Some notes on long range inventory problems.

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Introduction

An inventory system is a storage device which from time to time receives *inputs*, and from which *output* is removed. Usually the manager can control the inputs (when?, how much?) but there are situations where output is subject to control (water storage systems are the most common example). In either case there are associated costs and profits which must be optimized over some period. The difficulty of the decision lies in the interpretation af the term *some period*. Very frequently the process has no obvious point in time at which it will terminate and some mathematical device is required which will sum all future costs. *Dynamic Programming* is such a device, but in an infinite time period any replenishment policy will yield infite costs. There are two ways of keeping costs finite; the first is to apply a discount factor to future costs and the second is to consider the long run average costs per month. Realistically the discount factor must be close to one and the two appaoaches yield very similar policies in practice.

The Warehouse Problem

In what follows we will assume that decisions about replenishment are to be made once a month, and that there are costs associated with the quantity ordered (or produced). We also assume an income associated with sales and/or costs associated with holding stocks or failure to meet demand. As an example consider the problem of the owner of a warehouse of capacity H who has a stock of $s \leqslant H$ items. Suppose that in each of the months to come he can buy unlimited quantities or sell any

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amount up to his current inventory; the cost in month i being c_i and the sales price p_i. Let his planning horizon be n months and assume that replenishment ordered in each month is delivered after the sales have taken place. The problem of how much to buy and sell can be formulated as a linear programming problem as follows.

Let $x_i \geqslant 0$, $y_i \geqslant 0$ $(i = 1, 2, \ldots, n)$ be quantities bought and sold in month i, so the net profit P is given by:

$$P = \sum_{i=1}^{n} (p_i y_i - c_i x_i)$$

However x_i, y_i and subject to the constraints:

For $r = 1, 2, \ldots, n$ $\quad s + \sum_{i=1}^{r} (x_i - y_i) \leqslant H$ (capacity limitation)

For $r = 1, 2, \ldots, n$ $\quad s + \sum_{i=1}^{r-1} (x_i - y_i) - x_r \geqslant 0$ (no negative stocks).

Thus we have a problem in $2n$ variables with $2n$ constraints, and for $n = 12$, corresponding to a horizon of one year, we have a formidable calculation.

Let us define $f_t(u)$ to be the maximum profit starting with a stock of u in month $(n-t+1)$ and continuing for t months until month n. Then if we buy x and sell y in month n we have

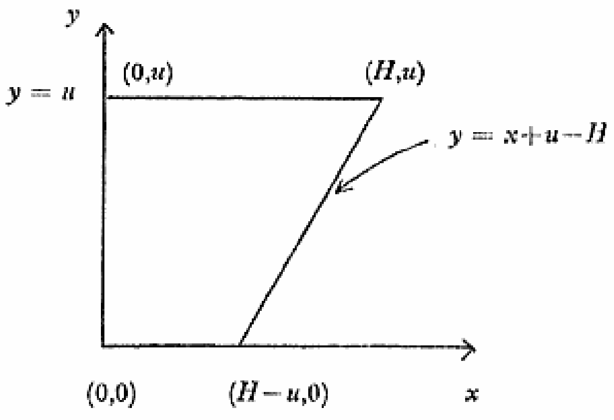
$$f_1(u) = \text{Max}\ \{p_n y - c_n x\}$$

where the maximum is over all x and y which satisfy

$$x \geqslant 0; \quad y \geqslant 0; \quad y \leqslant u; \quad u + x - y \leqslant H.$$

It is easy enough to see that the optimal values of x, y are one of the following: (See figure)

$$(0,0); \quad 0,u); \quad (H-u,0); \quad (H,u)$$



Since $p_n > 0$, $c_n > 0$ we see $x = 0$, $y = u$ and $f_1(u) = p_n u$.

Now if in month $(n-t+1)$ we start with u, buy x and sell y we start the following month with a stock of $u' = u + x - y$. Hence

$$f_t(u) = \text{Max}\ \{p_{n-t+1}\, y - c_{n-t+1}\, x + f_{t-1}\,(u + x - y)\}$$

where the maximum is over all x, y satisfying the previous restrictions. If we know $f_{t-1}(u')$ and it is linear, then the fuction to be maximized is linear and finding $f_{t-1}(u)$ is comparitively simple.

