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INVARIANTS OF LINKAGE OF MODULES

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Abstract

Let $(A, \mathfrak{m})$ be a Gorenstein local ring and let M, N be two Cohen-Macaulay A-modules with M linked to N via a Gorenstein ideal $\mathfrak{q}$. Let L be another finitely generated A-module. We show that $\operatorname{Ext}^i_A(L, M) = 0$ for all $i \gg 0$ if and only if $\operatorname{Tor}^A_i(L, N) = 0$ for all $i \gg 0$. If D is a Cohen-Macaulay module then we show that $\operatorname{Ext}^i_A(M, D) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_A(D^\dagger, N) = 0$ for all $i \gg 0$, where $D^\dagger = \operatorname{Ext}^r_A(D, A)$ and $r = \operatorname{codim} D$. As a consequence we get that $\operatorname{Ext}^i_A(M, M) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_A(N, N) = 0$ for all $i \gg 0$. We also show that $\operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) \cong (\operatorname{End}_A(N)/\operatorname{rad}\operatorname{End}_A(N))^{\mathrm{op}}$. We also give a negative answer to a question of Martsinkovsky and Strooker.

1. Introduction

Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Recall an ideal $\mathfrak{q}$ in A is said to be a *Gorenstein ideal* if $\mathfrak{q}$ is perfect and $A/\mathfrak{q}$ is a Gorenstein local ring. A special class of Gorenstein ideals are *CI* (*complete intersection*) *ideals*, i.e., ideals generated by an A-regular sequence. In this paper, a not necessarily perfect ideal $\mathfrak{q}$ such that $A/\mathfrak{q}$ is a Gorenstein ring will be called a *quasi-Gorenstein* ideal.

Two ideals I and J are linked by a Gorenstein ideal $\mathfrak{q}$ if $\mathfrak{q} \subseteq I \cap J$; $J = (\mathfrak{q} \colon I)$ and $I = (\mathfrak{q} \colon J)$. We write it as $I \sim_{\mathfrak{q}} J$. If $\mathfrak{q}$ is a complete intersection then we say I is CI-linked to J via $\mathfrak{q}$. If $\mathfrak{q}$ is a quasi-Gorenstein ideal then we say I is quasi-linked to J via $\mathfrak{q}$. Note that traditionally only CI-linkage used to be considered. However in recent times more general types of linkage are studied.

We say ideals I and J is in the *same linkage class* if there is a sequence of ideals $I_0, \dots, I_n$ in A and Gorenstein ideals $\mathfrak{q}_0 \dots, \mathfrak{q}_{n-1}$ such that

(i) $I_j \sim_{\mathfrak{q}_j} I_{j+1}$, for $j = 0, \dots, n-1$.
(ii) $I_0 = I$ and $I_n = J$.

If n is even then we say that I and J are *evenly linked*. We can analogously define CI-linkage class, quasi-linkage class, even CI-linkage class and even quasi-linkage class (of ideals).

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A natural question is that if I and J are in the same linkage class then what properties of I are shared by J. This was classically done when I and J are in the same CI-linkage class (or even CI-linkage class). In their landmark paper [13], Peskine and Szpiro proved that if I and J are in the same CI-linkage class and I is a Cohen-Macaulay ideal (i.e., A/I is a Cohen-Macaulay ring) then so is J. This can be proved more generally for ideals in a quasi-linkage class, see [11, Corollary 15, p. 616]. In another landmark paper [8, 1.14], Huneke proved that if I is in the CI-linkage class of a complete intersection then the Koszul homology $H_i(I)$ are Cohen-Macaulay for all $i \geq 0$. It is known that this result is not true in if I is linked to a complete intersection (via Gorenstein ideals and not-necessarily CI-ideals). If $A = k[[X_1, \ldots, X_n]]$ (where k is a field or a complete discrete valuation ring) then Huneke defined some invariants of even CI-linkage class of equidimensional unmixed ideals, see [9, 3.2]. Again these invariants are not stable under Gorenstein (even)-liason.

In a remarkable paper Martsinkovsky and Strooker, [11], introduced liaison for modules. See section two for definition. We note here that ideals I and J are linked as ideals if and only if A/I is linked to A/J as modules. One can analogously define linkage class of modules, even linkage of modules etc. We can also define CI linkage of modules, quasi-linkage of modules etc.

Thus a natural question arises: If M, N are in the same linkage class of modules (or same even linkage class of modules) then what properties of M are shared by N. The generalization of Peskine and Szpiro's result holds. If M is Cohen-Macaulay and N is quasi-linked to M then N is also Cohen-Macaulay, see [11, Corollary 15, p. 616]. To state another property which is preserved under linkage first let us recall the definition of Cohen-Macaulay approximation from [1]. A Cohen-Macaulay approximation of a finitely generated A-module M is a maximal Cohen-Macaulay A-module X such that there exists an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$$

where Y has finite projective dimension. Such a sequence is not unique but X is known to be unique up to a free summand and so is well defined in the stable category $\underline{\mathrm{CM}}(A)$ of maximal Cohen-Macaulay A-modules. We denote by $X(M)$ the maximal Cohen-Macaulay approximation of M. In [11, Theorem 13, p. 620], Martsinkovsky and Strooker proved that if M is evenly linked to N then $X(M) \cong X(N)$ in $\underline{\mathrm{CM}}(A)$. They also asked if this result holds for M and N are in the same even quasi-linkage class, see [11, Question 3, p. 623]. A motivation for this paper was to try and solve this question. We answer this question in the negative. We prove

THEOREM 1.1. *There exists a complete intersection A of dimension one and finite length modules M, N such that M is evenly quasi-linked to N but $X(M) \ncong X(N)$.*

To prove our next result we introduce a construction (essentially due to Ferrand) which is very useful, see section three. If E is an A-module then set $\operatorname{codim} E = \dim A - \dim E$. We prove:

THEOREM 1.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d. Let M be a Cohen-Macaulay A-module. Assume $M \sim_{\mathfrak{q}} N$ where $\mathfrak{q}$ is a Gorenstien ideal in A. Let L be a finitely generated A-module and let D be a Cohen-Macaulay A-module. Set $D^{\dagger} = \operatorname{Ext}_A^{\operatorname{codim} D}(D, A)$. Then*

(1) $\operatorname{Ext}_A^i(L, M) = 0$ *for all* $i \gg 0$ *if and only if* $\operatorname{Tor}_i^A(L, N) = 0$ *for all* $i \gg 0$.
(2) $\operatorname{Ext}_A^i(M, D) = 0$ *for all* $i \gg 0$ *if and only if* $\operatorname{Ext}_A^i(D^{\dagger}, N) = 0$ *for all* $i \gg 0$.

We should note that this result is new even in the case for cyclic modules. A remarkable consequence of Theorem 1.2 is the following result

COROLLARY 1.3. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d. Let M, C be Cohen-Macaulay A-modules. Assume $M \sim_{\mathfrak{q}} N$ and $C \sim_{\mathfrak{n}} D$ where $\mathfrak{q}, \mathfrak{n}$ are Gorenstein ideals in A. Then*

$$\operatorname{Ext}_A^i(M, C) = 0 \text{ for all } i \gg 0 \iff \operatorname{Ext}_A^i(D, N) = 0 \text{ for all } i \gg 0.$$

In particular

$$\operatorname{Ext}_A^i(M, M) = 0 \text{ for all } i \gg 0 \iff \operatorname{Ext}_A^i(N, N) = 0 \text{ for all } i \gg 0.$$

We also prove the following surprising invariant of quasi-linkage.

THEOREM 1.4. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring and let $M, N, N' \in \mathrm{CM}^g(A)$. Assume M is quasi-evenly linked to N and that it is quasi-oddly linked to N'. Then*

(1) $\operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M) \cong \operatorname{End}(N)/\operatorname{rad}\operatorname{End}(N)$.
(2) $\operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M) \cong \big(\operatorname{End}(N')/\operatorname{rad}\operatorname{End}(N')\big)^{\mathrm{op}}$.

