(w_{j,k}) = \mathbf{1}_{\Omega_k}\nu_j$. Then $w_{j,k} \searrow w_j$ as $k \to +\infty$. Applying Theorem 3.2 with $f = v = 0$, $T = H_m(h)$, and $u = w_{j,k}$ we get

$$\begin{aligned} \int_\Omega (-\widetilde{u})^m H_m(h) &\le \int_\Omega (-w_j)^m H_m(h) \\ &= \lim_{k\to+\infty} \int_\Omega (-w_{j,k})^m H_m(h) \\ &\le \lim_{k\to+\infty} m! \int_\Omega (-h) H_m(w_{j,k}) \\ &= \lim_{k\to+\infty} m! \int_\Omega (-h) \mathbf{1}_{\Omega_k}(1 - \mathbf{1}_{\Omega_j}) d\mu \\ &\le m! \int_\Omega (-h)(1 - \mathbf{1}_{\Omega_j}) d\mu. \end{aligned}$$

Letting $j \to +\infty$ we obtain $\int_\Omega (-\widetilde{u}) H_m(h) = 0$, which implies that $\widetilde{u} = 0$.

Step 2. We now treat the general case, i.e. f is not identically 0.

Since $f \in \mathcal{E}_m(\Omega)$, [19, Theorem 3.1] implies that there exists a decreasing sequence (f_k) in $\mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\bar{\Omega})$ that converges pointwise to f on Ω. Let (Ω_j) be a fundamental exhaustion of Ω. For each $k \in \mathbb{N}$, consider the sequence $f_k^j := (f_k)^j$ associated to f_k by the formula of Definition 2.6. Then $f_k^j \in \mathcal{E}_m^0(\Omega)$ and f_k^j is maximal on Ω_j. Let $\mu_j := \mathbf{1}_{\Omega_j}\mu$. For each $j \in \mathbb{N}$, μ_j is a Borel measure compactly supported on Ω that puts no mass on m-polar sets and we have that $\mu_j(\Omega_j) \le \mu(\Omega) < +\infty$. Then by [19, Theorem 1.2] there exists a unique $\varphi_j \in \mathcal{F}_m^a(\Omega_j) \subset \mathcal{N}_m^a(\Omega_j)$ such that $H_m(\varphi_j) = \mu_j$ on Ω_j. Moreover, from Theorem 4.6, for any $j, k \in \mathbb{N}$ there exists $u_{j,k} \in \mathcal{F}_m^a(\Omega_j, f_k^j)$ such that $H_m(u_{j,k}) = \mu_j$ on Ω_j. Since $H_m(u_{j,k}) = H_m(\varphi_j) \le H_m(\varphi_j + f_k^j)$, by the weak comparison principle we get for any $j, k \in \mathbb{N}$ $f_k^j \ge u_{j,k} \ge \varphi_j + f_k^j$ on Ω_j and for each fixed $j \in \mathbb{N}$, $(u_{j,k})_{k\in\mathbb{N}}$ is a decreasing sequence. Letting $k \to +\infty$, we obtain $f^j \ge u_j \ge \varphi_j + f^j$ on Ω_j, i.e., $u_j \in \mathcal{F}_m(\Omega_j, f^j) \subset \mathcal{N}_m(\Omega_j, f^j)$. Since f is maximal, we have $f^j = f$, thus $u_j \in \mathcal{N}_m(\Omega_j, f)$.

On the other hand, by the definition of μ_j and by the fact that (Ω_j) is an increasing sequence we have that $H_m(u_j) = \mu = H_m(u_{j+1})$ on Ω_j. Then

$(u_j)_j$ is decreasing and we get that $f \geq u_j \geq \varphi_j + f$ on Ω_j. By the first step one can find a function $v \in \mathcal{N}_m^a(\Omega)$ such that $\mu = H_m(v)$. Then by the weak comparison principle we get $f \geq u_j \geq \varphi_j + f \geq v + f$. Thus $u := (\lim_{j\to+\infty} u_j) \in \mathcal{N}_m(\Omega, f)$ with $H_m(u) = \mu$. This completes the proof.