Now, we know $f_1(u)$ and it is linear so we can find $f_2(u)$. Let us assume $f_{t-1}(u)$ is linear in u, say

$$f_{t-1}(u) = A_{t-1}\, u + B_{t-1}$$

Then $f_t(u)$ has one of the forms below:

If $x = 0,\ y = 0;\ f_t(u) = A_{t-1}\, u + B_{t-1}$

If $x = 0,\ y = u;\ f_t(u) = p_{n-t+1}\, u + B_{t-1}$

If $x = H - u,\ y = 0;\ f_t(u) = c_{n-t+1}\, u + H\,(A_{t-1} - c_{n-t+1}) + B_{t-1}$

If $x = H,\ y = u;\ f_t(u) = p_{n-t+1}\, u + H\,(A_{t-1} - c_{n-t+1}) + B_{t-1}$

Thus in all cases $f_t(u)$ is a linear function of u, and we have a complete system of recursive calculation. A numerical example is given in reference 4, pages 274–278.

More general problems do not assume that the system is deterministic; usually instead of unlimited demand, we consider demand in any month to be subject to a probability distribution and we study expected costs (or profits). Suppose that we made decisions every month and that if we start with a stock s and order $q = S - s$, then the expected costs during the month of holding inventory plus the penalty costs of failing to meet demand are $I\,(s,S)$. Let the procurement cost be $k + cq$ if $a > 0$ and zero if $q = 0$. We wish to find a policy for determining q, or what amounts to the same thing, for determining S.

Let $f_1(s)$ be the minimum expected cost over the month following the decision.

$$f_1(s) = \underset{S \geq s}{\text{Min}}\ \{\, k\,(1 - \delta_s^S) + c\,(S - s) + I\,(S, s)\}$$

$$\text{where } \delta_s^S = 0 \quad S \neq s$$
$$= 1 \quad S = s$$

Thus $f_1(s)$ can be determined (tabulated!) with comparitive ease.

Now assume that the density function for demand during any month is $\Phi\,(x)$, independent of the previous month's demand. Assume also that failure to meet a demand results in a lost sale. Let $f_t(s)$ be the minimum expected cost over t months. We obtain the following equation for $f_t(s)$ in terms of $f_{t-1}(s)$.

$$f_t(s) = \underset{S \geq s}{\text{Min}}\ k\,(1 - \delta_s^S) + c\,(S - s) + I\,(S, s)$$

$$+ \int_0^S f_{t-1}\,(S - x)\ \Phi\,(x)\, dx + f_{t-1}\ \ (0) \int_S^{\infty} \Phi\,(x)\, dx$$

[We assume that a total of $q + s = S$ becomes available during the current month].
Starting with $f_1(s)$ we can obtain f_2, f_3, ... successively. So long as t remains finite we can obtain the optimal procurement policy. If t becomes infinite both sides of the equation tend to infinity for all policies. To avoid this we can argue that next month's costs should be discounted by a factor $\alpha = 1/(1 + i)$ where i is the interest per month. We then define $f_t(s)$ as the minimum discounted costs and obtain:

$$f_t(s) = \underset{S \geq s}{\text{Min}} \{ k\,(1 - \delta_s^S) + c\,(S - s) + I\,(S, s)$$
$$+ \alpha \int_0^S f_{t-1}\,(S - x)\, \Phi\,(x)\, dx + \alpha\, f_{t-1}\,(0) \int_S^\infty \Phi\,(x)\, dx \}$$

It can be shown that so long as $0 \leqslant \alpha < 1$ the sequence of functions f_1, f_2, ... converge to a function $f(s)$ and moreover f is independent of f_1; further, f is the unique solution of the equation obtained by dropping the subscripts in the last equation. Equations of this type have been discussed extensively. (See references 1, 2, 4).

The alternative to using a discount factor is to study the asymptotic form of $f_t(s)$. It can be shown that for any given procurement policy the total costs over t months have the asymptotic form $v(s) + gt$ where v and g depend on the policy and g is independent of s. We woud choose the policy to minimize g – the long run average cost.

