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A GROUPOID PICTURE OF THE ELEK ALGEBRAS

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Abstract

We reformulate a construction by Gábor Elek, which associates C^*-algebras with uniformly recurrent subgroups, in the language of groupoid C^*-algebras. This allows us to simplify several proofs from the original paper and add the converse direction to Elek's characterisation of nuclearity, showing that his sufficient condition is in fact necessary. We furthermore relate our groupoids to the dynamics of the group acting on its uniformly recurrent subgroup.

1. Introduction

The constructions of (reduced) C^*-algebras associated with groups, or, more generally, crossed products associated with actions of groups on topological spaces, are well-known in the field of operator algebras, and for good reason: they provide a useful tool for constructing and understanding examples of C^*-algebras, as well as establishing ties to other branches of mathematics such as dynamics and geometric group theory. Defined by Glasner and Weiss [4], *uniformly recurrent subgroups*, or URSs for short, have recently drawn a lot of attention in the world of C^*-algebras since Kennedy [5] characterised C^*-simplicity of a discrete group Γ as the absence of non-trivial amenable uniformly recurrent subgroups. Another relation between uniformly recurrent subgroups and C^*-algebras was given by a construction of Elek [3], who defined a C^*-algebra closely tied to the dynamics of a finitely-generated discrete group Γ acting on one of its uniformly recurrent subgroups Z. His construction, the completion of the algebra of "local kernels" on the Schreier graph of a subgroup in the URS Z in a regular representation, takes more of the combinatorial nature of Z into account than the crossed product of Γ acting on $C(Z)$ does. This construction is thereby very well-suited to finding C^*-algebras with desired properties by rephrasing such properties at the combinatorial level of URSs. For example, Elek obtains a C^*-algebra with both a uniformly amenable and a nonuniformly amenable trace, by providing a Schreier graph with the corresponding properties for URSs.

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In this paper we recast Elek's construction from the viewpoint of groupoid C^*-algebras, which allows us to simplify and extend the ties between properties of the URS and its associated algebra. Using this new angle, we are able to prove that Elek's sufficient criterion for nuclearity of his C^*-algebras, the so-called *local property A*, is in fact also necessary, providing an equivalent characterisation of nuclearity of $C^*_{\mathrm{r}}(Z)$.

In addition to this introduction there are five sections. After recalling the necessary terminology in §2, we construct in §3 an étale groupoid for a given URS Z whose reduced C^*-algebra is canonically isomorphic to Elek's C^*-algebra $C^*_{\mathrm{r}}(Z)$. In §4 we relate this groupoid to the dynamics of the action of Γ on Z by conjugation. Finally, in §5, we use the new framework to give simpler proofs for some of Elek's results on simplicity and nuclearity of the associated C^*-algebras and provide the now equivalent characterisation of nuclearity.

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2. Preliminaries

We recall the definition of Elek's C^*-algebras associated with uniformly recurrent subgroups. Let Γ be a finitely-generated discrete group. Let $\mathrm{Sub}(\Gamma)$ be the space of its subgroups, equipped with the topology of pointwise convergence of the characteristic functions associated with the subsets and the left Γ-action $\gamma.H = \gamma H \gamma^{-1}$ by conjugation, where $H \in \mathrm{Sub}(\Gamma)$ and $\gamma \in \Gamma$. For a discrete group Γ, this topology is also known as the Fell topology on $\mathrm{Sub}(\Gamma)$. As convergence in the topology is a pointwise condition, and conjugation with a fixed element γ is simply a relabelling of Γ, the action of Γ on $\mathrm{Sub}(\Gamma)$ is continuous. Recall that a *uniformly recurrent subgroup* (URS) of Γ is a closed, Γ-invariant subspace of $\mathrm{Sub}(\Gamma)$ on which the action is *minimal*, that is, on which every orbit is dense. The URS is called *generic*, if the stabiliser of any subgroup $H \in Z$ is as small as possible, namely H itself.

Note that every normal subgroup of Γ trivially defines a URS that consists of a single point, and furthermore every closed, invariant subset of $\mathrm{Sub}(\Gamma)$ contains a URS by a compactness argument. The number of distinct URSs in a given countable discrete group Γ can vary wildly between just the trivial normal subgroups $\{e\}$ and Γ (for a "Tarski monster" group) and uncountably many non-isomorphic URSs (for $\mathbb{F}_2$, as shown by Glasner and Weiss [4, Example 1.9, Theorem 5.1]).

Fixing a finite, symmetric system of generators Q of Γ, to each $H \in \mathrm{Sub}(\Gamma)$ we may assign a rooted, labeled graph $S_\Gamma^Q(H)$ called its *Schreier graph*, which has vertex set Γ/H, root H, and for every $\gamma H \in \Gamma/H$ and $q \in Q$ an edge from γH to $q\gamma H$ labeled by q. The group Γ acts on the vertex set Γ/H of $S_\Gamma^Q(H)$ by left multiplication, or, equivalently, by following along the edges that spell out γ in terms of the generators Q. We denote the shortest-path metric on a graph S by d, or d_S if there is ambiguity, and likewise the balls of radius R around a vertex $x \in S$ by $B_R(x)$ or $B_R(S, x)$. On the space S_Γ^Q of Schreier graphs associated with subgroups of Γ, we introduce a metric by

$$d_{S_\Gamma^Q}(S_1, S_2) := 2^{-r} \tag{2.1}$$

for $S_1 = S_\Gamma^Q(H_1)$ and $S_2 = S_\Gamma^Q(H_2)$ two graphs in S_Γ^Q and r the largest integer such that $B_r(S_1, H_1)$ and $B_r(S_2, H_2)$ are root-label isomorphic. This space carries a left Γ-action, where $\gamma.S_\Gamma^Q(H) = S_\Gamma^Q(\gamma H \gamma^{-1})$, that is, γ acts by changing the root. Elek identifies the graphs $S_\Gamma^Q(H)$ for which the orbit closure of H forms a URS as those where any root-label isomorphism class of balls is repeated with at most bounded distance from any point in the graph (see [3, Proposition 2.1] for the full description), and the graphs for which it forms a generic URS as those where vertices with large isomorphic balls are sufficiently far apart (see [3, Proposition 2.3]).

To associate a C^*-algebra with a given URS Z, Elek considers its *local kernel algebra* $\mathbb{C}Z$ formed by the *local* kernels $K\colon \Gamma/H \times \Gamma/H \to \mathbb{C}$ of finite width on $S_\Gamma^Q(H)$ for some subgroup $H \in Z$. A kernel K on $S_\Gamma^Q(H)$ is of width R, if $K(p, q) = 0$ for any two vertices $p, q \in S_\Gamma^Q(H)$ with $d(p, q) > R$. It is furthermore *local* with width R, if $K(p, \gamma.p) = K(q, \gamma.q)$ for any p and q with root-label isomorphic R-balls and $\gamma \in \Gamma$ of length at most R. Equipped with pointwise addition, convolution $KL(p, q) = \sum_z K(p, z)L(z, q)$, and involution $K^*(p, q) = \overline{K(q, p)}$ for local kernels K and L and p, q, and z in Γ/H, the local kernel algebra forms a *-algebra. Up to isomorphism, this algebra does not depend on the choice of root $H \in Z$. A "regular" representation of $\mathbb{C}Z$ on $\ell^2(\Gamma/H)$ is given by $(Kf)(p) = \sum K(p, q)f(q)$ for $f \in \ell^2(\Gamma/H)$. The reduced C^*-algebra $C_r^*(Z)$ of Z is the completion of $\mathbb{C}Z$ in the norm induced by this representation.

Several C^*-algebraic properties of $C_r^*(Z)$ can be read off of the URS Z and its Schreier graph. Genericity of Z implies simplicity of $C_r^*(Z)$, as discussed in §5. Furthermore, a *local* version of Yu's property A for the Schreier graph $S_\Gamma^Q(H)$, for any subgroup H in the URS Z, is equivalent to nuclearity of $C_r^*(Z)$. Recall that a Schreier graph $S_\Gamma^Q(H)$ for $H \in Z \subseteq \Gamma$ has Elek's *local property*

A if there is a sequence of *local* functions

$$\rho^n\colon \Gamma/H \to \ell^2(\Gamma/H), \qquad p \mapsto \rho^n_p,$$

such that $\|\rho^n_p\|_2 = 1$, while $d(p,q) \leq n$ implies $\|\rho^n_p - \rho^n_q\| \leq 1/n$. As with kernels, *locality* of a function $\rho\colon \Gamma/H \to \ell^2(\Gamma/H)$ means that there is some $R > 0$ such that ρ_p is supported in the R-ball $B_R(p)$ centred at p, and whenever θ is a root-label isomorphism $B_R(p) \to B_R(q)$, we have $\rho_q \circ \theta = \rho_p$.

3. URS algebras as groupoid algebras

3.1. Construction of the groupoid

For a given uniformly recurrent subgroup, we construct a groupoid whose regular representation models Elek's construction on the local kernel algebra.