5. The subsolution theorem

In this section we will prove our main theorem. This result was proved in [2] by Åhag, Cegrell, Czyż, and Hiệp for psh functions. The results obtained in the previous sections will allow us to proceed similarly as in [2].

5.1. The Dirichlet problem for singular m-polar measures

Note that it follows from Theorem 2.15 that a Borel measure can be written as the sum of two measures, the first one is non m-polar while the second one is carried by an m-polar set. We are going to prove that these measures are both the Hessian of some functions in the class $\mathcal{E}_m(\Omega)$. Having this in hand, we also prove the following: given two functions $\psi, v \in \mathcal{E}_m(\Omega)$ with $\tilde{\psi} = 0$, $\tilde{v} = f$. If $H_m(\psi)$ is non m-polar and $H_m(v)$ is carried by an m-polar set, and if we add the assumption that any function belonging to $\mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is integrable with respect to $H_m(\psi) + H_m(v)$, then there exists $\varphi \in \mathcal{E}_m(\Omega)$ such that $H_m(\varphi) = H_m(\psi) + H_m(v)$ and $\tilde{\varphi} = f$.

The proof of our main theorem will be a consequence of the above decomposition whose proof requires several intermediate results

Let $u \in \mathcal{E}_m(\Omega)$. It follows from Theorem 2.15 that $H_m(u) = \alpha_u + \nu_u$, where $\alpha_u = f_u H_m(\psi_u)$ and ν_u is a positive measure carried by an m-polar set.

Lemma 5.1. *Let $u, v \in \mathcal{E}_m(\Omega)$. If there exists $\phi \in \mathcal{E}_m(\Omega)$ such that $H_m(\phi)$ is non m-polar and if $|u - v| \leq -\phi$, then $\nu_u = \nu_v$.*

Proof. We first assume that $u, v, \phi \in \mathcal{F}_m(\Omega)$. By [19, Lemma 3.10], it is enough to show that for any $\rho \in \mathcal{E}_m^0(\Omega)$

$$\int_\Omega (-\rho)\nu_u = \int_\Omega (-\rho)\nu_v, \tag{5.1}$$

Indeed, the assumption $|u - v| \leq -\phi$ yields that $v + \phi \leq u$. Therefore, it follows from Lemma 2.14 that

$$\int_\Omega (-\rho) H_m(u) \leq \int_\Omega (-\rho) H_m(v + \phi) < +\infty,$$

for all $\rho \in \mathcal{E}_m^0(\Omega)$. Now set $T_k = (dd^c v)^{m-k} \wedge \beta^{n-m}$. It follows from [15, Lem. 5.6] that $\sum_{k=1}^m C_m^k (dd^c\phi)^k \wedge T_k \ll C_m$, where C_m is the m-Hessian capacity with respect to Ω. Hence $\nu_{v+\phi} = \nu_v$ and $\alpha_{v+\phi} = \alpha_v + \sum_{k=1}^m C_m^k (dd^c\phi)^k \wedge$

T. Then the last inequality implies that

$$\int_\Omega (-\rho)\nu_u \le \int_\Omega (-\rho)(\nu_v + H_m(\phi)) < +\infty.$$

We claim that $\int_\Omega (-\rho)\nu_v = \int_\Omega (-\rho)\nu_u$. Indeed, let A be an m-polar set such that $\nu_u(\Omega \setminus A) = \nu_v(\Omega \setminus A) = 0$, then there exists $\varphi \in \mathcal{SH}_m^-(\Omega)$ such that $A \subset \{\varphi = -\infty\}$, if we take $\rho_k = \max\{\rho, \varphi/k\}$, then we still have

$$\int_\Omega (-\rho_k)\nu_u \le \int_\Omega (-\rho_k)(\nu_v + H_m(\phi)) < +\infty.$$

Letting $k \to +\infty$, we get

$$\int_{\{\varphi=-\infty\}} (-\rho)\nu_u \le \int_{\{\varphi=-\infty\}} (-\rho)(\nu_v + H_m(\phi)) < +\infty.$$

Since $H_m(\phi)$ is non m-polar then $\int_\Omega (-\rho)\nu_v \le \int_\Omega (-\rho)\nu_u$ for every $\rho \in \mathcal{E}_m^0(\Omega)$. As u and v play a symmetrical role, the equality (5.1) follows.