We have

$$f_t(s) = v(s) + gt = k\,(1 - \delta_s^S) + c\,(S - s) + I\,(S, s)$$
$$+ \int_0^S v\,(S - x)\, \Phi\,(x)\, dx + v(0) \int_S^\infty \Phi\,(x)\, dx + (t - 1)\, g$$

Thus $\quad v(s) + g = k\,(1 - \delta_s^S) + c\,(S - s) + I\,(S, s)$

$$+ \int_0^S v\,(S - x)\, \Phi\,(x)\, dx + v(0) \int_S^\infty \Phi\,(x)\, dx$$

Now a given policy expresses S as a function of s so in the theory we van solve this equation for $v(s)$ and g. Actually we can only determine $v(s)$ up to an additive constant. (If $v = u(s)$ is a solution then clearly so is $v = u(s) + c$). However we are mostly interested in g and we may assume for example that $v(0) = 0$. In practice some form of numerical calculation is required and the following system due to Howard is suggested. (Reference3).

Let the units of s be so defined that s takes the values of 0, 1, 2, $\dots$, k. The effect of a procurement decision is to change s to s' with a known probability. Thus if p_x is the probability of demand for x and our policy calls for procurement of $S - s$, the probability of s' is $p_{S-s'}$. (Of course S may be a function of s).

Let p_{ij}^A be the probability, that if we use policy A, a stock of i at the end of one month will become a stock of j at the end of the next month Let K_i^A be the expected costs, including procurement costs, incurred during a month starting with stock i; let $f_i(n)$ be the minimum costs over n months. Finally let $f_i^A(n)$ be the costs associated with policy A.

Then $$f_i^A(n) = K_i^A + \sum_{i=0} p_{ij}^A\, f_j^A\,(n-1) \qquad (1)$$

and $$f_i(n) = \operatorname*{Min}_B\ \{K_i^B + \sum_{j=0}^{k} p_{ij}^B\, f_j(n-1)\} \qquad (2)$$

where the minimum is over all policies.

Howard shows that $f_i^A(n)$ has the asymptotic form

$$f_i^A(n) = ng^A + v_i^A \qquad (3)$$

and it is clear that the optimal policy is the one which minimizes g^A (the long run average cost).

If we insert (3) in (1) we have

$$ng^A + v_i = K_i^A + \sum_{j=0}^{k} p_{ij}^A\,[(n-1)\,g^A + v_j^A]$$

or $$v_i^A + g^A = K_i^A + \sum_{j=0}^{k} p_{ij}^A\, v_j^A. \qquad (4)$$

From (4) we may determine the v_i^A (up to an additive constant) and g^A. As we are primarily interested in g^A we may arbitrarily set $v_0^A = 0$ and solve (4) for $v_1^A, \dots, v_k^A$ and g^A. Once this is done we can search for a better policy than A as follows:

Let $$F_i^B(n) = \operatorname*{Min}_B\ \{K_i^B + \sum_{j=0}^{k} p_{ij}^B\, f_j^A\,(n-1)\}$$

$$= \operatorname{Min}\ \{K_i^B + \sum_{j=0}^{k} p_{ij}^B\ [v_j^A + (n-1)\,g^A]\}$$

$$= \operatorname{Min}\ \{K_i^B + \sum_{j=0}^{k} p_{ij}^B\, v_j^A + (n-1)\,g^A\}$$

The minimum is achieved for that policy B which minimizes $K_i^B + \sum^{k} p_{ij}^B\, v_j^A$. Howard shows that if $B = A$ for all i, then policy A

is optimal, in the sense that $g^A \leqslant g^C$ for any $C \not\equiv A$. If $B \not\equiv A$ we can solve equations (4) for v_i^B and g^B and repeat the minimization procedure. If at any stage there is a value of i for which

$$K_i^A + \sum_{j=0}^{k} p_{ij}^A\, v_j^A \leqslant K_i^B + \sum_{j=0}^{k} p_{ij}^B\, v_j^A$$

we keep policy A for that i, even if some other policy B yields equality. It may be shown that this process must converge in the sense that we must eventually find $B \equiv A$.