Let Z be a uniformly recurrent subgroup of a discrete group Γ, fix $H \in Z$, and let $S = S^Q_\Gamma(H)$ be its Schreier graph. For any $n \in \mathbb{N}$ we define an equivalence relation on the vertices $V(S)$ of S by

$$p \sim_n q \iff B_n(S,p) \cong_{r,l} B_n(S,q);$$

that is, if the n-balls around p and q are isomorphic under an isomorphism preserving the roots and labels. Such a root-label isomorphism is necessarily unique, since the roots are unique and every vertex is uniquely described by the labels on any path from the root to said vertex. Likewise, if a root-label isomorphism of n-balls on a Schreier graph has a fixed point, the two balls coincide and the root-label isomorphism is the identity, since we may describe every vertex in the balls by paths relative to the fixed point. The equivalence relation on the vertices $p, q \in V(S)$ can alternatively be formulated in terms of the group elements γ_p and γ_q of Γ describing paths from the root of S to p and q, respectively: With the above we have $p \sim_n q$ if and only if $\gamma_p.S$ and $\gamma_q.S$ are 2^{-n}-close in the metric $d_{S^Q_\Gamma}$ on S^Q_Γ introduced in Equation (2.1). Let $E_n = V(S)/{\sim_n}$ denote the finite set of equivalence classes of $\sim_n$ equipped with the discrete topology and the obvious connecting maps $e_{n+1}\colon E_{n+1} \to E_n$. Note that these sets do not depend on the choice of $H \in Z$:

Proposition 3.1. *Let H and H' be two groups contained in a URS Z of a discrete, finitely-generated group Γ. Then H and H' have the same sets of equivalence classes of rooted, labeled n-balls in their associated Schreier graphs.*

Proof. Let Q be a generating set of Γ, and let E_n and E'_n be the sets of root-label equivalence classes of n-balls in $S = S^Q_\Gamma(H)$ and $S' = S^Q_\Gamma(H')$. We show that for every $p \in V(S)$ there is a $p' \in V(S')$ such that $B_n(S,p) \cong_{r,l}$

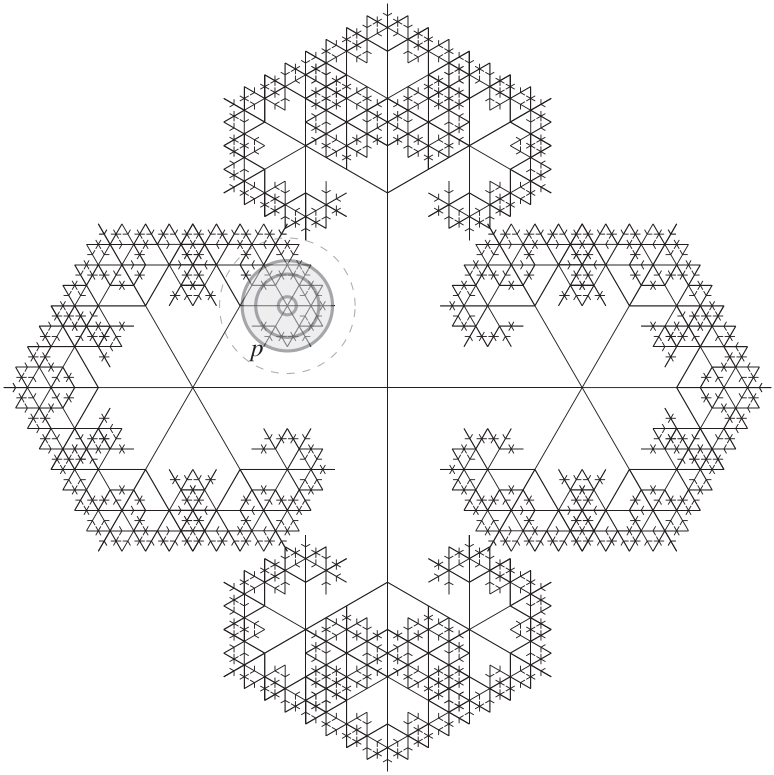

FIGURE 1. Elements of $\varprojlim E_n$ are described by a sequence of isomorphism classes of n-balls in some fixed Schreier graph $S_\Gamma^Q(H)$ for a group $H \in Z$. Writing $([p_0]_0, [p_1]_1, [p_2]_2, \ldots) \in \varprojlim E_n$, we denote by $[p_n]_n$ the equivalence class in E_n represented by the n-ball around a vertex $p_n \in S_\Gamma^Q(H)$. To form a valid element, the sequence of vertices $(p_n)_n$ has to be compatible in the sense that $[p_i]_i = [p_j]_i$ for all $i \leq j$. A particularly easy way to achieve this is by choosing a constant vertex p such that $p_n = p$ for all n. If a path from the root of $S_\Gamma^Q(H)$ to p is described by the group element γ, then the element $([p]_0, [p]_1, [p]_2, \ldots)$ is sent to the group $\gamma.H$ under the homeomorphism between $\varprojlim E_n$ and Z. This is reflected in the fact that changing the root of $S_\Gamma^Q(H)$ to p will yield the Schreier graph $S_\Gamma^Q(\gamma.H)$.

The Schreier graphs depicted are chosen for ease of drawing with no particular group in mind. Labels and loops are omitted for readability.

$B_n(S', p')$. If $p = \gamma H$, then since Z is uniformly recurrent, the subgroup $\gamma H \gamma^{-1}$ is in the orbit closure of H', and therefore $S_\Gamma^Q(\gamma H \gamma^{-1})$ is in the orbit closure of S'. Hence there is some $\gamma' \in \Gamma$ such that

$$B_n(S, p) \cong_{r,l} B_n(S_\Gamma^Q(\gamma H \gamma^{-1}), \gamma H \gamma^{-1}) \cong_{r,l} B_n(S', \gamma' H')$$

and $p' = \gamma' H'$ gives $E_n \subseteq E'_n$. The claimed equality follows by symmetry.

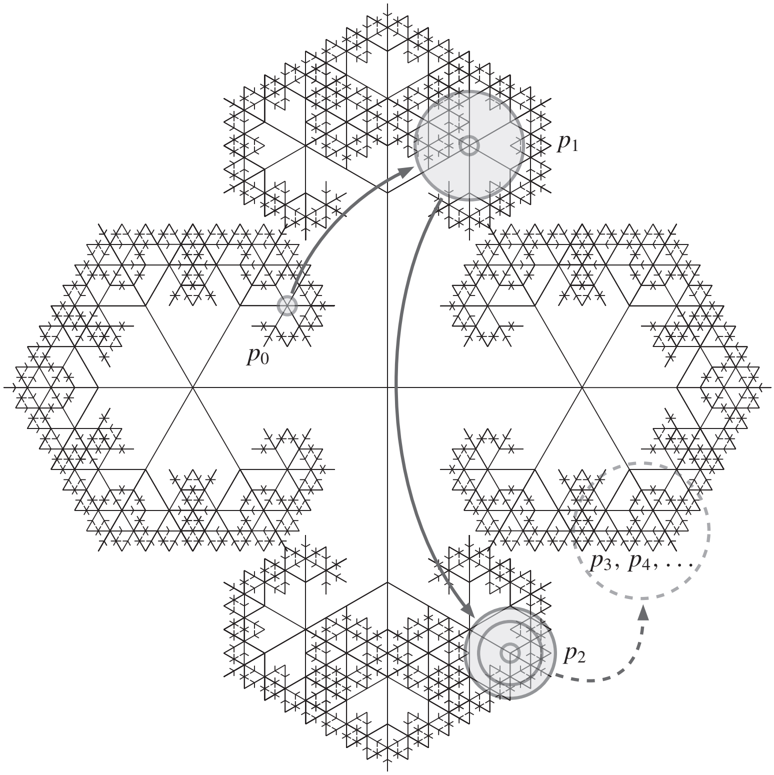

FIGURE 2. There might be elements $([p_0]_0, [p_1]_1, [p_2]_2, \ldots) \in \varprojlim E_n$ that are not described by a single, constant vertex $p \in V(S^Q_\Gamma(H))$, but by a choice of vertices p_n such that the i-ball around any p_j for $j \geq i$ is root-label-isomorphic to the i-ball around the vertex p_i, without eventually being constant. These isomorphism classes of balls assemble into a new Schreier graph, whose n-ball around the root is root-label-isomorphic to the n-ball around p_n. If $\gamma_n \in \Gamma$ describes a path from the root of $S^Q_\Gamma(H)$ to p_n, then $\gamma_n.H$ will converge to another group $K \in Z$ and the Schreier graph assembled from the balls around the vertices p_n is $S^Q_\Gamma(K)$. Consequently, $([p_0]_0, [p_1]_1, [p_2]_2, \ldots)$ is sent to K under the homeomorphism between $\varprojlim E_n$ and Z.

We endow the projective limit $\varprojlim E_n$ with the subspace topology of the product topology on $\prod_{n\in\mathbb{N}} E_n$. Then, as in [3, Lemma 6.1], it is easy to check that $\varprojlim E_n$ is homeomorphic to Z as a subspace of $\mathrm{Sub}(\Gamma)$, which in turn is either a Cantor space or a finite discrete set. It is noteworthy that under this identification, the orbit of H in Z is exactly described by those elements in $\varprojlim E_n$ that can be represented by the equivalence classes $[p]_n$ of a *fixed* vertex $p \in S$. This is depicted in Figure 1. The other elements, as in Figure 2, describe subgroups in the orbit closure, but not the orbit, of H.

We employ this description of the space Z to construct an ample Hausdorff étale groupoid $\mathscr{G}_Z$ with unit space $\mathscr{G}^0_Z$ homeomorphic to Z, whose reduced groupoid C^*-algebra is $C^*_{\mathrm{r}}(Z)$. As a set, the groupoid $\mathcal{G}_Z$ is identified with a subset of the projective limit $\varprojlim F_n$ of the finite, discrete sets

$$F_n = \bigsqcup_{[x_n]\in E_n} B_n(S, x_n) \sqcup \{\infty_n\}. \tag{3.1}$$

Here $B_n(S, x_n)$ denotes the n-ball in S that is determined uniquely by $[x_n]$, even if there is a choice in the representing vertex x_n. To avoid this choice of representing elements, a pair $([x_n], y)$ with $y \in B_n(S, x_n)$ as in Equation (3.1) can be more readily expressed as a pair $([x_n], \gamma)$ as in Equation (3.2) below, where γ is any chosen path from x_n to y inside $B_n(S, x_n)$, up to $\gamma \approx_{x_n} \gamma'$ if both paths lead to the same vertex in S; that is, if $\gamma.x_n = \gamma'.x_n$:

$$F_n \cong \bigsqcup_{[x_n]\in E_n} \{\gamma \in \Gamma \mid l(\gamma) \le n\}/{\approx_{x_n}} \sqcup \{\infty_n\}. \tag{3.2}$$