To deal with the general case, recall the following fact. If $w \in \mathcal{E}_m(\Omega)$, then by definition for any open set $\Omega' \Subset \Omega$, there exists $w' \in \mathcal{F}_m(\Omega)$ such that $w' = w$ on Ω'. Consider the greatest extension $\hat{w}$ of w defined as follows

$$\hat{w} := \sup\{\psi \in \mathcal{SH}_m^-(\Omega); \psi \le w \text{ on } \Omega'\}.$$

Then $\hat{w} \in \mathcal{SH}_m^-(\Omega)$, $w' \le \hat{w}$ on Ω and $\hat{w} = w$ on Ω'. Hence $\hat{w} \in \mathcal{F}_m(\Omega)$ and $\hat{w} = w$ on Ω'. Observe that by definition if $w_1 \le w_2$ on Ω' then $\hat{w}_1 \le \hat{w}_2$ on Ω. Applying this construction to u, v and ϕ on a fixed open set Ω' we obtain functions $\hat{u}$, $\hat{v}$ and $\hat{\phi}$ in $\mathcal{F}_m(\Omega)$ such that $|\hat{u} - \hat{v}| \le -\hat{\phi}$ on Ω. Applying the first case to these functions we conclude that $\nu_{\hat{u}} = \nu_{\hat{v}}$ on Ω. Since $\hat{u} = u$ and $\hat{v} = v$ on Ω' we conclude that $\nu_u = \nu_v$ on Ω'. As $\Omega' \Subset \Omega$ was arbitrary, this proves that $\nu_u = \nu_v$ on Ω.

The main theorem will be a consequence of the following two propositions.

PROPOSITION 5.2. *Let $v \in \mathcal{N}_m(\Omega)$ be such that $H_m(v)$ is carried by an m-polar set and $\int_\Omega (-\xi)H_m(v) < +\infty$ for all $\xi \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then the following function*

$$u := \sup\{\varphi \in \mathcal{SH}_m(\Omega) : \varphi \le \min(v, f)\}$$

satisfies $u \in \mathcal{N}_m(\Omega, f)$ and $H_m(u) = H_m(v)$ as measures on Ω.

PROOF. Observe that $f + v$ is m-subharmonic in Ω and $f + v \le \min(v, f)$. Therefore u is well defined and since $\min(v, f)$ is upper semicontinuous function, $u \in \mathcal{SH}_m(\Omega)$ and $f + v \le u \le f$. Hence $u \in \mathcal{N}_m(\Omega, f)$. By [19, Theorem 3.1] we can choose a decreasing sequence $(v_j)_{j\in\mathbb{N}}$ in $\mathscr{E}_m^0(\Omega)\cap\mathscr{C}(\overline{\Omega})$ that converges pointwise to v in Ω. By Theorem 4.7, for each $j\in\mathbb{N}$ there exists $\omega_j\in\mathscr{N}_m(\Omega,f)$ such that $H_m(\omega_j)=H_m(v_j)$. We claim that $v_j\ge\omega_j$ on Ω for any $j\in\mathbb{N}$. Indeed for any fixed j, $\max(v_j,\omega_j)\in\mathscr{E}_m^0(\Omega)$. Moreover, by [18, Theorem 3.1.16], we have

$$H_m(\max(v_j,\omega_j))\ge\mathbf{1}_{\{v_j>\omega_j\}}H_m(v_j)+\mathbf{1}_{\{\omega_j\ge v_j\}}H_m(\omega_j)=H_m(v_j),$$

in the sense of Borel measures on Ω. By the comparison principle [18, Corollary 1.3.14], it follows that $\max(v_j,\omega_j)\le v_j$ on Ω, which yields the required inequality $v_j\ge\omega_j$.