Here if Γ is a ring then Γ^{op} is its opposite ring.

We now describe in brief the contents of this paper. In section two we recall some preliminaries regarding linkage of modules as given in [11]. In section three we give a construction which is needed to prove our results. We prove Theorem 1.2(1) in section four and Theorem 1.2(2) in section five. We recall some facts regarding cohomological operators in section six. This is needed in section seven where we prove Theorem 1.1. Finally in section eight we prove Theorem 1.4.

2. Some preliminaries on liason of modules

Throughout all rings considered are commutative Noetherian and all modules under consideration are finitely generated. In this section we recall the definition of linkage of modules as given in [11]. Throughout $(A, \mathfrak{m})$ is a Gorenstein local ring of dimension d.

2.1. Let us recall the definition of transpose of a module. Let $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ be a minimal presentation of M. Let $(-)^* = \operatorname{Hom}(-, A)$. The *transpose* $\operatorname{Tr}(M)$ is defined by the exact sequence

$$0 \longrightarrow M^* \longrightarrow F_0^* \xrightarrow{\phi^*} F_1^* \longrightarrow \operatorname{Tr}(M) \longrightarrow 0.$$

Also let $\Omega(M)$ be the first syzygy of M.

DEFINITION 2.2. Two A-modules M and N are said to be *horizontally linked* if $M \cong \Omega(\operatorname{Tr}(N))$ and $N \cong \Omega(\operatorname{Tr}(M))$.

Next we define linkage in general.

DEFINITION 2.3. Two A-modules M and N are said to be linked via a Gorenstein ideal $\mathfrak{q}$ if

(1) $\mathfrak{q} \subseteq \operatorname{ann} M \cap \operatorname{ann} N$, and

(2) M and N are horizontally linked as $A/\mathfrak{q}$-modules.

We write it as $M \sim_{\mathfrak{q}} N$.

If $\mathfrak{q}$ is a complete intersection we say M is CI-linked to N via $\mathfrak{q}$. If $\mathfrak{q}$ is a quasi Gorenstein ideal then we say M is quasi-linked to N via $\mathfrak{q}$.

REMARK 2.4. It can be shown that ideals I and J are linked by a quasi-Gorenstein ideal $\mathfrak{q}$ (definition as in the introduction) if and only if the module A/I is quasi-linked to A/J by $\mathfrak{q}$, see [11, Proposition 1, p. 592].

2.5. We say M, N are in *same linkage class* of modules if there is a sequence of A-modules $M_0, \dots, M_n$ and Gorenstein ideals $\mathfrak{q}_0 \dots, \mathfrak{q}_{n-1}$ such that

(i) $M_j \sim_{\mathfrak{q}_j} M_{j+1}$, for $j = 0, \dots, n-1$.

(ii) $M_0 = M$ and $M_n = N$.

If n is even then we say that M and N are *evenly linked*. Analogously we can define the notion of CI-linkage class, quasi-linkage class, even CI-linkage class and even quasi-linkage class (of modules).

3. A construction

In this section we describe a construction essentially due to Ferrand. Throughout $(A, \mathfrak{m})$ is a Gorenstein local ring of dimension d. Let $\mathrm{CM}^g(A)$ be the full subcategory of Cohen-Macaulay A-modules of codimension g.

3.1. We note the following well-known result, see [5, 3.3.10]. Let $D \in \mathrm{CM}^g(A)$. Then $\mathrm{Ext}^i_A(D, A) = 0$ for $i \neq g$. Set $D^\dagger = \mathrm{Ext}^g_A(D, A)$. Then $D^\dagger \in \mathrm{CM}^g(A)$. Furthermore $(D^\dagger)^\dagger \cong D$.

The following result is well-known. However we give a proof as we do not have a reference.

Lemma 3.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let $M \in \mathrm{CM}^g(A)$. Let $\mathfrak{q}$ be a quasi-Gorenstein ideal of grade g contained in* $\operatorname{ann} M$. *Set $B = A/\mathfrak{q}$. Then* $\mathrm{Ext}^g_A(M, A) \cong \mathrm{Hom}_B(M, B)$.

Proof. Let $\mathbf{y} = y_1, \ldots, y_g \in \mathfrak{q}$ be a regular sequence. Set $C = A/(\mathbf{y})$. Then we have a natural ring homomorphism $C \to B$. As C, B are Gorenstein rings we have $\mathrm{Hom}_C(B, C) \cong B$, see [5, 3.3.7]. We now note that

$$\begin{aligned}
\mathrm{Ext}^g_A(M, A) &\cong \mathrm{Hom}_C(M, C), \qquad \text{see [5, 3.1.16]}\\
&= \mathrm{Hom}_C(M \otimes_B B, C),\\
&\cong \mathrm{Hom}_B(M, \mathrm{Hom}_C(B, C)),\\
&= \mathrm{Hom}_B(M, B).
\end{aligned}$$

This finishes the proof.

Construction 3.3. Let $M \in \mathrm{CM}^g(A)$ and let $\mathfrak{q}$ be a quasi-Gorenstein ideal in A of codimension g contained in $\operatorname{ann} M$. Let $M \sim_{\mathfrak{q}} N$. Let $\mathbf{P}$ be minimal free resolution of M over A and let $\mathbf{Q}$ be a minimal free resolution of $\mathbf{P}_0/\mathfrak{q}\mathbf{P}_0$ over A. We have a natural map $\mathbf{P}_0/\mathfrak{q}\mathbf{P}_0 \to M \to 0$. We lift this to a chain map $\phi\colon \mathbf{Q} \to \mathbf{P}$. We then dualize this map to get a chain map $\phi^*\colon \mathbf{P}^* \to \mathbf{Q}^*$. Let $\mathbf{C} = \mathrm{cone}(\phi^*)$.

Lemma 3.4.
$$H^i(\mathbf{C}) = \begin{cases} N, & \textit{if } i = g,\\ 0, & \textit{if } i \neq g.\end{cases}$$

Proof. We have an exact sequence of complexes

$$0 \longrightarrow \mathbf{Q}^* \longrightarrow \mathbf{C} \longrightarrow \mathbf{P}^*(-1) \longrightarrow 0.$$

Notice

$$H^i(\mathbf{P}^*) = \mathrm{Ext}^i_A(M, A) = \begin{cases} M^\dagger = \mathrm{Ext}^g_A(M, A), & \text{if } i = g,\\ 0, & \text{if } i \neq g.\end{cases}$$

We also have

$$H^i(\mathbf{Q}^*) = \operatorname{Ext}^i_A(\mathbf{P}_0/\mathfrak{q}\mathbf{P}_0, A) = 0 \quad \text{if } i \neq g.$$

It is now immediate that

$$H^i(\mathbf{C}) = 0 \quad \text{for } i \neq g-1, g.$$

Set $B = A/\mathfrak{q}$ and $\overline{P} = \mathbf{P}_0/\mathfrak{q}\mathbf{P}_0$. Then by 3.2 we have

$$\operatorname{Ext}^g_A(M, A) \cong \operatorname{Hom}_B(M, B), \quad \text{and} \quad \operatorname{Ext}^g_A(\overline{P}, A) \cong \operatorname{Hom}_B(\overline{P}, B).$$

We have an exact sequence of MCM B-modules

$$0 \longrightarrow K \longrightarrow \overline{P} \xrightarrow{\epsilon} M \longrightarrow 0.$$

Dualizing with respect to B we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_B(M, B) \xrightarrow{\epsilon^*} \operatorname{Hom}_B(\overline{P}, B) \longrightarrow K^* \longrightarrow 0,$$

since $\operatorname{Ext}^1_B(M, B) = 0$ as M is an MCM B-module. As $M \sim_{\mathfrak{q}} N$ we get that $N \cong K^*$. We note that the map $H^g(\mathbf{P}^*) \to H^g(\mathbf{Q}^*)$ is ϵ^*. As a consequence we obtain that

$$H^{g-1}(\mathbf{C}) = 0 \quad \text{and} \quad H^g(\mathbf{C}) = N.$$

The result follows.