Implicit in this description is our later identification of $\mathcal{G}_Z$ with a quotient of the transformation groupoid $Z \rtimes \Gamma$. In this picture, ∞_n fills in for choices of vertices that are not contained in $B_n(S, x_n)$, or, respectively, for those γ whose length as a word in the generators exceeds n. The connecting maps $F_{n+1} \to F_n$ are then given by

$$([x_{n+1}], \gamma) \mapsto \begin{cases} (e_{n+1}([x_{n+1}]), \gamma), & \text{if } d(x_{n+1}, \gamma.x_{n+1}) \le n, \\ \infty_n, & \text{otherwise,} \end{cases}$$

$$\infty_{n+1} \mapsto \infty_n.$$

Now let $\mathcal{G}_Z := \varprojlim F_n \setminus \{\infty\}$ with $\infty = (\infty_1, \infty_2, \ldots)$, equipped with the subspace topology of the projective limit, which itself is equipped with the subspace topology of the product topology on $\prod_{n\in\mathbb{N}} F_n$. Identifying E_n with a subset of F_n by sending $[x_n]$ to $([x_n], e)$, we may identify $Z \cong \varprojlim E_n$ with a subset of $\mathcal{G}_Z$. To simplify notation, we write $(x, \gamma) \in \varprojlim E_n \times \Gamma$ for the element of $\varprojlim F_n$ that it represents when making a choice of representing vertex $[x_n] \in E_n$ for every n while identifying $([x_n], \gamma)$ with ∞_n whenever $l(\gamma) > n$. Note that (x, γ) and (x, γ') with $l(\gamma') \ge l(\gamma)$, represent the same equivalence class in $\varprojlim F_n$ if $\gamma.x_{l(\gamma')} = \gamma'.x_{l(\gamma')}$. The topology on $\mathcal{G}_Z$ is then equivalently given by the metric $d_{\mathcal{G}_Z}$, where for x and y in Z and $\gamma, \gamma' \in \Gamma$,

$$d_{\mathcal{G}_Z}((x, \gamma), (y, \gamma')) = 2^{-N}$$

for N maximal such that $([x_N]_N, \gamma.x_N)$ and $([y_N]_N, \gamma'.y_N)$ coincide in F_N.

We define the range and source maps as $r(x, \gamma) = (x, e)$ and $s(x, \gamma) = (\gamma.x, e)$ respectively, with $\gamma.x := ([\gamma.x_{l(\gamma)}]_0, [\gamma.x_{l(\gamma)+1}]_1, \ldots)$. This is merely acting entry-wise with γ, but for the action to be well-defined we first need to left-shift the representing elements by $l(\gamma)$ steps because for two vertices p and q, $p \sim_n q$ does not imply that $\gamma.p \sim_n \gamma.q$, but $p \sim_{n+l(\gamma)} q$ does, as depicted in Figure 3. Clearly, such a left-shift of representing elements does not change the described element of $\varprojlim E_n$. If we define an action of Γ on $\varprojlim E_n$ by $\gamma.x = s(x, \gamma)$ as above, the mentioned homeomorphism between $\varprojlim E_n$ and Z is equivariant.

The range and source maps map $\varprojlim F_n$ onto the subspace where the group element is simply e, that is, onto the subspace $\varprojlim E_n$ in our earlier identification. The unit space $\mathcal{G}_Z^0 = r(\mathcal{G}_Z) = s(\mathcal{G}_Z)$ is therefore homeomorphic to Z. Elements of the range fibre $\mathcal{G}_Z^x$ have the form (x, γ) for $\gamma \in \Gamma$, and the identification of (x, γ) and (x, γ') with $l(\gamma') \geq l(\gamma)$ if $\gamma.x_{l(\gamma')} = \gamma'.x_{l(\gamma')}$ is exactly the definition of $\cong_{x_{l(\gamma')}}$. Note that, depending on the choice of γ, it might be that $d(x_{l(\gamma)}, \gamma.x_{l(\gamma)})$ is strictly less than $l(\gamma)$, in which case there is another $\gamma' \in \Gamma$ of length $d(x_{l(\gamma)}, \gamma.x_{l(\gamma)})$, such that (x, γ) and (x, γ') denote the same arrow. If there is no such γ', we call γ an element *of minimal length.*

If (x, γ) and (y, γ') are composable, that is, if $x = \gamma'.y$, then we define the composition $(y, \gamma')(x, \gamma)$ to be $(y, \gamma\gamma')$. Note that $s(y, \gamma\gamma') = s(x, \gamma)$, since $([x_0]_0, [x_1]_1, \ldots)$ coincides with the N-fold left-shift of representing elements $([x_N]_0, [x_{N+1}]_1, \ldots)$ in $\mathcal{G}_Z^0$ for any $N \in \mathbb{N}$. We see that any $x \in \mathcal{G}_Z^0$ is a unit in $\mathcal{G}_Z$ when written as (x, e) for $e \in \Gamma$ the neutral element, and consequently the equivalence class represented by (x, γ) has as its inverse the class represented by $(\gamma.x, \gamma^{-1})$.

Intuitively, an arrow with range x such that x is represented by the equivalence classes of a single, constant vertex in the Schreier graph is thought of as a path from x to $\gamma.x$, but $\gamma.x_n$ is only well-defined for $n \geq l(\gamma)$, in which case $\gamma.x_n$ determines a unique isomorphism class of $(n-d)$-balls and thereby a class in E_{n-d}, where $d = d(x_n, \gamma.x_n)$ as in Figure 3.

An element of $\mathcal{G}_Z^x$ is therefore represented by a pair (x, γ) with γ of minimal length, and we can describe (x, γ) as consistent choices of vertices $\gamma.x_n$ in the balls described by the classes $[x_n]_n$.

This structure indeed gives rise to a groupoid:

PROPOSITION 3.2. *Let Z be a uniformly recurrent subgroup of a discrete, finitely-generated group Γ. Then the associated construction of $\mathcal{G}_Z$ gives rise to an ample minimal Hausdorff étale groupoid with unit space homeomorphic to Z.*

PROOF. It is easy to see that the range and source maps,

$$r(x, \gamma) = x, \qquad s(x, \gamma) = \gamma.x,$$

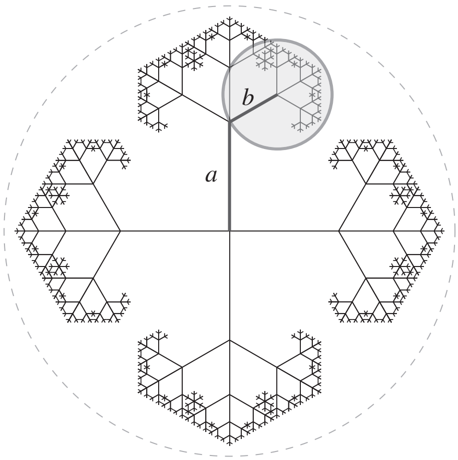

FIGURE 3. If x_n is a class in E_n, then γx_n determines a class in $E_{n-l(\gamma)}$. Here, $n = 5$ is depicted with $\gamma = ba$ for a, b two generators.

together with the composition and inversion

$$(y, \gamma')(x, \gamma) = (y, \gamma\gamma'), \qquad (x, \gamma)^{-1} = (\gamma.x, \gamma^{-1}),$$

indeed turn $\mathscr{G}_Z$ into a groupoid. Equipping $\mathscr{G}_Z$ with the locally compact Hausdorff subspace topology of $\varprojlim F_n$, we have to check that the defined operations are continuous. Consider the basic open sets

$$U_{e_N,\gamma} = \{(x, \gamma) \in \mathscr{G}_Z \mid [x_N]_N = e_N\}$$

that fix $\gamma \in \Gamma$ and $e_N \in E_N$ for some N. The range map is obviously continuous, as any basic open set of $\varprojlim E_n$ can be turned into a union of basic open sets of $\varprojlim F_n$ by letting γ vary. Inversion is continuous since the action of Γ on Z is, which in turn makes the source map continuous. To see that the composition is continuous, we fix $e_N \in E_N$ and $\gamma \in \Gamma$ and find the preimage of $U_{e_N,\gamma}$ under the composition map. For $(x, \eta)(\eta^{-1}.x, \eta')$ to be contained in $U_{e_N,\gamma}$, we first need that $x \in U_{e_N,\eta}$, and secondly that $\eta' = \eta^{-1}\gamma$. Hence the desired preimage is described by the intersection of

$$\bigcup_{\eta \in \Gamma} U_{e_N,\eta} \times U_{e_0,\eta^{-1}\gamma}$$

with the subspace $\mathscr{G}_Z^{(2)}$ of composable pairs in $\mathscr{G}_Z \times \mathscr{G}_Z$ and therefore open in $\mathscr{G}_Z^{(2)}$. Here we use e_0 to denote the unique equivalence class in E_0.

To see that the range map is a local homeomorphism, note that it restricts to a homeomorphism onto its image on every basic open set $U_{e_N,\gamma}$, and $\mathscr{G}_Z$

is therefore étale. Finally, $\mathscr{G}_Z$ is ample as it is an étale groupoid with totally disconnected unit space.

As the orbits of Z and $\mathscr{G}_Z^{(0)}$ coincide, $\mathscr{G}_Z$ is minimal, concluding the proof.

By Proposition 3.2, the groupoid $\mathscr{G}_Z$ does not depend on a choice of subgroup $H \in Z$ to construct the underlying Schreier graph.

3.2. Identification of the associated algebras

In the following, we show that Elek's algebra associated with a URS Z of a discrete, finitely-generated group Γ is isomorphic to the reduced C^*-algebra of our associated groupoid $\mathscr{G}_Z$ via a canonical isomorphism. We first identify the local kernel algebra $\mathbb{C}Z$ of Z with a dense subset of $\mathscr{C}_c(\mathscr{G}_Z)$ and then show that Elek's representation of $\mathbb{C}Z$ arises as the restriction of the regular representations of $\mathscr{G}_Z$ along this embedding.

Let $K \in \mathbb{C}Z$ be a local kernel, seen as a function on $V(S) \times V(S)$, where $S = S_\Gamma^Q(H)$ is the Schreier graph of a group $H \in Z$. Recall that there is a minimal $N \in \mathbb{N}$ called the *width* of K such that K vanishes on $(p, q) \in \Gamma/H \times \Gamma/H$ if the distance between p and q is more than N and such that $B_N(S, p) \cong_{r,l} B_N(S, q)$ implies that $K(p, \gamma p) = K(q, \gamma q)$ if $l(\gamma) \leq N$; that is, K only depends on the root-label isomorphism class of N-balls.