Now fix $j\in\mathbb{N}$ and define the following function:

$$u_j:=\sup\{\varphi\in\mathscr{SH}_m(\Omega):\varphi\le\min(v_j,f)\}.$$

Then $u_j\in\mathscr{E}_m^0(\Omega,f)$ and $u_j\ge\omega_j$. By Lemma 2.14 we get

$$\int_\Omega(-\xi)H_m(u_j)\le\int_\Omega(-\xi)H_m(\omega_j),$$

for all $\xi\in\mathscr{E}_m^0(\Omega)\cap\mathscr{C}(\overline{\Omega})$. Moreover Proposition 2.13 yields

$$\int_\Omega(-\xi)H_m(u)\le\int_\Omega(-\xi)H_m(v)\tag{5.2}$$

for all $\xi\in\mathscr{E}_m^0(\Omega)\cap\mathscr{C}(\overline{\Omega})$.

On the other hand, since $u\le v$, it follows from [15, Lemma 5.2] that

$$\mathbf{1}_{\{v=-\infty\}}H_m(v)\le\mathbf{1}_{\{u=-\infty\}}H_m(u),$$

in the sense of Borel measures on Ω. Hence

$$\begin{aligned}\int_\Omega(-\xi)H_m(v)&=\int_{\{v=-\infty\}}(-\xi)H_m(v)\\&\le\int_{\{u=-\infty\}}(-\xi)H_m(u)\\&\le\int_\Omega(-\xi)H_m(u),\end{aligned}$$

for all $\xi\in\mathscr{E}_m^0(\Omega)\cap\mathscr{C}(\overline{\Omega})$. This inequality together with (5.2) yield that $\int_\Omega(-\xi)H_m(u)=\int_\Omega(-\xi)H_m(v)$ for all $\xi\in\mathscr{E}_m^0(\Omega)\cap\mathscr{C}(\overline{\Omega})$.

Since $\int_\Omega(-\xi)H_m(v)<+\infty$ for all $\xi\in\mathscr{E}_m^0(\Omega)\cap\mathscr{C}(\overline{\Omega})$, it follows from [19, Lemma 3.10] that $H_m(u)=H_m(v)$ as Borel measures on Ω. Thus, the proof is completed.

To state the next proposition we need the following notation.

$$\mathcal{F}(H_m(\psi), v) := \{\varphi \in \mathcal{E}_m(\Omega) : H_m(\psi) \leq H_m(\varphi) \text{ and } \varphi \leq v\}.$$

PROPOSITION 5.3. *Let $\psi \in \mathcal{N}_m^a(\Omega)$ and $v \in \mathcal{N}_m(\Omega, f)$ be such that $H_m(v)$ is carried by an m-polar set and for all $\xi \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega})$,*

$$\int_\Omega (-\xi)(H_m(\psi) + H_m(v)) < +\infty.$$

Then the following function

$$u := \sup\{\varphi : \varphi \in \mathcal{F}(H_m(\psi), v)\},$$

satifies $u \in \mathcal{N}_m(\Omega, f)$ and $H_m(u) = H_m(\psi) + H_m(v)$ in the sense of measures on Ω.

PROOF. It follows from Choquet's lemma that $u^* = (\sup_{j\geq 1} \phi_j)^*$ for some sequence $(\phi_j)_{j\in\mathbb{N}}$ belonging to $\mathcal{F}\big(H_m(\psi), v\big)$. Observe that $u^* \in \mathcal{E}_m(\Omega)$ and we can assume that the sequence (ϕ_j) increases to u^* almost everywhere in Ω. By continuity of the Hessian operator for increasing sequences, we have $H_m(\phi_j) \to H_m(u^*)$ in the weak sense on Ω. As $H_m(\phi_j) \geq H_m(\psi)$, it follows that $H_m(u^*) \geq H_m(\psi)$. Moreover since $u^* \leq v$, it follows that $u^* \in \mathcal{F}\big(H_m(\psi), v\big)$, which yields that $u^* = u$. On the other hand, $\omega + v \in \mathcal{F}(H_m(\psi), v)$, hence $u \in \mathcal{N}_m(\Omega, f)$.