REMARK 3.5.

(1) If A is regular local, $M = A/I$, $\mathfrak{q}$ is a complete intersection and $I \sim_{\mathfrak{q}} J$, then this construction was used by Ferrand to give a projective resolution of A/J, see [13, Proposition 2.6].

(2) If M is perfect A-module of codimension g and $\mathfrak{q}$ is a Gorenstein ideal then this construction was used by Martsinkovsky and Strooker to give a projective resolution of N, see [11, Proposition 10, p. 597].

Our interest in this construction is due to the following:

OBSERVATION 3.6. Let $B^g(\mathbf{C})$ be the module of g-boundaries of $\mathbf{C}$ and $Z^g(\mathbf{C})$ be the module of g-cocycles of $\mathbf{C}$. Then $\operatorname{projdim}_A B^g(\mathbf{C})$ is finite and $Z^g(\mathbf{C})$ is a maximal Cohen-Macaulay A-module. Thus the sequence

$$0 \longrightarrow B^g(\mathbf{C}) \longrightarrow Z^g(\mathbf{C}) \longrightarrow N \longrightarrow 0,$$

is a maximal Cohen-Macaulay approximation of N.

PROOF. This follows from Lemma 3.4.

4. Proof of Theorem 1.2(1)

In this section we prove Theorem 1.2(1). It is an easy consequence of the following result:

THEOREM 4.1. *Let* $(A, \mathfrak{m})$ *be a Gorenstein local ring of dimension* d. *Let* $M \in \mathrm{CM}^g(A)$. *Assume* $M \sim_{\mathfrak{q}} N$ *where* $\mathfrak{q}$ *is a Gorenstien ideal in A. Let L be a maximal Cohen-Macaulay A-module. Let* $\mathbf{P}$ *be minimal free resolution of N and let* $\mathbf{Q}$ *be a minimal free resolution of* $\mathbf{P}_0/\mathfrak{q}\mathbf{P}_0$. *We do the construction as in* 3.3 *and let* $\mathbf{C} = \mathrm{cone}(\phi^*)$. *Set* $X = Z^g(\mathbf{C})$ *and* $Y = B^g(\mathbf{C})$. *For* $s \geq 1$, *let* X_s *be the image of the map* $\mathbf{P}^*_{s-1} \to \mathbf{P}^*_s$. *Let* $\mathbf{x} = x_1, \ldots, x_d$ *be a maximal regular A-sequence. Set* $\overline{A} = A/(\mathbf{x})$ *and* $\overline{L} = L/\mathbf{x}L$. *Then the following assertions are equivalent:*

(i) $\mathrm{Ext}^i_A(L, M) = 0$ *for all* $i \gg 0$.
(ii) $\mathrm{Ext}^i_A(L, X) = 0$ *for all* $i \gg 0$.
(iii) *For all* $s \gg 0$ *and for all* $i \geq 1$ *we have* $\mathrm{Ext}^i_A(L, X_s) = 0$.
(iv) $H^i(\mathrm{Hom}_A(L, \mathbf{P}^*)) = 0$ *for all* $i \gg 0$.
(v) $H^i(\mathrm{Hom}_A(L \otimes_A \mathbf{P}, A)) = 0$ *for all* $i \gg 0$.
(vi) $H^i(\mathrm{Hom}_{\overline{A}}(\overline{L} \otimes_A \mathbf{P}, \overline{A})) = 0$ *for all* $i \gg 0$.
(vii) $\mathrm{Tor}^A_i(\overline{L}, N) = 0$ *for all* $i \gg 0$.
(viii) $\mathrm{Tor}^A_i(L, N) = 0$ *for all* $i \gg 0$.

We need a few preliminary results before we are able to prove Theorem 4.1.

4.2. Let $\mathbf{K}\colon \cdots \to K_n \xrightarrow{d_n} K_{n+1} \to \cdots$ be a co-chain complex. Let s be an integer. By $\mathbf{K}_{\geq s}$ we mean the co-chain complex

$$0 \longrightarrow K_s \xrightarrow{d_s} K_{s+1} \longrightarrow \cdots \longrightarrow K_n \xrightarrow{d_n} K_{n+1} \longrightarrow \cdots$$

Clearly $H^i(\mathbf{K}_{\geq s}) = H^i(\mathbf{K})$ for all $i \geq s+1$.

4.3. Let $\mathbf{P}, \mathbf{Q}, \mathbf{C}$ be as in Theorem 4.1. As $\mathfrak{q}$ is a Gorenstein ideal, it has in particular finite projective dimension. It follows that for $i \geq d+1$, we get $\mathbf{C}_i = \mathbf{P}^*_{i+1}$ and the map $\mathbf{C}_i \to \mathbf{C}_{i+1}$ is same as the map $\mathbf{P}^*_{i+1} \to \mathbf{P}^*_{i+2}$. In particular if $s \geq d+1$ then $\mathbf{C}_{\geq s} = \mathbf{P}^*_{\geq s+1}$.

We need the following:

LEMMA 4.4. *Let* $(A, \mathfrak{m})$ *be a Gorenstein local ring and let* $\mathbf{D}$ *be a chain complex with* $\mathbf{D}_n = 0$ *for* $n \leq -1$. *Assume that* $\mathbf{D}_n$ *is a maximal Cohen-Macaulay A-module for all* $n \geq 0$. *Let* $x \in \mathfrak{m}$ *be A-regular. Let* $\overline{A} = A/(x)$ *and let* $\overline{\mathbf{D}}$ *be the complex* $\mathbf{D} \otimes_A \overline{A}$. *Let* $\mathbf{D}^*$ *be the complex* $\mathrm{Hom}_A(\mathbf{D}, A)$ *and let* $\overline{\mathbf{D}}^*$ *be the complex* $\mathrm{Hom}_{\overline{A}}(\overline{\mathbf{D}}, \overline{A})$. *Then the following are equivalent:*

(i) $H^i(\mathbf{D}^*) = 0$ *for* $i \gg 0$.
(ii) $H^i(\overline{\mathbf{D}}^*) = 0$ *for* $i \gg 0$.

PROOF. Let E be a maximal Cohen-Macaulay A-module. Notice x is E-regular. Furthermore $\operatorname{Hom}_A(E, A)$ is a maximal Cohen-Macaulay A-module and $\operatorname{Ext}^i_A(M, A) = 0$ for $i \geq 1$. Set $\overline{E} = E/xE$.

The exact sequence $0 \to A \xrightarrow{x} A \to \overline{A} \to 0$ induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(E, A) \xrightarrow{x} \operatorname{Hom}_A(E, A) \longrightarrow \operatorname{Hom}_{\overline{A}}(\overline{E}, \overline{A}) \longrightarrow 0.$$

Thus we have an exact sequence of co-chain complexes of A-modules

$$0 \longrightarrow \mathbf{D}^* \xrightarrow{x} \mathbf{D}^* \longrightarrow \overline{\mathbf{D}}^* \longrightarrow 0.$$

This in turn induces a long-exact sequence in cohomology

$$\cdots \longrightarrow H^i(\mathbf{D}^*) \xrightarrow{x} H^i(\mathbf{D}^*) \longrightarrow H^i(\overline{\mathbf{D}}^*) \longrightarrow H^{i+1}(\mathbf{D}^*) \longrightarrow \cdots \quad (4.4.1)$$

We now prove (i) $\Rightarrow$ (ii). This follows from (4.4.1).

(ii) $\Rightarrow$ (i). From (4.4.1) we get that for all $i \gg 0$ the map $H^i(\mathbf{D}^*) \xrightarrow{x} H^i(\mathbf{D}^*)$ is surjective. The result now follows from Nakayama's Lemma. So our result follows.