Lemma 3.3. *Let Z be a URS of a discrete, finitely-generated group Γ, and let $\mathscr{G}_Z$ be the associated groupoid. The local kernel algebra $\mathbb{C}Z$ of Z is in bijection with the dense subalgebra of locally constant functions in $\mathscr{C}_c(\mathscr{G}_Z)$. Equipped with convolution and involution, this is an isomorphism of *-algebras.*

Proof. Since $\mathscr{G}_Z$ is equipped with the subspace topology of a projective limit of discrete sets, we may describe an open set of $\mathscr{G}_Z$ by fixing a group element γ and the first n coordinates $[x_n] \in E_n$ while letting $[x_m] \in E_m$ for $m > n$ vary. Together, these form a basis of the topology of $\mathscr{G}_Z$. If a kernel K on $\Gamma/H \times \Gamma/H$ for a subgroup $H \in Z$ is local of width N, this means that K only depends on $(\gamma H, \gamma' H)$ up to the equivalence class of γH in E_N. Therefore, even though the range and source of an element $([x_n], \gamma) \in G$ might not be described by a single, constant choice of vertices in $S_\Gamma^Q(H)$ like in Figure 1, we can obtain a function f_K on $\mathscr{G}_Z$ from a kernel K by picking a representing vertex in G/H corresponding to the component in E_N as in the next paragraph and this choice is made uniformly on a neighbourhood as described above. This gives a way to translate back and forth between kernels in $\mathbb{C}Z$ and locally constant functions in $\mathscr{C}_c(\mathscr{G}_Z)$, and indeed we built $\mathscr{G}_Z$ such that the algebraic operations are compatible.

Let K be a local kernel of width N, and define a function $f_K \in \mathscr{C}_c(\mathscr{G}_Z)$ given by $f_K((x, \gamma)) = K(x_M, \gamma x_M)$, where $M = \max\{N, l(\gamma)\}$. If γ is chosen of minimal length, we may pick $M = N$. Equivalently, f_K evaluates (x, γ) at its component $([x_N]_N, \gamma)$ in F_N and assigns $K(x_N, \gamma x_N)$ to it, where $K(\infty_N) = 0$. This is well-defined, as the vertex x_N is given up to root-label isomorphism of N-balls. The function f_K is continuous, because it is uniformly locally constant: It is constant on any 2^{-N}-ball in $\mathcal{G}_Z$. To see that f_K is compactly supported, note that the embedding $\mathcal{G}_Z \hookrightarrow \varprojlim F_n$ is the one-point compactification of $\mathcal{G}_Z$. Therefore, a set $U \subseteq \mathcal{G}_Z$ is relatively compact exactly if there is some $M \in \mathbb{N}$ such that no element of U has F_M-component ∞_M. Equivalently, U as a subset of $\varprojlim F_n$ does not intersect the 2^{-M}-ball centred at ∞. As f_K is supported outside of the 2^{-N}-ball centred at ∞ for N the width of K, it is compactly supported.

Conversely, any locally constant function $f \in \mathscr{C}_c(\mathcal{G}_Z)$ defines a kernel K_f in $\mathbb{C}Z$. As f is locally constant, for each $g \in \mathcal{G}_Z$, we can pick a ball centred at g on which f is constant. Then finitely many of such balls cover the support of f, and we may pick $N \in \mathbb{N}$ such that these have radius at least 2^{-N}. Since two 2^{-N}-balls in $\mathcal{G}_Z$ or $\varprojlim F_n$ are either disjoint or equal, f is constant on any 2^{-N}-ball and supported outside of the 2^{-N}-ball of ∞. Given such f, we may now define a local kernel K_f with width at most N as follows: For a fixed vertex $p \in V(S)$, consider the unit $[\![p]\!] := ([p]_0, [p]_1, \ldots) \in \mathcal{G}_Z^0$ and define $K_f(p, \gamma p) := f([\![p]\!], \gamma)$. If $\gamma p = \gamma' p$, then $([\![p]\!], \gamma) = ([\![p]\!], \gamma')$, so K_f is a well-defined kernel. As the F_N-component of $([\![p]\!], \gamma)$ is ∞_N if $d(p, \gamma p) > N$, we have $K_f(p, \gamma p) = 0$ in that case. Furthermore, if the N-balls $B_N(S, p)$ and $B_N(S, q)$ are root-label-isomorphic for $p, q \in V(S)$ and $l(\gamma) \leq N$, then $([\![p]\!], \gamma)$ and $([\![q]\!], \gamma)$ are 2^{-N}-close, since their first N components coincide. As f is constant on 2^{-N}-balls, we have $K_f(p, \gamma p) = K_f(q, \gamma q)$, so K_f is local of width N.

It is easy to check that $f_{K_f} = f$ and $K_{f_K} = K$, so the local kernel algebra $\mathbb{C}Z$ of Z is in bijection with the subset of locally constant functions in $\mathscr{C}_c(\mathcal{G}_Z)$, which is dense in $\mathscr{C}_c(\mathcal{G}_Z)$, as $\mathcal{G}_Z$ is totally disconnected.

Furthermore, the *-algebra structure of $\mathbb{C}Z$ is preserved under the described inclusion: For $p \in V(S)$ and $\gamma \in \Gamma$, we have $s([\![p]\!], \gamma) = [\![\gamma.p]\!]$ and the orbit of $[\![p]\!]$ in $\mathcal{G}_Z$ is $\{[\![q]\!] \mid q \in V(S)\}$. We calculate

$$\begin{aligned} f_{L*K}([\![p]\!], \gamma) &= L * K(p, \gamma p) = \sum_{q \in V(S)} L(p, q) K(q, \gamma p) \\ &= \sum_{([\![p]\!], \gamma') \in \mathcal{G}_Z^{[\![p]\!]}} f_L([\![p]\!], \gamma') f_K([\![\gamma' p]\!], \gamma(\gamma')^{-1}) = f_L * f_K([\![p]\!], \gamma) \end{aligned}$$

and

$$(f_K)^*([\![p]\!], \gamma) = \overline{f_K([\![\gamma p]\!], \gamma^{-1})} = \overline{K(\gamma p, p)} = K^*(p, \gamma p) = f_{K^*}([\![p]\!], \gamma).$$

We conclude that $f_{L*K} = f_L * f_K$ and $(f_K)^* = f_{K^*}$ on the subset

$$\{([\![p]\!], \gamma) \mid p \in V(S), \gamma \in \Gamma\} \subseteq \mathcal{G}_Z$$

of arrows in $\mathcal{G}_Z$, whose range is given by the equivalence classes of a single, constant vertex p in S. This subset is dense, as the orbit of S is dense in $S_\Gamma^Q(Z)$, and as the functions in question are continuous, they coincide on all of $\mathcal{G}_Z$. Hence $\mathbb{C}Z$ embeds into $C_c(\mathcal{G}_Z)$ by a *-homomorphism onto the sub-*-algebra of locally constant functions as claimed.

Recall that Elek's algebra $C^*_\mathrm{r}(Z)$ is the completion of $\mathbb{C}Z$ in the norm arising from the faithful representation π on $\mathcal{B}(\ell^2(\Gamma/H))$ given by

$$(\pi(K)h)(\gamma H) = \sum_{\gamma' H \in \Gamma/H} K(\gamma H, \gamma' H)h(\gamma' H)$$

for $h \in \ell^2(\Gamma/H)$. We show that this norm arises as the restriction of the reduced norm on $\mathcal{C}_c(\mathcal{G}_Z)$ along the canonical embedding, and furthermore that $\mathbb{C}Z$ is dense in $\mathcal{C}_c(\mathcal{G}_Z)$ with respect to the reduced norm. Consequently, $C^*_\mathrm{r}(Z)$ is canonically isomorphic to the reduced groupoid C^*-algebra $C^*_\mathrm{r}(\mathcal{G}_Z)$.

THEOREM 3.4. *Let Z be a uniformly recurrent subgroup of a discrete, finitely-generated group Γ and $\mathcal{G}_Z$ the associated groupoid. The C^*-algebras $C^*_\mathrm{r}(Z)$ and $C^*_\mathrm{r}(\mathcal{G}_Z)$ are isomorphic via the isomorphism extending the canonical embedding of the local kernel algebra $\mathbb{C}Z$ as the subalgebra of locally constant functions in $\mathcal{C}_c(\mathcal{G}_Z)$.*

Recall that for an étale groupoid $\mathcal{G}_Z$, the reduced norm on $\mathcal{C}_c(\mathcal{G}_Z)$ is given by $\|f\| = \sup_x \|\pi_x(f)\|$ with the representations $\pi_x\colon \mathcal{C}_c(\mathcal{G}_Z) \to \mathcal{B}(\ell^2((\mathcal{G}_Z)_x))$ defined by

$$(\pi_x(f)\xi)(g) = \sum_{g' \in (\mathcal{G}_Z)_{s(g)}} f(g(g')^{-1})\xi(g')$$

for $x \in \mathcal{G}_Z^0$, $g \in (\mathcal{G}_Z)_x$, and $\xi \in \ell^2((\mathcal{G}_Z)_x)$.