By Theorem 2.15, there exists α_u, ν_u two positive measures on Ω such that $H_m(u) = \alpha_u + \nu_u$ with α_u is non m-polar and ν_u is carried by an m-polar set. Since $|u - v| \leq -\psi$, then by Lemma 5.1, $\nu_u = \nu_v = H_m(v)$. Note that $\alpha_u \geq H_m(\psi)$. Indeed, we have $H_m(u) = \alpha_u + \nu_u \geq H_m(\psi)$. Since $H_m(\psi)$ is non m-polar then for a Borel subset $B \subset \Omega$ we get

$$\begin{aligned}\int_B \alpha_u = \int_{B\setminus\{u=-\infty\}} \alpha_u &= \int_{B\setminus\{u=-\infty\}} H_m(u) \\ &\geq \int_{B\setminus\{u=-\infty\}} H_m(\psi) = \int_B H_m(\psi).\end{aligned}$$

By Proposition 2.12 there exists a decreasing sequence $(v_j)_{j\in\mathbb{N}}$ in $\mathcal{E}_m^0(\Omega, f)$ that converges pointwise to v on Ω i.e., there exists $\gamma_j \in \mathcal{E}_m^0(\Omega)$ such that $\gamma_j + f \leq v_j \leq f$. It follows then from [15, Proposition 5.6] that

$$\int_A H_m(v_j) \leq \int_A H_m(\gamma_j + f) \leq C(m)\Big(\int_A H_m(\gamma_j)\Big)^{1/m}\Big(\int_A H_m(f)\Big)^{1/m} = 0,$$

for every m-polar set $A \subset \Omega$, which implies that $H_m(v_j)$ vanishes on m-polar sets. Furthermore, since

$$\int_\Omega (-\xi)(H_m(\psi) + H_m(v_j)) < +\infty, \quad \text{for all } \xi \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega}),$$

it follows from Theorem 4.7 that there exists $\omega_j \in \mathcal{N}_m(\Omega, f)$ such that

$$H_m(\omega_j) = H_m(v_j) + H_m(\psi).$$

According to the weak comparison principle applied to ω_j and v_j we get $\omega_j \in \mathcal{F}(H_m(\psi), v_j)$. Hence we can consider the function

$$u_j := \sup\{\varphi : \varphi \in \mathcal{F}(H_m(\psi), v_j)\}.$$

The sequence (u_j) decreases pointwise to u, as $j \to +\infty$. Furthermore, Lemma 2.14 yields

$$\int_\Omega (-\xi) H_m(u_j) \le \int_\Omega (-\xi) H_m(\omega_j) = \int_\Omega (-\xi)(H_m(\psi) + H_m(v_j)).$$

Letting $j \to +\infty$ and using Proposition 2.13, we get

$$\int_\Omega (-\xi) H_m(u) = \int_\Omega (-\xi)(\alpha_u + \nu_u) \le \int_\Omega (-\xi)(H_m(\psi) + \nu_u),$$

for all $\xi \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Since $\alpha_u \ge H_m(\psi)$, it follows that $\int_\Omega \xi \alpha_u = \int_\Omega \xi H_m(\psi)$ for all $\xi \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Hence $\alpha_u = H_m(\psi)$ and the proof is completed.

5.2. Proof of the subsolution theorem

We give here the proof of the main theorem from the introduction.

Proof of the main theorem. By Theorem 2.15 there exists $\phi \in \mathcal{E}_m^0(\Omega)$ and a function $0 \le \ell \in L^1_{\mathrm{loc}}(H_m(\phi))$ such that $\mu = \ell H_m(\phi) + \nu$.