As an easy consequence of 4.4 we get the following:

COROLLARY 4.5 (with same hypotheses as in Lemma 4.4). *Let* $\mathbf{x} = x_1, \ldots, x_d$ *be a maximal regular sequence in* A. *Set* $B = A/(\mathbf{x})$, $\mathbf{K} = \mathbf{D} \otimes_A B$, *Let* $\mathbf{K}^*$ *be the complex* $\operatorname{Hom}_B(\mathbf{K}, B)$. *Then the following are equivalent:*

(i) $H^i(\mathbf{D}^*) = 0$ *for* $i \gg 0$.
(ii) $H^i(\mathbf{K}^*) = 0$ *for* $i \gg 0$.

We now give

PROOF OF THEOREM 4.1. (i) $\Leftrightarrow$ (ii). By 3.6 we get that X is a maximal Cohen-Macaulay A-module, $\operatorname{projdim}_A Y$ is finite and we have an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0.$$

As A is Gorenstein we have that $\operatorname{injdim}_A Y$ is finite. It follows that $\operatorname{Ext}^i_A(L, X) \cong \operatorname{Ext}^i_A(L, M)$ for all $i \geq d + 1$. The result follows.

(ii) $\Leftrightarrow$ (iii). For $s \geq 1$ let $\widetilde{X_s} = \operatorname{image}(\mathbf{C}_s \to \mathbf{C}_{s+1})$. For $s \geq g + 1$ by Lemma 3.4 we get that $\widetilde{X_s}$ is maximal Cohen-Macaulay. By Lemma 3.4 we also have an exact sequence

$$0 \longrightarrow X \longrightarrow \mathbf{C}_g \longrightarrow \mathbf{C}_{g+1} \longrightarrow \cdots \longrightarrow \mathbf{C}_s \longrightarrow \widetilde{X_s} \longrightarrow 0.$$

It follows that if $s \gg 0$ then $\mathrm{Ext}^i_A(L, X) = 0$ for $i \gg 0$ is equivalent to $\mathrm{Ext}^i_A(L, \widetilde{X_s}) = 0$ for $i \geq 1$. The result now follows as $\mathfrak{q}$ is a Gorenstein ideal; so we have $\widetilde{X_s} = X_{s+1}$ for $s \geq d+1$, see 4.3.

(iii) $\Rightarrow$ (iv). Suppose $\mathrm{Ext}^i_A(L, X_s) = 0$ for all $i \geq 1$ and all $s \geq c$. Set $a = \max\{g+1, c\}$. As $H^i(\mathbf{P}^*) = \mathrm{Ext}_i^A(N, A) = 0$ for $i \geq g+1$ we have an exact sequence

$$0 \longrightarrow X_a \longrightarrow \mathbf{P}^*_{a+1} \longrightarrow \mathbf{P}^*_{a+2} \longrightarrow \cdots \longrightarrow \mathbf{P}^*_n \longrightarrow \mathbf{P}^*_{n+1} \longrightarrow \cdots$$

As $\mathrm{Ext}^i_A(L, X_s) = 0$ for all $i \geq 1$ and all $s \geq a$ we get that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(L, X_a) \longrightarrow \mathrm{Hom}_A(L, \mathbf{P}^*_{a+1}) \longrightarrow \cdots$$

is exact. It follows that $H^i(\mathrm{Hom}_A(L, \mathbf{P}^*_{\geq a+1})) = 0$ for $i \geq a+2$. Thus by 4.2 we get that $H^i(\mathrm{Hom}_A(L, \mathbf{P}^*)) = 0$ for $i \geq a+2$.

(iv) $\Rightarrow$ (iii). Suppose $H^i(\mathrm{Hom}_A(L, \mathbf{P}^*)) = 0$ for $i \geq r$. If $s \geq r$ then $H^i(\mathrm{Hom}_A(L, \mathbf{P}^*_{\geq s})) = 0$ for all $i \geq s+1$. Now let $a = \max\{g+1, r\}$. Let $s \geq a$. As argued before we have an exact sequence

$$0 \longrightarrow X_s \longrightarrow \mathbf{P}^*_{s+1} \longrightarrow \mathbf{P}^*_{s+2} \longrightarrow \cdots \longrightarrow \mathbf{P}^*_n \longrightarrow \mathbf{P}^*_{n+1} \longrightarrow \cdots$$

As $H^i(\mathrm{Hom}_A(L, \mathbf{P}^*_{\geq s})) = 0$ for all $i \geq s+1$ we get that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(L, X_s) \longrightarrow \mathrm{Hom}_A(L, \mathbf{P}^*_{s+1}) \longrightarrow \cdots \quad \text{is exact.}$$

In particular $\mathrm{Ext}^A_1(L, X_s) = 0$. Notice $\mathrm{Ext}^2_A(L, X_s) \cong \mathrm{Ext}^1_A(L, X_{s+1})$. The latter module is zero by the same argument. Iterating we get $\mathrm{Ext}^i_A(L, X_s) = 0$ for all $i \geq 1$.

(iv) $\Leftrightarrow$ (v). We have an isomorphism of complexes

$$\mathrm{Hom}_A(L, \mathbf{P}^*) \cong \mathrm{Hom}_A(L \otimes_A \mathbf{P}, A).$$

The result follows.

(v) $\Leftrightarrow$ (vi). $\mathbf{D} = L \otimes_A \mathbf{P}$ is a complex of maximal Cohen-Macaulay A-modules since L is maximal Cohen-Macaulay and $\mathbf{P}_n$ is a free A-module. Also $\mathbf{D}_n = 0$ for $n \leq -1$. Set $\mathbf{K} = \mathbf{D} \otimes_A \overline{A} = \overline{L} \otimes_A \mathbf{P}$. The result now follows from Corollary 4.5.

(vi) $\Leftrightarrow$ (vii). We note that $\overline{A}$ is a zero-dimensional Gorenstein local ring and so is injective as an $\overline{A}$-module. Furthermore $\overline{A}$ is the injective hull of its residue field. It follows that

$$H^i(\mathrm{Hom}_{\overline{A}}(\overline{L} \otimes_A \mathbf{P}, \overline{A})) \cong \mathrm{Hom}_{\overline{A}}(H_i(\overline{L} \otimes_A \mathbf{P}), \overline{A}) = \mathrm{Hom}_{\overline{A}}(\mathrm{Tor}^A_i(\overline{L}, N), \overline{A}).$$

Therefore by [5, 3.2.12] we get the result.

(vii) $\Leftrightarrow$ (viii). As L is a maximal Cohen-Macaulay A-module we get that $\mathbf{x}$ is an L-regular sequence. Set $L_j = L/(x_1, \dots, x_j)$ for $j = 1, \dots, d$. Note $L_d = \overline{L}$.

We have an exact sequence $0 \to L \xrightarrow{x_1} L \to L_1 \to 0$. This induces a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_i^A(L, N) \xrightarrow{x_1} \operatorname{Tor}_i^A(L, N) \longrightarrow \operatorname{Tor}_i^A(L_1, N) \longrightarrow \operatorname{Tor}_{i-1}^A(L, N) \longrightarrow \cdots$$

Clearly if $\operatorname{Tor}_i^A(L, N) = 0$ for $i \gg 0$ then $\operatorname{Tor}_i^A(L_1, N) = 0$ for all $i \gg 0$. Conversely if $\operatorname{Tor}_i^A(L_1, N) = 0$ for all $i \gg 0$ then the map $\operatorname{Tor}_i^A(L, N) \xrightarrow{x_1} \operatorname{Tor}_i^A(L, N)$ is surjective for all $i \gg 0$. By Nakayama's Lemma we get $\operatorname{Tor}_i^A(L, N) = 0$ for $i \gg 0$.