PROOF. We show that the norm on $\mathbb{C}Z$ defined by Elek coincides with the reduced norm on $\mathcal{C}_c(\mathcal{G}_Z)$ restricted to the subset of locally constant functions, which we identified with $\mathbb{C}Z$ in Lemma 3.3. In fact, we even show that Elek's representation π of $\mathbb{C}Z$ and the groupoid representation π_x of any $x \in \mathcal{G}_Z^0$ are equivalent. Let us first consider the easiest case of $x = [\![H]\!]$, the unit represented by the root in $S = S_\Gamma^Q(H)$, the Schreier graph of $H \leq \Gamma$. We obtain a map $V(S) \to (\mathcal{G}_Z)_{[\![H]\!]}$ by $\gamma H \mapsto ([\![\gamma H]\!], \gamma^{-1})$, mapping a vertex γH in S to the arrow in $\mathcal{G}_Z$ that is described by any path from H to γH. This is obviously surjective and is well-defined, as the arrows $([\![\gamma H]\!], \gamma^{-1})$ and $([\![\gamma' H]\!], (\gamma')^{-1})$ are identified if $\gamma H = \gamma' H$. It is furthermore injective: If

$([\![\gamma H]\!], \gamma^{-1}) = ([\![\gamma' H]\!], (\gamma')^{-1})$, then the N-balls centred at γH and $\gamma' H$ are root-label isomorphic via an isomorphism mapping H to H if $N > l(\gamma)+l(\gamma')$. But a root-label-isomorphism of N-balls in Schreier graphs is the identity if it has a fixed point; hence, $\gamma H = \gamma' H$. We have thus established a bijection between $(\mathscr{G}_Z)_{[\![H]\!]}$ and $\Gamma/H = V(S)$. This yields a unitary T between $\ell^2(V(S))$ and $\ell^2((\mathscr{G}_Z)_{[\![H]\!]})$ that intertwines the representations π and $\pi_{[\![H]\!]}$ on $\mathbb{C}Z$ as follows. Let $h \in \ell^2(V(S))$. Then

$$\pi(K)h(\gamma H) = \sum_{\gamma' H \in V(S)} K(\gamma H, \gamma' H)h(\gamma' H),$$

and so

$$\begin{aligned} T(\pi(K)h)([\![\gamma H]\!], \gamma^{-1}) &= \sum_{\gamma' H \in V(S)} K(\gamma H, \gamma' H)h(\gamma' H) \\ &= \sum_{\gamma' H \in V(S)} f_K([\![\gamma H]\!], \gamma'\gamma^{-1})h(\gamma' H) \\ &= \sum_{\gamma' H \in V(S)} f_K([\![\gamma H]\!], \gamma'\gamma^{-1})(Th)([\![\gamma' H]\!], (\gamma')^{-1}) \\ &= \sum_{g \in (\mathscr{G}_Z)_{[\![H]\!]}} f_K(([\![\gamma H]\!], \gamma^{-1})g^{-1})(Th)(g) \\ &= \big(\pi_{[\![H]\!]}(f_K)Th\big)([\![\gamma H]\!], \gamma^{-1}), \end{aligned}$$

and therefore the representations π and $\pi_{[\![H]\!]}$ define identical reduced norms on the local kernel algebra $\mathbb{C}Z$ as a subalgebra of $\mathscr{C}_c(\mathscr{G}_Z)$.

Morally, this already implies that all source-fibre representations π_x for $x \in \mathscr{G}_Z^0$ are unitarily equivalent to π, since the groupoid $\mathscr{G}_Z$ does not depend on the choice of $H \in Z$ in its construction, and for every $x \in \mathscr{G}_Z^0$ there is a unique subgroup in Z which is mapped to x under the homeomorphism $Z \cong \mathscr{G}_Z^0$. For completeness we nevertheless show that all reduced representations of $\mathscr{G}_Z$ are equivalent.

As in any groupoid, the representations π_x are unitarily equivalent for any two units x that share an orbit. Therefore we only need to consider π_x for $x \in \mathscr{G}_Z^0$ that corresponds to a subgroup H' in $Z \setminus \Gamma.H$. For fixed $K \in \mathbb{C}Z$ of width R and $\epsilon > 0$, we may pick $h \in \ell^2((\mathscr{G}_Z)_{[\![H]\!]})$ of norm one such that

$$\|\pi_{[\![H]\!]}(f_K)h\|_2 > \|\pi_{[\![H]\!]}(f_K)\| - \epsilon$$

and h is supported on arrows $([\![\gamma H]\!], \gamma^{-1})$ with $l(\gamma) \leq N$ for some $N \in \mathbb{N}$. As the orbit of H' in Z is dense, there is a unit y in the orbit of x in $\mathscr{G}_Z^0$ such that y and $[\![H]\!]$ are $2^{-(N+R)}$-close; that is, their E_{N+R}-components coincide. Hence, for every arrow $([\![\gamma H]\!], \gamma^{-1}) \in (\mathcal{G}_Z)_{[\![H]\!]}$ with $l(\gamma) \leq N + R$, there is a *unique* arrow $(\gamma.y, \gamma^{-1}) \in (\mathcal{G}_Z)_y$ that is described by the same γ^{-1}, since two paths of length less than $N + R$ starting at H end in the same vertex exactly if the analogous paths in the isomorphic $(N+R)$-ball of y do. This yields a bijection between the subspaces of $(\mathcal{G}_Z)_{[\![H]\!]}$ and $(\mathcal{G}_Z)_y$ described by elements of Γ with length at most $N + R$. Extending by zero, we transport $h \in \ell^2((\mathcal{G}_Z)_{[\![H]\!]})$ along this bijection to a function $h' \in \ell^2((\mathcal{G}_Z)_y)$ with $1 = \|h\|_2 = \|h'\|_2$. Noting that $\pi_{[\![H]\!]}(f_K)h \in \ell^2((\mathcal{G}_Z)_{[\![H]\!]})$ is supported on $([\![\gamma H]\!], \gamma^{-1})$ with $l(\gamma) \leq N + R$, we may likewise transport this to a function in $\ell^2((\mathcal{G}_Z)_y)$ of the same norm. It is now easy to see that this function will just be $\pi_y(f_K)h'$, whence

$$\|\pi_y(f_K)\| \geq \|\pi_y(f_K)h'\|_2 > \|\pi_{[\![H]\!]}(f_K)\| - \epsilon,$$

and by unitary equivalence

$$\|\pi_x(f_K)\| > \|\pi_{[\![H]\!]}(f_K)\| - \epsilon.$$

By symmetry we obtain the converse direction, implying that all norms on $\mathcal{C}_c(\mathcal{G}_Z)$ induced by reduced representations are identical and coincide on $\mathbb{C}Z \subseteq \mathcal{C}_c(\mathcal{G}_Z)$ with the reduced norm of $\mathbb{C}Z$ as given by Elek's representation π.

To conclude that the completion of $\mathbb{C}Z$ in this norm is the same as the completion of $\mathcal{C}_c(\mathcal{G}_Z)$ in the reduced norm, and hence $C^*_{\mathrm{r}}(\mathcal{G}_Z)$, we finally show that $\mathbb{C}Z$ is dense in $\mathcal{C}_c(\mathcal{G}_Z)$ in the reduced norm. Let $f \in \mathcal{C}_c(\mathcal{G}_Z)$. As f is compactly supported, it vanishes on the 2^{-R}-ball around $\infty \in \varprojlim F_n$ for some R, so that $f([\![\eta H]\!], \gamma^{-1}) = 0$ if γ is chosen of minimal length and yet $l(\gamma) > R$. We therefore find for $h \in \ell^2((\mathcal{G}_Z)_{[\![H]\!]})$ that

$$\begin{aligned}
\left\|\pi_{[\![H]\!]}(f)h\right\|_2^2 &= \sum_{\gamma H \in \Gamma/H} \left|\big(\pi_{[\![H]\!]}(f)h\big)([\![\gamma H]\!], \gamma^{-1})\right|^2 \\
&= \sum_{\gamma H \in \Gamma/H} \left| \sum_{\gamma' H \in B_R(S, \gamma H)} f([\![\gamma H]\!], \gamma'\gamma^{-1}) h([\![\gamma' H]\!], (\gamma')^{-1}) \right|^2 \\
&\leq \|f\|_\infty^2 \sum_{\gamma H \in \Gamma/H} \left| \sum_{\gamma' H \in B_R(S, \gamma H)} h([\![\gamma' H]\!], (\gamma')^{-1}) \right|^2 \\
&\leq \|f\|_\infty^2 \sum_{\gamma H \in \Gamma/H} \left| \sum_{\gamma' H \in B_R(S, \gamma H)} |h([\![\gamma' H]\!], (\gamma')^{-1})| \right|^2 \\
&\leq \|f\|_\infty^2 \sum_{\gamma H \in \Gamma/H} \left| \sum_{\eta \in \Gamma, l(\eta) \leq R} h^{(\eta)}([\![\gamma H]\!], \gamma^{-1}) \right|^2 \\
&\leq (|Q| + 1)^{2R} \|f\|_\infty^2 \|h\|_2^2,
\end{aligned}$$

with $h^{(\eta)}([\![\gamma H]\!], \gamma^{-1}) = |h([\![\eta\gamma H]\!], (\eta\gamma)^{-1})|$ denoting copies of $|h|$ shifted by η. Note that passing to the absolute value is necessary, since while $\eta\gamma H$ for $\{\eta \in \Gamma \mid l(\eta) \leq R\}$ cover $B_r(S, \gamma G)$, they might not do so uniquely. The last inequality stems from the fact that there are fewer than $(|Q|+1)^R$ different η of length at most R in Γ, and $\|h^{(\eta)}\|_2 = \|h\|_2$. Hence $\|f\|_r \leq (|Q|+1)^{2R}\|f\|_\infty$ for all $f \in \mathscr{C}_c(\mathscr{G}_Z)$ supported outside of the 2^{-R}-ball around ∞ with $\|f\|_r = \|\pi_{[\![H]\!]}(f)\|$ the unique reduced norm. For any $\epsilon > 0$, by partitioning $\mathscr{G}_Z$ into open 2^{-N}-balls for large N, we may approximate f in $\mathscr{C}_c(\mathscr{G}_Z)$ up to ϵ by a function f_ϵ that is constant on every 2^{-N}-ball. As two balls are either disjoint or one is contained in the other, we may choose $f_\epsilon(g)$ supported outside of the 2^{-R}-ball of ∞ for large N, such that $\|f - f_\epsilon\|_r \leq (|Q|+1)^{2R}\|f - f_\epsilon\|_\infty = \epsilon(|Q|+1)^{2R}$.

As $\mathbb{C}Z$ is dense in $\mathscr{C}_c(\mathscr{G}_Z)$ in the reduced norm, which restricts to Elek's norm, the embedding extends to a *-isomorphism as claimed.

In summary, for any uniformly recurrent subgroup Z we have constructed an ample minimal étale Hausdorff groupoid with unit space homeomorphic to Z, such that the reduced C^*-algebras $C^*_{\mathrm{r}}(Z)$ of Z and $C^*_{\mathrm{r}}(\mathscr{G}_Z)$ of $\mathscr{G}_Z$ coincide. This enables us to examine $C^*_{\mathrm{r}}(Z)$ using tools for groupoid C^*-algebras.