We claim that one can find $\Phi, \Psi \in \mathcal{E}_m(\Omega)$ such that $\Phi, \Psi \ge \omega$, $H_m(\Psi) = \ell H_m(\phi)$, $H_m(\Phi) = \nu$, and ν is carried by $\{\Phi = -\infty\}$. Indeed, since $\mu \le H_m(\omega)$, it follows from the Radon-Nikodym decomposition theorem that

$$\ell H_m(\phi) = \tau \mathbf{1}_{\{\omega > -\infty\}} H_m(\omega) \quad \text{and} \quad \nu = \tau \mathbf{1}_{\{\omega = -\infty\}} H_m(\omega),$$

where $0 \le \tau \le 1$ is a measurable function. For each j, use [19, Theorem 5.1] to find $\Psi_j \in \mathcal{E}_m^0(\Omega)$ such that

$$H_m(\Psi_j) = \min(\ell, j) H_m(\phi) =: \mu_j.$$

By Corollary 3.3 $\Psi_j \geq \omega$ and (Ψ_j) is a decreasing sequence. Therefore,

$$\Psi = \lim_{j \to +\infty} \Psi_j \in \mathscr{E}_m(\Omega) \quad \text{and} \quad H_m(\Psi) = \ell H_m(\phi).$$

On the other hand, $\tau \mathbf{1}_{\{\omega=-\infty\}} H_m(\omega)$ is a positive measure vanishing outside $\{\omega = -\infty\}$, then by [15, Proposition 5.8] there exists $\Phi \in \mathscr{E}_m(\Omega)$ such that $H_m(\Phi) = \nu$ and $\Phi \geq \omega$. This proves the claim.

Let now $(g_j)_{j\in\mathbb{N}}$ be an increasing sequence of measurable functions on Ω with compact support which converges to $g := \mathbf{1}_{\{\omega=-\infty\}}\tau$. It follows from [15, Proposition 5.7] that there exists a decreasing sequence $(\omega^{g_j})_{j\in\mathbb{N}}$ in $\mathscr{F}_m(\Omega)$ that converges pointwise to ω^g such that $H_m(\omega^{g_j}) = g_j\, H_m(\omega)$. Moreover $\omega^g \geq \omega$. Hence it follows from the continuity of the Hessian operator along decreasing sequences that $H_m(\omega^g) = \mathbf{1}_{\{\omega=-\infty\}}\tau H_m(\omega) = \nu$. Set for each $j \in \mathbb{N}$,

$$v_j := \sup\{\varphi \in \mathscr{SH}_m(\Omega) : \varphi \leq \min(\omega^{g_j}, f)\}.$$

Then it follows from Proposition 5.2 that $v_j \in \mathscr{N}_m(\Omega, f)$ and $H_m(v_j) = H_m(\omega^{g_j})$ for each $j \in \mathbb{N}$. On the other hand set for each $j \in \mathbb{N}$,

$$u_j := \sup\{\varphi : \varphi \in \mathscr{F}(H_m(\Psi_j), \min(\omega^{g_j}, f))\}.$$

Then $(u_j)_{j\in\mathbb{N}}$ is a decreasing sequence which converges to some m-sh function u. By Proposition 5.3 $u_j \in \mathscr{N}_m(\Omega, f)$ and $H_m(u_j) = H_m(\Psi_j) + H_m(\omega^{g_j})$ for each $j \in \mathbb{N}$. Furtheremore, we have $\omega + f \leq u_j \leq f$ for any j. Hence we get the desired result by letting $j \longrightarrow +\infty$. Note that if $\omega \in \mathscr{N}_m(\Omega)$ then $u \in \mathscr{N}_m(\Omega, f)$. This completes the proof of main theorem.