We also have an exact sequence $0 \to L_1 \xrightarrow{x_2} L_1 \to L_2 \to 0$. A similar argument gives that $\operatorname{Tor}_i^A(L_1, N) = 0$ for $i \gg 0$ if and only if $\operatorname{Tor}_i^A(L_2, N) = 0$ for all $i \gg 0$. Combining with the previous result we get that $\operatorname{Tor}_i^A(L, N) = 0$ for $i \gg 0$ if and only if $\operatorname{Tor}_i^A(L_2, N) = 0$ for all $i \gg 0$.

Iterating this argument yields the result.

We now give

PROOF OF THEOREM 1.2(1). Let $0 \to W \to E \to L \to 0$ be a maximal Cohen-Macaulay approximation of L. As $\operatorname{projdim}_A W$ is finite we get that $\operatorname{Ext}_A^i(L, M) \cong \operatorname{Ext}_A^i(E, M)$ for $i \geq d+2$ and $\operatorname{Tor}_i^A(L, N) \cong \operatorname{Tor}_i^A(E, N)$ for $i \geq d+2$. The result now follows from Theorem 4.1.

5. Proof of Theorem 1.2(2) and Corollary 1.3

In this section we prove Theorem 1.2(2). We need a few preliminary facts to prove this result.

5.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let E be the injective hull of its residue field. If G is an A-module then set $G^\vee = \operatorname{Hom}_A(G, E)$. Let $\ell(G)$ denote the length of G. The following result is known. We give a proof as we are unable to find a reference.

LEMMA 5.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and let E be the injective hull of its residue field. Let M be an A-module. Let L be an A-module with $\ell(L) < \infty$. Then for all $i \geq 0$ we have an isomorphism*

$$\operatorname{Ext}_A^i(M, L) \cong (\operatorname{Tor}_i^A(M, L^\vee))^\vee.$$

PROOF. We note that $(L^\vee)^\vee \cong L$, see [5, 3.2.12]. Let $\mathbf{P}$ be a minimal projective resolution of M. We have the following isomorphism of complexes:

$$\operatorname{Hom}_A(\mathbf{P} \otimes_A L^\vee, E) \cong \operatorname{Hom}_A(\mathbf{P}, \operatorname{Hom}_A(L^\vee, E)) \cong \operatorname{Hom}_A(\mathbf{P}, L).$$

We have $H^i(\operatorname{Hom}_A(\mathbf{P}, L)) = \operatorname{Ext}_i^A(M, L)$. Notice as E is an injective A-module we have

$$\begin{aligned} H^i(\operatorname{Hom}_A(\mathbf{P} \otimes_A L^\vee, E)) &\cong \operatorname{Hom}_A(H_i(\mathbf{P} \otimes_A L^\vee), E) \\ &\cong \operatorname{Hom}_A(\operatorname{Tor}_i^A(M, L^\vee), E) \\ &= (\operatorname{Tor}_i^A(M, L^\vee))^\vee. \end{aligned}$$

This finishes the proof.

The following result is also known. We give a proof as we are unable to find a reference. If A is Gorenstein and U is a Cohen-Macaulay A-module then set $U^\dagger = \operatorname{Ext}_A^{\operatorname{codim} U}(U, A)$.

LEMMA 5.3. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let E be the injective hull of its residue field. Let $D \in \mathrm{CM}^r(A)$ and let $\mathbf{x} = x_1, \dots, x_c \in \mathfrak{m}$ be a D-regular sequence (note $c \leq d - r$). Then*

(1) *$\mathbf{x}$ is a $D^\dagger$-regular sequence.*

(2) $(D/\mathbf{x}D)^\dagger \cong D^\dagger/\mathbf{x}D^\dagger$.

(3) *If $r = d$ (and so $\ell(D) < \infty$) then*

$$D^\dagger \cong D^\vee (= \operatorname{Hom}_A(D, E)).$$

(4) *If T is another A-module then*

(a) $\operatorname{Ext}_A^i(T, D) = 0$ *for* $i \gg 0$ *if and only if* $\operatorname{Ext}_A^i(T, D/\mathbf{x}D) = 0$ *for* $i \gg 0$.

(b) $\operatorname{Ext}_A^i(D, T) = 0$ *for* $i \gg 0$ *if and only if* $\operatorname{Ext}_A^i(D/\mathbf{x}D, T) = 0$ *for* $i \gg 0$.

PROOF. (1) and (2). Let x be D-regular. Then notice D/xD is a Cohen-Macaulay A-module of codimension $r + 1$. We have a short-exact sequence $0 \to D \xrightarrow{x} D \to D/xD \to 0$. Applying the functor $\operatorname{Hom}_A(-, A)$ yields a long exact sequence which after applying 3.1 reduces to a short-exact sequence

$$0 \longrightarrow \operatorname{Ext}_A^r(D, A) \xrightarrow{x} \operatorname{Ext}_A^r(D, A) \longrightarrow \operatorname{Ext}_A^{r+1}(D/xD, A) \longrightarrow 0.$$

It follows that x is $D^\dagger$-regular and $D^\dagger/xD^\dagger \cong (D/xD)^\dagger$.

Iterating this argument yields (1) and (2).

(3). Let $\mathbf{y} = y_1, \dots, y_d \subset \operatorname{ann}_A D$ be a maximal A-regular sequence. Set $B = A/(\mathbf{y})$. Then

$$D^\dagger = \operatorname{Ext}^n_A(D, A) \cong \operatorname{Hom}_B(D, B).$$

We have

$$D^\vee = \operatorname{Hom}_A(D, E) = \operatorname{Hom}_A(D \otimes_A B, E) \cong \operatorname{Hom}_B(D, \operatorname{Hom}_A(B, E)).$$

Now $\operatorname{Hom}_A(B, E)$ is an injective B-module of finite length. As B is an Artin Gorenstein local ring we get that $\operatorname{Hom}_A(B, E)$ is free as a B-module. As $\ell(\operatorname{Hom}_A(B, E)) = \ell(B)$ (see [5, 3.2.12]) it follows that $\operatorname{Hom}_A(B, E) = B$. The result follows.

(4). It suffices to prove for $c = 1$. Set $D_1 = D/x_1 D$.

4(a) (sketch). The exact sequence $0 \to D \xrightarrow{x_1} D \to D_1 \to 0$ induces a long exact sequence

$$\begin{aligned} \cdots \operatorname{Ext}^i_A(T, D) &\xrightarrow{x_1} \operatorname{Ext}^i_A(T, D) \\ &\longrightarrow \operatorname{Ext}^i_A(T, D_1) \longrightarrow \operatorname{Ext}^{i+1}_A(T, D) \longrightarrow \cdots \end{aligned}$$

The proof now follows from an argument similar to Lemma 4.4

4(b). This is similar to 4(a).

Thus our result follows.

We now give

Proof of Theorem 1.2(2). Let $\mathbf{x} = x_1, \dots, x_{d-r}$ be a maximal D-regular sequence. Then by 5.3, $\mathbf{x}$ is also a $D^\dagger$ regular sequence. Also by 5.3, we get $D^\dagger/\mathbf{x}D^\dagger = (D/\mathbf{x}D)^\dagger$. Let E be the injective hull of the residue field of A.

We have

$$\begin{aligned} &\operatorname{Ext}^i_A(M, D) = 0 \text{ for all } i \gg 0 \\ &\iff \operatorname{Ext}^i_A(M, D/\mathbf{x}D) = 0 \text{ for all } i \gg 0; \text{ see 5.3}, \\ &\iff \operatorname{Tor}^A_i(M, (D/\mathbf{x}D)^\vee) = 0 \text{ for all } i \gg 0; \text{ see 5.2 and [5, 3.2.12]}, \\ &\iff \operatorname{Tor}^A_i(M, D^\dagger/\mathbf{x}D^\dagger) = 0 \text{ for all } i \gg 0; \text{ see 5.3}, \\ &\iff \operatorname{Ext}^A_i(D^\dagger/\mathbf{x}D^\dagger, N) = 0 \text{ for all } i \gg 0; \text{ see Theorem 1.2(1)}, \\ &\iff \operatorname{Ext}^A_i(D^\dagger, N) = 0 \text{ for all } i \gg 0; \text{ see 5.3}. \end{aligned}$$

The proof is now complete.