4. The transformation groupoid

In this section we establish a relationship between our newly-defined groupoid $\mathscr{G}_Z$ associated with a uniformly recurrent subgroup Z of a finitely-generated discrete group Γ and the transformation groupoid $Z \rtimes \Gamma$ associated with the action of Γ on Z by conjugation.

As a space, the transformation groupoid $Z \rtimes \Gamma$ is simply the Cartesian product $Z \times \Gamma$ equipped with the product topology. The unit space is given by the subspace $Z \times \{e\}$ and identified with Z, the range and source of an arrow (H, γ) are respectively given by H and $\gamma^{-1}.H$, while the product of two composable arrows is $(H, \gamma)(\gamma^{-1}.H, \eta) = (H, \gamma\eta)$. This turns $Z \rtimes \Gamma$ into a Hausdorff étale groupoid, which for a URS Z is furthermore ample and minimal.

To distinguish our groupoid $\mathscr{G}_Z$ from $Z \rtimes \Gamma$, we describe the range fibres $\mathscr{G}_Z^H$ further. Recall that the homeomorphism between Z and $\varprojlim E_n$ describes every group $H \in Z$ as a sequence of isomorphism classes of balls in $S^Q_\Gamma(H')$ for an arbitrary $H' \in Z$. In particular, the isomorphism class in E_n associated with H is given by the n-ball around the root in $S^Q_\Gamma(H)$. Two arrows $([\![H]\!], \gamma)$ and $([\![H]\!], \eta)$ in $\mathscr{G}_Z$ will then coincide if and only if the paths in $S^Q_\Gamma(H)$ that start at the root and are described by γ and η end at the same vertex. That is, they coincide if $\gamma H = \eta H$, or equivalently $\eta^{-1}\gamma \in H$, and hence the range fibre $\mathcal{G}_Z^H$ at H is in bijection with Γ/H via $([\![H]\!], \gamma) \mapsto \gamma H$. To simplify this notation and remove the ambiguity in the choice of γ, we denote $([\![H]\!], \gamma)$ as $(H, \gamma H)$ for the remainder of this section. From this identification we obtain a surjective map $q\colon Z \rtimes \Gamma \to \mathcal{G}_Z$ via $(H, \gamma) \mapsto (H, \gamma^{-1}H)$. Note that the range fibres $(Z \rtimes \Gamma)^H$ of $Z \rtimes \Gamma$ are simply given by Γ. Then the map q is a quotient map of topological groupoids:

Proposition 4.1. *Let Z be a URS of a finitely-generated discrete group Γ, and let $\mathcal{G}_Z$ be the associated groupoid. Let $q\colon Z \rtimes \Gamma \to \mathcal{G}_Z$ be the map given by $(H, \gamma) \mapsto (H, \gamma^{-1}H)$ for a subgroup $H \in Z$ and a group element $\gamma \in \Gamma$. Then q is a continuous, open, and surjective groupoid homomorphism. In particular, $\mathcal{G}_Z$ is a quotient of the transformation groupoid $Z \rtimes \Gamma$.*

Proof. We first check that q is a groupoid homomorphism. Indeed, q preserves sources and ranges, since

$$\begin{aligned} q(s(H,\gamma)) &= q(\gamma^{-1}.H, e) = (\gamma^{-1}.H, \gamma^{-1}.H) \\ &= s(H, \gamma^{-1}H) = s(q(H,\gamma)), \end{aligned}$$

and

$$q(r(H,\gamma)) = q(H,e) = (H,H) = r(H,\gamma^{-1}H) = r(q(H,\gamma)).$$

Hence the images of two composable arrows (H, γ) and $(\gamma^{-1}.H, \eta)$ in $Z \rtimes \Gamma$ are composable in $\mathcal{G}_Z$ and we calculate

$$\begin{aligned} q(H,\gamma)q(\gamma^{-1}.H,\eta) &= (H,\gamma^{-1}H)(\gamma^{-1}.H, \eta^{-1}(\gamma^{-1}.H)) \\ &= (H, \eta^{-1}\gamma^{-1}H) \\ &= q(H, \gamma\eta) \\ &= q((H,\gamma),(\gamma^{-1}.H,\eta)). \end{aligned}$$

Furthermore, q is surjective, since any arrow $(H, \gamma H)$ in $\mathcal{G}_Z$ clearly has preimage (H, γ^{-1}) in $Z \rtimes \Gamma$.

We continue by showing that q is continuous and open. Consider the basis of the topology of $\mathcal{G}_Z$ given by the open sets

$$U_{H,N,\gamma} = \left\{(K, \gamma K) \mid d_{S_\Gamma^Q}(S_\Gamma^Q(H), S_\Gamma^Q(K)) \le 2^{-N}\right\}$$

indexed by a subgroup $H \in Z$, an element $\gamma \in \Gamma$ and $N \in \mathbb{N}$ with $l(\gamma) \le N$, and consisting of all arrows $(K, \gamma K)$ such that the N-balls around the root in $S_\Gamma^Q(H)$ and $S_\Gamma^Q(K)$ coincide. Equivalently, a group element $\eta \in \Gamma$ with

$l(\eta) \leq 2N$ is contained in K if and only if it is contained in H. Likewise, we fix a basis

$$V_{H,N,\gamma} = \left\{(K, \gamma) \mid d_{S_\Gamma^Q}(S_\Gamma^Q(H), S_\Gamma^Q(K)) \leq 2^{-N}\right\}$$

of the topology of $Z \rtimes \Gamma$.

It is now easy to see that q is open: Let $(K, \gamma^{-1}K) \in q(V_{H,N,\gamma})$. Then $U_{K,M,\gamma^{-1}}$ for $M = \max\{N, l(\gamma)\}$ is a neighbourhood of $(K, \gamma^{-1}K)$ contained in $q(V_{H,N,\gamma})$.

Conversely,

$$q^{-1}(U_{H,N,\gamma^{-1}}) = \left\{(K, \gamma k) \mid d_{S_\Gamma^Q}(S_\Gamma^Q(H), S_\Gamma^Q(K)) \leq 2^{-N},\ k \in K\right\},$$

and we claim that this is open in $Z \rtimes \Gamma$. Indeed, for every $(K, \gamma k) \in q^{-1}(U_{H,N,\gamma^{-1}})$ the neighbourhood $V_{K,M,\gamma k}$ is contained in $q^{-1}(U_{H,N,\gamma^{-1}})$, where $M = \max\{N, l(k)\}$, so that any subgroup in the 2^{-M}-ball around K is guaranteed to contain the element k. We conclude that q is the desired quotient map, ending the proof.

REMARK 4.2. Up to the inversion of γ, the map q can be thought of as dividing out the subgroup H from its range fibre $(Z \rtimes \Gamma)^H$ at every $H \in Z$. The inversion is necessary, since we chose to define our groupoid $\mathscr{G}_Z$ with the opposite conventions of those commonly used for transformation groupoids, as this matches Elek's definition more closely. If we had defined the groupoid $\mathscr{G}_Z$ as the opposite of the current definition, the groupoid homomorphism q in the proof of Proposition 4.1 would not need the inversion of γ, but to identify $\mathbb{C}Z$ with a subalgebra of $\mathscr{C}_c(\mathscr{G}_Z)$ in Lemma 3.3 we would first need to pass to the opposite to cancel the effects. In particular, the isomorphism $K \mapsto f_K$ of Lemma 3.3 would satisfy $f_{L*K} = f_K * f_L$ rather than $f_{L*K} = f_L * f_K$. An alternative way of defining q in Proposition 4.1 would therefore be to first pass to the *opposite groupoid* $(Z \rtimes \Gamma)^{\mathrm{op}}$ via the canonical groupoid isomorphism and then divide out the base group from every source fibre without inversion. We thank the anonymous reviewer for pointing this out.

Note that the quotient map q can only be injective if Z is the trivial uniformly recurrent subgroup containing only the trivial subgroup, and in that case $C_{\mathrm{r}}^*(Z)$ coincides with the reduced group algebra of Γ. In general, if Z is a singleton consisting of a normal subgroup N, the C^*-algebras $C_{\mathrm{r}}^*(Z)$ and $C_{\mathrm{r}}^*(\mathscr{G}_Z)$ will be isomorphic to the reduced group algebra of the quotient group Γ/N, since $\mathscr{G}_Z$ will be the transformation groupoid of Γ/N acting on a singleton. In every nontrivial case, when Z is not just the trivial subgroup, the associated groupoid $\mathscr{G}_Z$ is different from the transformation groupoid.

5. Simplicity and nuclearity

5.1. Characterisations of simplicity and nuclearity

As before let Z be a URS of a finitely-generated discrete group Γ, and let $\mathscr{G}_Z$ be the associated groupoid. We employ our description of the Elek algebras as groupoid algebras to give simplified proofs of Elek's characterisations, explaining why they arise in the language of groupoids.

PROPOSITION 5.1. *Let Z be a uniformly recurrent subgroup of a finitely-generated, discrete group. If Z is generic, then the associated groupoid $\mathscr{G}_Z$ is principal.*

PROOF. Supposing $\mathscr{G}_Z$ is not principal, there is a unit $x = ([x_0]_0, [x_1]_1, \ldots)$ and an arrow in the isotropy $(\mathscr{G}_Z)_x^x$ that is not a unit. Therefore there is a group element $\gamma \in \Gamma$ such that $x = \gamma.x$ but $(x, e) \neq (x, \gamma)$. In particular, $\gamma.x_N \neq x_N$ for large N, while $B_{N-l(\gamma)}(S, x_N) \cong_{r,l} B_{N-l(\gamma)}(S, \gamma.x_N)$ and $d(x_N, \gamma.x_N) \leq l(\gamma)$. By [3, Proposition 2.3], Z is not generic, concluding the proof.

As an immediate corollary, we reproduce [3, Theorem 7].

COROLLARY 5.2. *Let Z be a uniformly recurrent subgroup of a finitely-generated, discrete group. If Z is generic, then its reduced C^*-algebra $C^*_{\mathrm{r}}(Z)$ is simple.*

PROOF. By Proposition 5.1, $\mathscr{G}_Z$ is a minimal principal étale groupoid, and every such groupoid has a simple reduced C^*-algebra by [7, Proposition 4.3.7].