REMARK 5.4. (1). The main theorem fails in general if f is not maximal. Indeed, let $\mu = H_m(\omega)$ with $\omega \in \mathscr{E}_m^0(\Omega)$ and let $f \in \mathscr{E}_m(\Omega)$ such that $H_m(f) > H_m(\omega)$ (take for example $f = \omega + (\|z\|^2 - A)$ with $A > 0$ is such that $f \leq 0$ in Ω), suppose that there exists $u \in \mathscr{E}_m(\Omega)$ such that $\omega + f \leq u \leq f$ in Ω and $H_m(u) = \mu$. Since $\lim_{z\to\xi}(u - f)(z) \geq \lim_{z\to\xi}(\omega + f - f)(z) = 0$ for all $\xi \in \partial\Omega$ and $H_m(u)$ is non m-polar, then by using the weak comparison principle we get $u \geq f$. This implies that $u = f$ in Ω, which contradicts the fact that $H_m(f) > \mu$.

(2). Assume that $\omega \in \mathscr{E}_m^0(\Omega)$ with $m = 1$, μ is a positive measure with finite total mass, and f is a subharmonic function with continuous boundary values. Then the solution to the Dirichlet problem for the Poisson equation $\Delta u = \mu$ with boundary values f can be written $u = \tilde{f} + g_\mu$, where $\tilde{f}$ is the harmonic extension of f from the boundary to Ω (the least hamonic majorant of u) and g_μ is the Green potential of the Borel measure μ, so that $f + \omega \leq u \leq \tilde{f}$.

In the general case, under the sole assumption that $f \in \mathscr{E}_m(\Omega)$, the statement of the main theorem is still valid with the conclusion that $f + \omega \leq u \leq \tilde{f}$, where $\tilde{f}$ is the least maximal m-subharmonic majorant of f. If f is maximal then $\tilde{f} = f$ and we obtain the conclusion of the main theorem. Therefore, the hypothesis of maximality of f in the statement of the main theorem is natural and its conclusion can be seen as a non-linear version of the Riesz decomposition theorem in classical potential theory.

The following example shows that in general a solution does not exist if the Borel measure μ is not dominated by the Hessian measure of a function in the class $\mathcal{E}_m(\Omega)$, even if $f = 0$ and μ has finite mass.

EXAMPLE 5.5. Let $\Omega = B \subset \mathbb{C}^3$ be the unit ball and fix $0 < r < 1$. From [18, Lemma 1.4.3.], the relative extremal 2-subharmonic function of the $\bar{B}(0, r)$ of radius r is given by

$$u_r = u_{2,\bar{B}(r),B} = \max\left\{-\frac{1 - \|z\|^{-1}}{1 - r^{-1}}, -1\right\}.$$

By [18, Theorem 1.4.12 and Example 1.4.14] we have

$$\int_B (dd^c u_r)^2 \wedge \beta = \frac{2}{3}\frac{r^2}{(1-r)^2}.$$

Also from the fact that $u_r \equiv -1$ on $\bar{B}(r)$ and by [18, Theorem 1.4.11] we have $(dd^c u_r)^2 \wedge \beta = 0$ on $B \setminus S(r)$.

Set for $j \geq 2$, $a_j = 1/j^{1/2}$, $b_j = 1/(j^{1/2} \log j)$, and $r_j = 1/(1 + b_j)$. Then $r_j \nearrow 1$, $\sum_2^{+\infty} a_j^2 = +\infty$ and $\sum_2^{+\infty} a_j b_j = +\infty$.

Observe that for $j \geq 2$, one can show that $a_j \sum_{k=2}^{j} a_k < 2$. Indeed

$$\sum_{k=2}^{j} a_k = \sum_{k=2}^{j} \frac{1}{\sqrt{k}} \leq \sum_{k=2}^{j} \int_{k-1}^{k} \frac{dx}{\sqrt{x}} = \int_1^j \frac{dx}{\sqrt{x}} < 2\sqrt{j}.$$

Set $\varphi_j = a_j(1 - r_j)/(r_j)u_{r_j} = a_j b_j u_{r_j}$ for $j \geq 2$ and $u_N := \sum_{j=2}^{N} \varphi_j$ for $N \geq 2$. Then $u_N \in \mathcal{E}_2^0(B)$ and for any N and the sequence $(u_n)_{N\geq 2}$ decreases to $-\infty$. Indeed, if $z \in B$, there exists j_0 such that $z \in B(r_{j_0})$, then $u_{r_j} = -1$ for $j \geq j_0$, so that $\varphi_j(z) = -a_j b_j$, for any $j \geq j_0$. This implies that $u_N(z) = \sum_{j=2}^{N} \varphi_j(z) \leq -\sum_{j \geq j_0}^{N} a_j b_j \searrow -\infty$ as $N \to +\infty$.