We now give

PROOF OF COROLLARY 1.3. By Theorem 1.2(2) we get that

$$\operatorname{Ext}^i_A(M, C) = 0 \text{ for all } i \gg 0 \iff \operatorname{Ext}^i_A(C^\dagger, N) = 0 \text{ for all } i \gg 0.$$

We note that $C^\dagger = \operatorname{Ext}^r_A(C, A) \cong \operatorname{Hom}_{A/\mathfrak{n}}(C, A/\mathfrak{n})$, see 3.2. As $C \sim_{\mathfrak{n}} D$ we have an exact sequence $0 \to C^\dagger \to G \to D \to 0$, where G is a free $A/\mathfrak{n}$-module. As $\mathfrak{n}$ is a Gorenstein ideal in A we get $\operatorname{projdim}_A G$ is finite. The result follows.

6. Some preliminaries to prove Theorem 1.1

In this section we discuss a few preliminaries which will enable us to prove Theorem 1.1.

6.1. Throughout $(Q, \mathfrak{n})$ is a local Noetherian ring. Let $\mathbf{f} = f_1, \dots, f_c$ be a regular sequence. We assume $\mathbf{f} \subseteq \mathfrak{n}^2$. Set $I = (\mathbf{f})$ and $A = Q/I$. Recall that all modules considered will be finitely generated.

6.2. We need to recall the notion of complexity of a module. This notion was introduced by Avramov in [2]. Let $\beta_i^A(M) = \ell(\operatorname{Tor}_i^A(M, k))$ be the i^{th} Betti number of M over A. The complexity of M over A is defined by

$$\operatorname{cx}_A M = \inf\left\{ b \in \mathbb{N} \;\middle|\; \varlimsup_{n\to\infty} \frac{\beta_n^A(M)}{n^{b-1}} < \infty \right\}.$$

If A is a local complete intersection of codim c then $\operatorname{cx}_A M \le c$. Furthermore all values between 0 and c occur.

6.3. Let $(A, \mathfrak{m})$ be a complete intersection of codimension c and dimension d. Let E be an A-module. Then

(1) $\operatorname{cx}_A E = 0$ if and only if $\operatorname{projdim}_A E$ is finite.

(2) $\operatorname{cx}_A E = 1$ if and only if the sequence of Betti numbers of E is bounded and E has infinite projective dimension. In this case, by a result of Eisenbud in fact $\operatorname{Syz}^A_{d+1}(E)$ is a periodic module with period ≤ 2, see [7, 4.1].

6.4. Suppose $A = B/(f)$ where $(B, \mathfrak{n})$ is local complete intersection and $f \in \mathfrak{m}^2$ is a regular element. Suppose E is an A-module with an exact sequence of B-modules $0 \to F \to G \to E \to 0$; where F, G are free B-modules. By Auslander-Buchsbaum formula we get that E is a maximal Cohen-Macaulay A-module. We note that the Betti numbers of E as an A-module are bounded. To see this just tensor the above exact sequence with A to obtain an exact sequence

$$0 \longrightarrow E \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow E \longrightarrow 0 \qquad (*)$$

where $\overline{F}, \overline{G}$ are free A-modules (use $\operatorname{Tor}_i^B(A, E) = E$ for $i = 0, 1$). Splicing the exact sequence $(*)$ we get a (not-necessarily minimal) free resolution of E

$$\cdots \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow E \longrightarrow 0.$$

Thus the Betti numbers of E are bounded. Also note that as E is a maximal Cohen-Macaulay A-module we get that $\operatorname{projdim}_A E$ is finite if and only if E is free (by Auslander-Buchsbaum formula).

6.5. Let x be $M \oplus A$-regular. Then $\operatorname{cx}_A M/xM = \operatorname{cx}_A M$; see [3, 4.2.4].

6.6. Let Q is regular local with infinite residue field and let M be an A-module with $\operatorname{cx}_A(M) = r$. The surjection $Q \to A$ factors as $Q \to R \to A$, with the kernels of both maps generated by regular sequences, $\operatorname{projdim}_R M < \infty$ and $\operatorname{cx}_A M = \operatorname{projdim}_R A$ (see [2, 3.9]).

We need the following:

Proposition 6.7. *Let $Q = k[x, y, z]_{(x,y,z)}$ where k is an infinite field. Let $\mathfrak{m}$ be the maximal ideal of Q. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary Gorenstein ideal such that $\mathfrak{q}$ is not a complete intersection. Suppose $\mathfrak{q} \supseteq (u, v)$ where $u, v \in \mathfrak{m}^2$ is an Q-regular sequence. Set $A = Q/(u, v)$ and $\overline{\mathfrak{q}} = \mathfrak{q}/(u, v)$. Then $\operatorname{cx}_A A/\overline{\mathfrak{q}} = 2$.*

Proof. As $\operatorname{codim} A = 2$ we have that $\operatorname{cx}_A A/\overline{\mathfrak{q}} = 0, 1$ or 2. We prove $\operatorname{cx} A/\overline{\mathfrak{q}} \neq 0, 1$.

If $\operatorname{cx}_A A/\overline{\mathfrak{q}} = 0$ then $A/\overline{\mathfrak{q}}$ has finite projective dimension over A. So by Auslander-Buchsbaum formula $\operatorname{projdim}_A A/\overline{\mathfrak{q}} = 1$. Therefore $\overline{\mathfrak{q}}$ is a principal ideal. It follows that $\mathfrak{q}$ is a complete intersection, a contradiction.

If $\operatorname{cx}_A A/\overline{\mathfrak{q}} = 1$ then by 6.6, the surjection $Q \to A$ factors as $Q \to R \to A$, with the kernels of both maps generated by regular sequences, $\operatorname{projdim}_R A/\overline{\mathfrak{q}} < \infty$ and $\operatorname{cx}_A A/\overline{\mathfrak{q}} = \operatorname{projdim}_R A = 1$. Note R is Cohen-Macaulay. Also by Auslander-Buchsbaum formula we get $\operatorname{depth} R = 2$. Thus $\dim R = 2$. So $R = Q/(h)$ for some h.

As $A/\overline{\mathfrak{q}}$ has finite length, by Auslander-Buchsbaum formula $\operatorname{projdim}_R A/\overline{\mathfrak{q}} = 2$. Consider the minimal resolution of $A/\overline{\mathfrak{q}}$ over R:

$$0 \longrightarrow R^b \longrightarrow R^a \longrightarrow R \longrightarrow A/\overline{\mathfrak{q}} \longrightarrow 0.$$

As $A/\overline{\mathfrak{q}}$ is a Gorenstein ring we have $b = 1$ and so $a = 2$. Thus there exists $\alpha, \beta \in R$ with $A/\overline{\mathfrak{q}} = R/(\alpha, \beta) = Q/(h, \alpha, \beta)$. It follows that $\mathfrak{q}$ is a complete intersection ideal, a contradiction. This finishes the proof.

We will also need the next result to prove Theorem 1.1.

Lemma 6.8. *Let $(Q, \mathfrak{n})$ be a regular local ring and let $\mathbf{f} = f_1, \ldots, f_c \in \mathfrak{n}^2$ be a regular sequence. Set $A = Q/(\mathbf{f})$. Let $M \in \operatorname{CM}^g(A)$. Also let $M \sim_{\mathfrak{q}} N$ where $\mathfrak{q}$ is a quasi-Gorenstein ideal in A. Assume $\operatorname{cx}_A M = 1$. Then*

(1) *If* $\mathfrak{q}$ *is a Gorenstein ideal then* $\operatorname{cx}_A N = 1$.

(2) *If* $\operatorname{projdim} A/\mathfrak{q} = \infty$ *and* $\operatorname{cx}_A A/\mathfrak{q} \geq 2$ *then* $\operatorname{cx}_A N \geq 2$.