In many cases sales demand has a seasonable pattern. This may be included in the model by writing p_{ijt}^A for the probability that using policy A a stock of i at the start of month t becomes j at the start of month $t+1$, (mod 12). Of course in place of K_i^A and v_i^A we have K_{it}^A and v_{it}^A. $(t = 1, 2, \ldots, 12)$

We now define the following matrices and vectors:

$\boldsymbol{P}_t^A = [p_{ijt}^A]$ ($k+1$ sided square matrix).

$\boldsymbol{K}_t^A = [K_{0t}^A;\ K_{1t}^A;\ \ldots;\ K_{kt}^A]^T$ (column vector).

$\boldsymbol{v}_t^A = [v_{0t}^A;\ v_{1t}^A;\ \ldots;\ v_{kt}^A]^T$ (column vector).

$\boldsymbol{g}^A = [g^A;\ g^A;\ \ldots;\ g^A]^T$ (column vector).

In place of equations (4) we have

$$\boldsymbol{v}_t^A + \boldsymbol{g}^A = \boldsymbol{K}_t^A + \boldsymbol{P}_t^A \boldsymbol{v}_{t+1}^A \qquad t = 1, 2, \ldots, 11 \tag{5}$$

$$\boldsymbol{v}_{12}^A + \boldsymbol{g}^A = \boldsymbol{K}_{12}^A + \boldsymbol{P}_{12}^A\, \boldsymbol{v}_1^A \tag{6}$$

We can substitute for $\boldsymbol{v}_{12}^A$ from (6) into the last of (5) and then substitute for $\boldsymbol{v}_{11}^A$ in the next equation of (5) and so on, until we obtain:

$$\begin{aligned} \boldsymbol{v}_1^A &= \boldsymbol{K}_1^A + \boldsymbol{P}_1^A\, \boldsymbol{K}_2^A + \boldsymbol{P}_1^A\, \boldsymbol{P}_2^A\, \boldsymbol{K}_3^A + \ldots + \boldsymbol{P}_1^A\, \boldsymbol{P}_2^A \ldots \boldsymbol{P}_1^A\, \boldsymbol{K}_{12}^A \\ &- [\boldsymbol{I} + \boldsymbol{P}_1^A + \boldsymbol{P}_1^A\, \boldsymbol{P}_2^A + \ldots + \boldsymbol{P}_1^A\, \boldsymbol{P}_2^A \ldots \boldsymbol{P}_{11}^A]\, \boldsymbol{g}^A \\ &+ \boldsymbol{P}_1^A\, \boldsymbol{P}_2^A \ldots \boldsymbol{P}_{12}^A\, \boldsymbol{v}_1^A \end{aligned} \tag{7}$$

Now since each of $\boldsymbol{P}_t^A$ $(t = 1, \ldots, 12)$ is a stochastic matrix with rows adding to one, so are the products $\boldsymbol{P}_1^A\, \boldsymbol{P}_2^A \ldots \boldsymbol{P}_{12}^A$. As $\boldsymbol{g}^A$ is a column vector of indentical elements, g^A, we see that

$$[\boldsymbol{P}_1^A\, \boldsymbol{P}_2^A \ldots \boldsymbol{P}_{12}^A]\ \boldsymbol{g}^A = \boldsymbol{g}^A.$$

Thus (7) may be written

$$\boldsymbol{v}_1^A + 12\, \boldsymbol{g}^A = \boldsymbol{K}^A + \boldsymbol{P}^A\ \boldsymbol{v}_1^A \tag{8}$$

where $K_A = K_1^A + P_1^A\ K_2^A + P_1^A\ P_2^A\ K_3^A + \ldots. + P_1^A\ P_2^A \ldots. P_{11}^A\ K_{12}^A$
and $P^A = P_1^A\ P_2^A \ldots. P_{12}^A$

Equation (8) has exactly the same form as (4). It may be solved for v_1^A and g^A provided we first set $v_{1,1}^A = 0$. Once v_1^A is known, $v_{12}^A\ v_{11}^A \ldots$ v_2^A may be found by successive substitutions in equations (6) and (5).

The procedure for finding the optimizing policy is carried out by successive improvements on policy A, using the technique described for the non-seasonal case.

References

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where $K_A = K_1^A + P_1^A\ K_2^A + P_1^A\ P_2^A\ K_3^A + \ldots. + P_1^A\ P_2^A \ldots. P_{11}^A\ K_{12}^A$
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