Since $(dd^c \varphi_j)^2 \wedge \beta = 0$ in $B(r_j)$ for any j, it follows that for $k \leq j$ we have

$$\begin{aligned}\int_B (\log|z|)^2 dd^c\varphi_j \wedge dd^c\varphi_k \wedge \beta &= \int_{B\setminus B(r_j)} (\log|z|)^2 dd^c\varphi_j \wedge dd^c\varphi_k \wedge \beta \\ &\le (\log r_j)^2 \int_B dd^c\varphi_j \wedge dd^c\varphi_k \wedge \beta \\ &= (\log r_j)^2 a_j b_j a_k b_k \int_B dd^c u_{r_j} \wedge dd^c u_{r_k} \wedge \beta.\end{aligned}$$

From mixed inequalities (see [9]), it follows that

$$\begin{aligned}&\int_B (\log|z|)^2\, dd^c\varphi_j \wedge dd^c\varphi_k \wedge \beta \\ &\quad \le (\log r_j)^2 a_j b_j a_k b_k \left(\int_B H_2(u_{r_j})\right)^{\frac{1}{2}} \left(\int_B H_2(u_{r_k})\right)^{\frac{1}{2}} \\ &\quad = (\log r_j)^2 a_j b_j a_k b_k \frac{2}{3}\frac{1}{b_j}\frac{1}{b_k} \\ &\quad = \frac{2}{3}(\log(1+b_j))^2 a_j a_k.\end{aligned}$$

Recall that $u_N = \sum_{j=2}^N \varphi_j$ and set $\mu_N := H_2(u_N)$ for $N \in \mathbb{N}$ with $N \ge 2$. Then we have

$$\begin{aligned}\int_B (\log|z|)^2 d\mu_N &= \int_B (\log|z|)^2 \left(dd^c \sum_{j=2}^N \varphi_j\right)^2 \wedge \beta \\ &= \sum_{j,k=2}^N \int_B (\log|z|)^2\, dd^c\varphi_j \wedge dd^c\varphi_k \wedge \beta \\ &\le 2\sum_{j=2}^N \sum_{k=2}^j \int_B (\log|z|)^2\, dd^c\varphi_j \wedge dd^c\varphi_k \wedge \beta \\ &\le \frac{4}{3}\sum_{j=2}^N (\log 1 + b_j)^2 a_j \sum_{k=2}^j a_k \\ &\le \frac{8}{3}\sum_{j=2}^N (\log(1+b_j))^2 < +\infty,\end{aligned}$$

since $a_j \sum_{k=2}^j a_k \le 2$. The last series converges because

$$(\log(1+b_j))^2 \simeq b_j^2 = \left(\sqrt{j}\log(j)\right)^{-2}.$$

Finally it follows from the last estimate that the sequence of Borel measures (μ_N) have locally uniformly bounded masses in B. Since it is increasing, it converges to a Borel measure μ on B such that for any $N \in \mathbb{N}$, $\mu_N \leq \mu$.

We claim that for the Borel measure μ, there is no function $u \in \mathscr{E}_2^-(B)$ such that $\mu = (dd^c u)^2 \wedge \beta$. Indeed, assume that there exists $u \in \mathscr{E}_2^-(B)$ such that $\mu = (dd^c u)^2 \wedge \beta$. Then since $u_N \in \mathscr{E}_m^0(B)$, the Borel measure μ_N is non m-polar for any N and we can apply the weak comparison principle Corollary to conclude that $u \leq u_N$ in B for any N. Since $u_N \searrow -\infty$ as $N \to +\infty$, it follows that $u \equiv -\infty$ in B, which is a contradiction.

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