PROOF. Set $B = A/\mathfrak{q}$. Let

$$M^\dagger = \operatorname{Ext}^g_A(M, A) \cong \operatorname{Hom}_B(M, B),$$

by Lemma 3.2. As $M \sim_{\mathfrak{q}} N$ we have an exact sequence

$$0 \longrightarrow M^\dagger \longrightarrow G \longrightarrow N \longrightarrow 0; \quad \text{where } G \text{ is free } B\text{-module.} \tag{6.8.1}$$

By [4, 3.3] we get $\operatorname{cx}_A M^\dagger = \operatorname{cx}_A M = 1$.

(1) If $\mathfrak{q}$ is a Gorenstein ideal then $\operatorname{projdim}_A G$ is finite. So the Betti numbers of $M^\dagger$ and N are asymptotically identical. It follows that $\operatorname{cx}_A N = 1$.

(2) If $\operatorname{cx}_A N \leq 1$ then N has bounded Betti numbers. As $\operatorname{cx}_A M^\dagger = 1$ we get that $M^\dagger$ has bounded betti numbers. From 6.8.1 we get that $A/\mathfrak{q}$ has bounded Betti numbers which implies that $\operatorname{cx}_A A/\mathfrak{q} \leq 1$. This is a contradiction. Therefore $\operatorname{cx}_A N \geq 2$. The proof is now complete.

7. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first make the following:

CONSTRUCTION 7.1. Let $Q = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and let $\mathfrak{m}$ be its maximal ideal. We construct a $\mathfrak{m}$-primary Gorenstein ideal $\mathfrak{q}$ in Q such that

(1) $\mathfrak{q}$ is not a complete intersection,

(2) $(x^8, y^8, u) \subseteq \mathfrak{q} \subseteq (u, x^2, y^2)$, where $u = x^2 + y^2 + z^2$.

Note x^2, y^2, u is an Q-regular sequence. It follows that x^8, y^8, u is also an Q-regular sequence, see [12, 16.1].

Let $R = Q/(u)$, $\mathfrak{n} = \mathfrak{m}/(u)$. Then note (x, y) is a reduction of $\mathfrak{n}$ and $\mathfrak{n}^2 = (x, y)\mathfrak{n}$. It follows that x^7, y^7 is a regular sequence in R. So $I = (x^7, y^7) \colon (x + y + z)$ is a Gorenstein ideal in R (see [10, 8.3]). Using Singular, [6], it can be shown that I has 4 minimal generators and $I \subseteq \mathfrak{n}^7$. By Singular we get $I = (g_1, g_2, g_3, g_4)$ where

$$g_1 = x^7, \quad g_2 = y^7,$$
$$g_3 = zy^5x - y^6x - zy^4x^2 + y^4x^3 + zy^2x^4 - y^3x^4 - zyx^5 + yx^6,$$

and

$$g_4 = zy^6 - y^6x - zy^4x^2 + y^5x^2 + zy^3x^3 - y^3x^4 - zyx^5 + y^2x^5 + zx^6.$$

In particular I is not a complete intersection in R. Also note that $\mathfrak{n}^7 \subseteq (x, y)^6 \subseteq (x^2, y^2)$. Let $\mathfrak{q}$ be an ideal in Q containing u such that $\mathfrak{q}/(u) = I$. Then by

Singular $\mathfrak{q}$ has 5 minimal generators namely g_1, g_2, g_3, g_4, u. So $\mathfrak{q}$ is not a complete intersection. Also clearly $\mathfrak{q}$ is $\mathfrak{m}$-primary. One can also check using Singular that the minimal resolution of $\mathfrak{q}$ is

$$0 \longrightarrow Q \longrightarrow Q^5 \longrightarrow Q^5 \longrightarrow Q \longrightarrow 0.$$

So again it follows that $\mathfrak{q}$ is a Gorenstein ideal which is not a complete intersection, see [10, 3.2]. Note

$$(x^8, y^8, u) \subseteq (x^7, y^7, u) \subseteq \mathfrak{q} \subseteq (x^2, y^2, u).$$

We now give

Proof of Theorem 1.1. Let $Q = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and let $\mathfrak{m}$ be its maximal ideal. We make the construction as in 7.1. Let $E = Q/(x)$ considered as a $Q/(x^2)$-module. Note E is a non-free stable maximal Cohen-Macaulay $Q/(x^2)$-module. Let

$$0 \longrightarrow Q \longrightarrow Q \longrightarrow E \longrightarrow 0$$

be a minimal resolution of E as a Q-module. Note $u = x^2 + y^2 + z^2$ is $Q/(x^2)$-regular and so E-regular. Thus we have an exact sequence

$$0 \longrightarrow Q/(u) \longrightarrow Q/(u) \longrightarrow E/uE \longrightarrow 0. \qquad (**)$$

Set $A = Q/(u, x^8)$ and $\overline{\mathfrak{q}} = \mathfrak{q}/(u, x^8)$. Then note that $\overline{\mathfrak{q}}$ is a quasi-Gorenstein ideal in A which is not a complete intersection. By 6.7 we get $\operatorname{cx}_A A/\overline{\mathfrak{q}} = 2$. Furthermore notice that $E/uE = Q/(u, x)$ is a maximal Cohen-Macaulay $Q/(x^2, u)$-module and so a maximal Cohen-Macaulay A-module. Notice that E/uE is not a free A-module. By the exact sequence $(**)$ it follows that $\operatorname{cx}_A E/uE = 1$; see 6.4.

We now note that y^2 is A-regular and so E/uE-regular. Set

$$M = E/(u, y^2)E = Q/(x, u, y^2).$$

Thus M is an A-module of finite length. Also $\operatorname{cx}_A M = 1$ by 6.5. Furthermore

$$\mathfrak{q} \subseteq (x^2, y^2, u) \subseteq \operatorname{ann}_Q M.$$

So we have $\overline{\mathfrak{q}} \subseteq \operatorname{ann}_A M$. Thus we have finite length module M of complexity one and a quasi-Gorenstein ideal $\overline{\mathfrak{q}}$ of complexity two as A-modules. Set $B = A/\overline{\mathfrak{q}}$. Clearly M does not have B as a direct summand. Set $N = \Omega_B(\operatorname{Tr}_B(M))$. Then M is horizontally linked to N as B-modules, see [11, Proposition 8, p. 596]. So $M \sim_{\overline{\mathfrak{q}}} N$ as A-modules. Note $y^2 \in \operatorname{ann}_A M$ is an A-regular element.

Furthermore $M = Q/(x, u, y^2)$ does not have $C = A/(y^2) = Q/(u, x^8, y^2)$ as a direct-summand. Set $L = \Omega_C(\operatorname{Tr}_C(M))$. Then M is horizontally linked to L as C-modules, see [11, Proposition 8, p. 596]. So $M \sim_{(y^2)} L$ as A-modules.

By Lemma 6.8 we have:

$$\operatorname{cx}_A N \geq 2 \quad \text{and} \quad \operatorname{cx}_A L = 1.$$

As $\operatorname{cx}_A N \leq \operatorname{codim} A = 2$; we get $\operatorname{cx}_A N = 2$.

This finishes the proof as for any module P the complexity of a maximal Cohen-Macaulay approximation of P is equal to complexity of P. We note that N is evenly linked to L and $\operatorname{cx}_A X(N) = \operatorname{cx}_A N = 2$ while $\operatorname{cx}_A X(L) = \operatorname{cx}_A L = 1$. It follows that $X(N)$ is not stably isomorphic to $X(L)$. Our result follows.

8. Proof of Theorem 1.4

In this section we prove Theorem 1.4. We need to prove several preliminary results first.