Regarding nuclearity, we are able to add the converse direction to Elek's characterisation [3, Theorem 8]. We first describe when our groupoids are amenable:

THEOREM 5.3. *Let Z be a uniformly recurrent subgroup of the finitely-generated discrete group Γ, and let $\mathscr{G}_Z$ be the groupoid associated with Z. The Schreier graph $S^Q_\Gamma(H)$ of any group $H \in Z$ has local property A if and only if $\mathscr{G}_Z$ is (topologically) amenable.*

PROOF. We first show that local property A of the Schreier graph $S^Q_\Gamma(H)$ implies topological amenability of $\mathscr{G}_Z$ by building functions f_n in $\mathscr{C}_c(\mathscr{G}_Z)$ from the functions ρ^n witnessing local property A much like from kernels in the proof of Lemma 3.3. These functions will then witness topological amenability. Let $\mathscr{G}_Z$ be constructed from $H \in Z$ and let $\rho^n \colon \Gamma/H \to \ell^2(\Gamma/H)$ implement local property A of $S^Q_\Gamma(H)$. Let R_n describe the locality of ρ^n as in §2. Let x be a unit of $\mathscr{G}_Z$ and let x_{R_n} denote any element of the equivalence class of E_{R_n} forming the R_n-component of x. Define $f_n \in \mathscr{C}_c(\mathscr{G}_Z)$ by $f_n(x, \gamma) = \rho^n_{x_{R_n}}(\gamma x_{R_n})$. This is independent of the choice of x_{R_n}, as the ρ^n are locally defined and of width at most R_n. In addition, ρ^n is continuous as it is constant on 2^{-R_n}-balls and compactly supported as it vanishes on the 2^{-R_n}-ball of ∞. Then, using the fact that $\rho^n_{x_{R_n}}$ is supported in the R_n-ball centred at x_{R_n},

$$\sum_{(x,\gamma)\in\mathcal{G}_Z^x} |f_n(x,\gamma)|^2 = \sum_{y\in B_{R_n}(x_{R_n})} |\rho^n_{x_{R_n}}(y)|^2 = \|\rho^n_{x_{R_n}}\|_2^2 = 1.$$

Furthermore, we calculate

$$f_n * f_n^*(x,\gamma) = \sum_{(\gamma.x,\gamma')\in\mathcal{G}_Z^{\gamma.x}} f_n((x,\gamma)(\gamma.x,\gamma'))\overline{f_n}(\gamma.x,\gamma'),$$

where we may restrict to $l(\gamma') \le R_n$, as $f_n(\gamma.x,\gamma')$ vanishes otherwise:

$$\begin{aligned} f_n * f_n^*(x,\gamma) &= \sum_{(\gamma.x,\gamma')\in\mathcal{G}_Z^{\gamma.x}, l(\gamma')\le R_n} \rho^n_{x_{R_n+l(\gamma)}}(\gamma'\gamma x_{R_n+l(\gamma)})\overline{\rho^n_{\gamma x_{R_n+l(\gamma)}}}(\gamma'\gamma x_{R_n+l(\gamma)}) \\ &= \langle \rho^n_{x_{R_n+l(\gamma)}}, \rho^n_{\gamma x_{R_n+l(\gamma)}}\rangle, \end{aligned}$$

using again the assumptions on the support of $\rho^n_{\gamma x_{R_n+l(\gamma)}}$. However, for fixed $\gamma H, \gamma' H \in \Gamma/H$ we have

$$\begin{aligned} |1 - \langle \rho^n_{\gamma H}, \rho^n_{\gamma H}\rangle| &= |\langle \rho^n_{\gamma H} - \rho^n_{\gamma H}, \rho^n_{\gamma H}\rangle| \\ &\le \|\rho^n_{\gamma H} - \rho^n_{\gamma H}\|_2 \cdot \|\rho^n_{\gamma H}\|_2 \overset{(\dagger)}{\le} 1/n \cdot 1 \xrightarrow{n\to\infty} 0, \end{aligned}$$

where the estimate (†) holds for large n such that $d(u,v) \le n$. But as $l(\gamma)$ is bounded on compact sets, the same estimate may be used uniformly on any compact set for sufficiently large n, so that $f_n * f_n^*$ converges to 1 uniformly on compact subsets. These functions witness the (topological) amenability of $\mathcal{G}_Z$ as in the original definition by Renault [6, p. 92].

Conversely, suppose that $\mathcal{G}_Z$ is topologically amenable. By an equivalent characterisation of amenability [1, Proposition 2.2.13], we may assume that there is a sequence $f_n \in \mathscr{C}_c(\mathcal{G}_Z)$, such that

$$\sum_{(x,\gamma)\in\mathcal{G}_Z^x} |f_n(x,\gamma)|^2 \xrightarrow{n\to\infty} 1 \tag{5.1}$$

uniformly on compact subsets of $\mathcal{G}_Z^0$ as a function of $x \in \mathcal{G}_Z^0$, and

$$\sum_{h\in\mathcal{G}_Z^{r(g)}} |f_n(g^{-1}h) - f_n(h)|^2 \xrightarrow{n\to\infty} 0 \tag{5.2}$$

uniformly on compact subsets of $\mathscr{G}_Z$ as a function of g. We use the existence of such a sequence of functions to obtain local maps ρ_x^n as in the definition of local property A, analogous to the previous part of the proof, albeit in reverse. As before, the translation only makes sense for locally constant functions, so we first, for any $\epsilon > 0$, approximate f_n by (uniformly) locally constant functions $f_{n,\epsilon} \in \mathscr{C}_c(\mathscr{G}_Z)$ with $\| f_n - f_{n,\epsilon}\|_\infty < \epsilon$, choosing $f_{n,\epsilon}$ to be zero on the largest possible ball centred at ∞.

We next construct a sequence $\epsilon_n \searrow 0$ such that the diagonal sequence f_{n,ϵ_n} satisfies Equations (5.1) and (5.2), thereby obtaining a sequence of locally constant functions witnessing amenability of $\mathscr{G}_Z$. As for fixed n there is $T_n \in \mathbb{N}$ such that both f_n and $f_{n,\epsilon}$ for every $\epsilon > 0$ vanish on the 2^{-T_n}-ball of ∞, we may restrict summations like in Equation (5.1) and Equation (5.2) to arrows described by group elements γ with length $l(\gamma)$ at most T_n, such that the respective sums become finite with fewer than $(|Q|+1)^{T_n}$ terms. First, observe that

$$\begin{aligned}
&\Bigg| \sum_{(x,\gamma)\in\mathscr{G}_Z^x} |f_n(x,\gamma)|^2 - \sum_{(x,\gamma)\in\mathscr{G}_Z^x} |f_{n,\epsilon}(x,\gamma)|^2 \Bigg| \\
&\qquad \le \sum_{\substack{(x,\gamma)\in\mathscr{G}_Z^x,\\ l(\gamma)\le T_n}} (|f_n(x,\gamma)| + \epsilon)^2 - |f_n(x,\gamma)|^2 \\
&\qquad \le (|Q|+1)^{T_n}(2\|f_n\|_\infty + \epsilon)\epsilon \xrightarrow{\epsilon\to 0} 0,
\end{aligned}$$

with the convergence uniform in $x \in \mathscr{G}_Z^0$. Then, to verify the condition of Equation (5.2), we calculate

$$\begin{aligned}
&\sum_{h\in\mathscr{G}_Z^{r(g)}} |f_{n,\epsilon}(g^{-1}h) - f_{n,\epsilon}(h)|^2 \\
&\quad \le \sum_{h\in\mathscr{G}_Z^{r(g)}, l(\gamma)\le T_n} (|f_n(g^{-1}h) - f_n(h)| + 2\epsilon)^2 \\
&\quad \le (4\epsilon^2 + 8\epsilon\|f_n\|_\infty)(|Q|+1)^{T_n} + \sum_{h\in\mathscr{G}_Z^{r(g)}} |f_n(g^{-1}h) - f_n(h)|^2 \\
&\quad \xrightarrow{\epsilon\to 0} \sum_{h\in\mathscr{G}_Z^{r(g)}} |f_n(g^{-1}h) - f_n(h)|^2,
\end{aligned}$$

with the convergence uniform in $g \in \mathscr{G}_Z$. Picking $\epsilon_n \to 0$ such that

$$\begin{aligned}
1/n &\ge (|Q|+1)^{T_n}(2\|f_n\|_\infty + \epsilon_n)\epsilon_n \qquad \text{and} \\
1/n &\ge (4\epsilon_n^2 + 8\epsilon_n\|f_n\|_\infty)(|Q|+1)^{T_n}
\end{aligned}$$

does the trick. To such f_{n,ϵ_n} we may assign $\rho^n\colon \Gamma/H \to \ell^2(\Gamma/H)$ given by $\rho^n_{\gamma H}(\gamma'\gamma H) = f_{n,\epsilon_n}([\![\gamma H]\!], \gamma')$.

Picking R_n such that f_{n,ϵ_n} is constant on any 2^{-R_n}-ball, we see that $\rho^n_{\gamma H}$ is supported on $B_{R_n}(\gamma H)$, since f_{n,ϵ_n} vanishes on the 2^{-R_n}-neighbourhood of ∞. Similarly, $\rho^n_{\gamma H}(\gamma'\gamma H) = \rho^n_{\eta H}(\gamma'\eta H)$ for $l(\gamma') \le R_n$ if the R_n-balls around γH and ηH are isomorphic, as $(\gamma H, \gamma')$ and $(\eta H, \gamma')$ are 2^{-R_n}-close in that case, and we conclude that ρ_n is local of width R_n.