8.1. Let M, N be A-modules. By $\beta(M, N)$ we mean the subset of $\operatorname{Hom}_A(M, N)$ which factor through a free A-module. We first prove the following

Proposition 8.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Let M be a maximal Cohen-Macaulay A-module with no free summands. Then $\beta(M, M) \subseteq$ $\operatorname{rad}\operatorname{End}(M)$.*

Proof. Let $f \in \beta(M, M)$. Say $f = v \circ u$ where $u: M \to F$, $v: F \to M$ and $F = A^n$. Let $u = (u_1, \ldots, u_n)$ where $u_i: M \to A$. As M does not have a free summand we get that $u_i(M) \subseteq \mathfrak{m}$ for each i. Thus $u(M) \subseteq \mathfrak{m}F$. It follows that $f(M) \subseteq \mathfrak{m}M$. We claim that $f \in \operatorname{rad}\operatorname{End}(M)$. Let $g \in \operatorname{End}(M)$. Then notice $g \circ f(M) \subseteq \mathfrak{m}M$. It follows that $1 - g \circ f$ is surjective and so an isomorphism. Thus $f \in \operatorname{rad}\operatorname{End}(M)$.

8.3. It can be easily seen that $\beta(M, M)$ is a two-sided ideal in $\operatorname{End}(M)$. Set $\underline{\operatorname{End}}(M) = \operatorname{End}(M)/\beta(M, M)$. Assume A is Gorenstein and M is maximal Cohen-Macaulay with no free summands. Let

$$0 \longrightarrow N \xrightarrow{u} F \xrightarrow{\pi} M \longrightarrow 0,$$

be a minimal presentation. Note that N is also maximal Cohen-Macaulay with no free-summands. We construct a ring homomorphism

$$\sigma: \underline{\operatorname{End}}(M) \longrightarrow \underline{\operatorname{End}}(N)$$

as follows: Let $\theta \in \operatorname{End}(M)$. Let $\delta \in \operatorname{End}(N)$ be a lift of θ. We first note that if δ' is another lift of θ then it can be easily shown that there exists $\xi\colon F \to N$ such that $\xi \circ u = \delta - \delta'$. Thus we have a well defined element $\sigma(\theta) \in \underline{\operatorname{End}}(N)$. So we have a map

$$\widetilde{\sigma}\colon \operatorname{End}(M) \longrightarrow \underline{\operatorname{End}}(N)$$

It is easy to see that $\widetilde{\sigma}$ is a ring homomorphism. We prove

PROPOSITION 8.4 (with hypotheses as in 8.3). *If $f \in \beta(M, M)$ then $\widetilde{\sigma}(f) = 0$.*

PROOF. Let $\mathbb{F}$ be a minimal resolution of M with $\mathbb{F}_0 = F$. Suppose $f = \psi \circ \phi$ where $\psi\colon G \to M$, $\phi\colon M \to G$ and $G = A^m$ for some m. We may take $\mathbb{G}$ be a minimal resolution of G with $\mathbb{G}_0 = G$ and $\mathbb{G}_n = 0$ for $n > 0$. We can construct a lift $\widetilde{f}\colon \mathbb{F} \to \mathbb{F}$ of f by composing a lift of v with that of u. It follows that $\widetilde{f}_n = 0$ for $n \geq 1$. An easy computation shows that for this lift $\widetilde{f}$ the corresponding map $\delta\colon N \to N$ is infact *zero*. Thus $\widetilde{\sigma}(f) = 0$. Our result follows.

So we have a ring homomorphism $\sigma\colon \underline{\operatorname{End}}(M) \to \underline{\operatorname{End}}(N)$. Our next result is

PROPOSITION 8.5 (with hypotheses as in 8.3). *σ is an isomorphism.*

PROOF. We construct a ring homomorphism $\tau\colon \underline{\operatorname{End}}(N) \to \underline{\operatorname{End}}(M)$ which we show is the inverse of σ.

Let $\theta\colon N \to N$ be A-linear. Then $\theta^*\colon N^* \to N^*$ is also A-linear. We dualize the exact sequence $0 \to N \to F \to M \to 0$ to get an exact sequence

$$0 \longrightarrow M^* \xrightarrow{\pi^*} F^* \xrightarrow{u^*} N^* \longrightarrow 0.$$

We can lift θ^* to an A-linear map $\delta\colon M^* \to M^*$. Also if δ' is another lift then as before it is easy to see $\delta - \delta' \in \beta(M^*, M^*)$. We define $\widetilde{\tau}\colon \operatorname{End}(N) \to \underline{\operatorname{End}}(M)$ by $\widetilde{\tau}(\theta) = \delta^*$. It is clear that $\widetilde{\tau}$ is a ring homomorphism. Finally as in Proposition 8.4, it can be easily proved that if $\theta \in \beta(N, N)$ then $\widetilde{\tau}(\theta) = 0$. Thus we have a ring homomorphism $\tau\colon \underline{\operatorname{End}}(M) \to \underline{\operatorname{End}}(N)$. Finally it is tautological that

$$\tau \circ \sigma = 1_{\underline{\operatorname{End}}(M)} \quad \text{and} \quad \sigma \circ \tau = 1_{\underline{\operatorname{End}}(N)}.$$

This finishes our proof.

8.6. If $\phi\colon R \to S$ is an isomorphism of rings then it is easy to see that $\phi(\operatorname{rad} R) = \operatorname{rad} S$ and thus we have an isomorphism of rings $R/\operatorname{rad} R \cong S/\operatorname{rad} S$. As a corollary we obtain

COROLLARY 8.7 (with hypotheses as in 8.3).

$$\operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M) \cong \operatorname{End}(N)/\operatorname{rad}\operatorname{End}(N).$$

PROOF. By Proposition 8.5 we have an isomorphism $\sigma\colon \underline{\operatorname{End}}(M) \to \underline{\operatorname{End}}(N)$. By 8.2 we have that $\beta(M, M) \subseteq \operatorname{rad}\operatorname{End}(M)$. It follows that

$$\operatorname{rad}\underline{\operatorname{End}}(M) = \operatorname{rad}\operatorname{End}(M)/\beta(M, M).$$

Similarly $\operatorname{rad}\underline{\operatorname{End}}(N) = \operatorname{rad}\operatorname{End}(N)/\beta(N, N)$. The result follows from 8.6.

We now give

PROOF OF THEOREM 1.4. It suffices to consider to prove that if $M_0 \sim_{\mathfrak{q}} M_1$ then

$$\operatorname{End}_A(M_1)/\operatorname{rad}\operatorname{End}_A(M_1) \cong (\operatorname{End}_A(M_0)/\operatorname{rad}\operatorname{End}_A(M_0))^{\mathrm{op}}.$$

By assumption $M_0, M_1 \in \mathrm{CM}^g(A)$. It follows that $\mathfrak{q}$ is a codimension g quasi-Gorenstein ideal [11, Lemma 14, p. 616]. Set $B = A/\mathfrak{q}$. Then B is a Gorenstein ring. Notice M_0, M_1 are maximal Cohen-Macaulay B-modules. Furthermore they are stable B-modules, see [11, Proposition 3, p. 593]. Notice $\operatorname{Hom}_A(M_i, M_i) = \operatorname{Hom}_B(M_i, M_i)$ for $i = 0, 1$.

As M_0 is horizontally linked to M_1 we get that $M_0^* \cong \Omega(M_1)$. By 8.7 we get that

$$\operatorname{End}(M_1)/\operatorname{rad}\operatorname{End}(M_1) \cong \operatorname{End}(M_0^*)/\operatorname{rad}\operatorname{End}(M_0^*).$$

Furthermore it is easy to see that $\operatorname{End}_B(M_0) \cong \operatorname{End}_B(M_0^*)^{\mathrm{op}}$ and this is preserved when we go mod radicals. Thus

$$\operatorname{End}(M_1)/\operatorname{rad}\operatorname{End}(M_1) \cong \big(\operatorname{End}(M_0)/\operatorname{rad}\operatorname{End}(M_0)\big)^{\mathrm{op}}.$$

Our result follows.

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