Furthermore, we compute that

$$\|\rho^n_{\gamma H}\|_2^2 = \sum_{\gamma' H \in \Gamma/H} |\rho^n_{\gamma H}(\gamma' H)|^2 = \sum_{([\![\gamma H]\!],\gamma') \in \mathscr{G}_Z^{[\![\gamma H]\!]}} |f_{n,\epsilon_n}([\![\gamma H]\!], \gamma')|^2 \xrightarrow{n\to\infty} 1$$

uniformly on $\mathscr{G}_Z^0$. Likewise,

$$\begin{aligned}
\|\rho^n_{\gamma H} - \rho^n_{\gamma' H}\|_2^2 &= \sum_{\eta H \in \Gamma/H} |\rho^n_{\gamma H}(\eta H) - \rho^n_{\gamma' H}(\eta H)|^2 \\
&= \sum_{\eta H \in \Gamma/H} |f_{n,\epsilon_n}([\![\gamma H]\!], \eta\gamma^{-1}) - f_{n,\epsilon_n}([\![\gamma' H]\!], \eta(\gamma')^{-1})|^2 \\
&= \sum_{\eta H \in \Gamma/H} \big| f_{n,\epsilon_n}\big(([\![\gamma H]\!], \gamma'\gamma^{-1})([\![\gamma' H]\!], \eta(\gamma')^{-1})\big) \\
&\qquad - f_{n,\epsilon_n}([\![\gamma' H]\!], \eta(\gamma')^{-1})\big|^2 \\
&= \sum_{h \in \mathscr{G}_Z^{[\![\gamma' H]\!]}} \big| f_{n,\epsilon_n}\big(([\![\gamma H]\!], \gamma'\gamma^{-1})h\big) - f_{n,\epsilon_n}(h)\big|^2 \xrightarrow{n\to\infty} 0
\end{aligned}$$

uniformly for $([\![\gamma' H]\!], \gamma(\gamma')^{-1})$ in compact subsets of $\mathscr{G}_Z$. In particular, the convergence is uniform when the pair (γ, γ') is taken from a subset with bounded difference; that is, if there is some $N > 0$ such that $l(\gamma(\gamma')^{-1}) < N$. More importantly, this means that for any $N \in \mathbb{N}$, we may choose $n_0 \in \mathbb{N}$ such that $\|\rho^n_{\gamma H} - \rho^n_{\gamma' H}\|_2^2 \le 1/N$, whenever $d_{S^Q_\Gamma(H)}(\gamma H, \gamma' H) \le N$ and $n \ge n_0$. The analogous statement holds after replacing ρ^n_x with the normed function $\hat\rho^n_x = \rho^n_x / \|\rho^n_x\|_2$, since

$$\|\hat\rho^n_{\gamma H} - \hat\rho^n_{\gamma' H}\|_2 \le \frac{1}{\|\rho^n_{\gamma H}\|_2}\big(\|\rho^n_{\gamma H} - \rho^n_{\gamma' H}\|_2 + \big|\|\rho^n_{\gamma H}\|_2 - \|\rho^n_{\gamma' H}\|_2\big|\big),$$

while $\|\rho^n_{\gamma H}\|_2$ and $\big|\|\rho^n_{\gamma H}\|_2 - \|\rho^n_{\gamma' H}\|_2\big|$ converge uniformly to 1 and 0, respectively. Then, after relabelling, $\hat\rho^n$ witnesses local property A of $S^Q_\Gamma(H)$, concluding the proof.

For étale groupoids there is a clear relation between amenability and nuclearity of their reduced C^*-algebras, which directly translates to our case. See for example [8, §2] for a brief overview of the different notions of groupoid amenability. This implies the converse direction of [3, Theorem 8]:

COROLLARY 5.4. *Let Z be a uniformly recurrent subgroup and $H \in Z$. The graph $S^Q_\Gamma(H)$ has local property A if and only if the C^*-algebra $C^*_{\mathrm{r}}(Z)$ is nuclear.*

PROOF. As the groupoid $\mathscr{G}_Z$ associated with Z is étale, $C^*_{\mathrm{r}}(\mathscr{G}_Z)$ is nuclear if and only if $\mathscr{G}_Z$ is (topologically) amenable by [1, Corollary 6.2.14]. By Theorem 5.3 this is the case if and only if the Schreier graph $S^Q_\Gamma(H)$ has local property A.

5.2. Applications of groupoid simplicity

As non-trivial uniformly recurrent subgroups are difficult to construct, Elek's description of which Schreier graphs come from uniformly recurrent subgroups (see [3, Proposition 2.1]) gives an interesting new way to construct examples combinatorially. Elek gives such examples (see [3, §§5.2, 10.1]), all of which give rise to simple C^*-algebras, since they are generic.

Below we provide examples of non-generic URSs, whose associated C^*-algebras can nonetheless be proven to be simple via the groupoid picture. To this end, we employ a sufficient criterion for simplicity of groupoid C^*-algebras that the author established in [2, Corollary 6.9]: If a minimal groupoid with compact unit space has at least one C^*-simple isotropy group, the reduced groupoid C^*-algebra is simple.

For ease of notation, we describe our examples as quotients of the transformation groupoids as described in §4, but skip passing to the opposite, since a groupoid and its opposite are isomorphic and hence so are their reduced C^*-algebras: For a discrete group Γ with a subgroup $H \leq \Gamma$ such that its orbit closure $Z := \overline{\Gamma.H}$ forms a uniformly recurrent subgroup, we consider the transformation groupoid $Z \rtimes \Gamma$ associated with the action of Γ on Z and divide out by the equivalence relation given by

$$(\gamma, K) \sim (\eta, K) \iff \gamma K = \eta K$$

for two group elements $\gamma, \eta \in \Gamma$ and a subgroup $K \in Z$. We write $\mathscr{G}'_Z := (Z \rtimes \Gamma)/\sim$ so that $C^*_{\mathrm{r}}(\mathscr{G}_Z) \cong C^*_{\mathrm{r}}(\mathscr{G}'_Z)$.

Note that the isotropy group of $Z \rtimes \Gamma$ at unit (e, K) for $K \in Z$ is given by the normaliser $N(K)$ of K, that is, exactly these group elements that fix K under conjugation. After passing to the quotient $\mathscr{G}'_Z$, the isotropy group at (e, K) is $N(K)/K$, which is sometimes called the *Weyl group* of K. Any uniformly recurrent subgroup Z for which a contained subgroup $H \in Z$ has C^*-simple Weyl group will therefore give rise to a C^*-simple Elek algebra by the aforementioned result [2, Corollary 6.9].

Consider the dihedral group $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$, which arises as the semidirect product of the action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$, where the non-trivial element $[1]_2$ acts by inversion. The following arguments also work slightly more generally with $\mathbb{Z}_3$ replaced by $\mathbb{Z}_{2k+1}$ for any $k \in \mathbb{N}$. Let F be any non-trivial C^*-simple discrete group, for example, the non-abelian free group in two generators. Then let $\Gamma = (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$ and $H = \mathbb{Z}_2 \subseteq \Gamma$ be the canonical copy of $\mathbb{Z}_2$ inside $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$, which sits inside Γ. In other words, $H = \{(([0]_3, x), e_F) \in \Gamma \mid x \in \mathbb{Z}_2\}$ with $[0]_3$ the neutral element of $\mathbb{Z}_3$ and e_F the neutral element of F.

Calculating the action of an arbitrary element of Γ on H as

$$((n, x), f)H((n, x), f)^{-1} = \{e_\Gamma, ((2n, [1]_2), e_F)\}$$

for $n \in \mathbb{Z}_3$, $x \in \mathbb{Z}_2$, and $f \in F$, we see that the normaliser of H in Γ is $\mathbb{Z}_2 \times F$ seen as a subset of Γ. The orbit of H in $\mathrm{Sub}(\Gamma)$ contains exactly three distinct subgroups, described by the three choices of $n \in \mathbb{Z}_3$. As it is finite, the orbit is closed in $\mathrm{Sub}(\Gamma)$, whence the orbit of H is a URS of Γ. The Weyl group of $H \leq \Gamma$ is simply $(\mathbb{Z}_2 \times F)/\mathbb{Z}_2 \cong F$, which is simple by assumption on F. We conclude that $H \leq \Gamma$ provides an example of a simple Elek algebra that neither arises from a *generic* URS, nor as the group C^*-algebra of a C^*-simple group, as would be the case if H were normal in Γ.

In the above example we avoided the subtleties of finding a uniformly recurrent subgroup by providing a group with *finite* orbit under conjugation. However, the example can be adapted to have infinite orbit without too much work, but will no longer be finitely-generated and therefore not be covered by Elek's framework as discussed in [3]. Nonetheless, it stands to reason that the appropriate quotient $\mathscr{G}'_Z$ of the transformation groupoid gives rise to a generalised Elek algebra associated with a URS Z of a not-necessarily finitely-generated discrete group Γ.

Let Γ be the infinite direct sum

$$\Gamma := \bigoplus_{\infty} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$$

with subgroup $H := \bigoplus_\infty \mathbb{Z}_2$, again for a non-trivial C^*-simple group F. Since the conjugation is component-wise, the normaliser of H is readily identified as $\bigoplus_\infty \mathbb{Z}_2 \times F$, and its Weyl group as $\bigoplus_\infty F$, which is C^*-simple. We merely have to verify that H is uniformly recurrent in Γ. First note that the convergence in the Chabauty topology of $\mathrm{Sub}(\Gamma)$ is component-wise in $\mathrm{Sub}((\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F)$, and $\mathbb{Z}_2$ has finite, discrete orbit in $(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$. Further, note that the orbit of H

in Γ is given by a component-wise choice of conjugate of $\mathbb{Z}_2$ in $(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$. That is, the orbit contains exactly the subgroups of the form $\bigoplus_{i=1}^{\infty} K_i$ with each K_i one of the three conjugates of $\mathbb{Z}_2$ in $(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$ while $K_i = \mathbb{Z}_2$ for all but finitely many i. As the set $\mathscr{K}$ of all subgroups of Γ that are of the form $\bigoplus_{i=1}^{\infty} K_i$ for K_i arbitrary conjugates of $\mathbb{Z}_2$ is compact in $\mathrm{Sub}(\Gamma)$, it contains the orbit closure of H, and as the orbit of H is clearly dense, they coincide. Since any group of the form $\bigoplus_{i=1}^{\infty} K_i$ is conjugate to $\bigoplus_{i=1}^{\infty} K_i'$ if and only if $K_i = K_i'$ for all but finitely many i, any subgroup contained in $\mathscr{K}$ again has dense orbit in $\mathscr{K}$, so the orbit closure of H is minimal and therefore a uniformly recurrent subgroup. In conclusion, the orbit closure of $H \leq \Gamma$ is an example of a non-generic, infinite URS giving rise to a simple (generalised) Elek algebra.

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