On the Development of Utility Spaces for Multi-Goal Systems.

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As operations research becomes a more highly developed science and attempts to tackle significant problems of large, multi-goal systems, theory is required which can provide a framework for the optimization of such systems. This article attempts to contribute to such theory. In particular, the article discusses the development of utility functions which express the preferences of those responsible for the system, when these preferences or goals are expressed in several units of measure. A series of theorems is presented which state the necessary and sufficient conditions for the utility functions to have certain desirable forms. The article closes with a discussion of the practical application of the theory developed.

Introduction.

The operations researcher is often confronted with the problem of recommending a course of action which will optimize a utility function expressing the preferences of the decision-makers responsible for a system. Optimizing with respect to a system thus compels the operations researcher to consider the objectives associated with components of the system, objectives which may be in apparent conflict with each other, and which may be expressed in different units of measure. Furthermore, even the sub-systems may be characterized by multi-goal objectives, and thus by measures of performance which are expressed

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in different units. A decision-maker may specify, for example, that he is motivated by a profit incentive, but that he also pays much attention to certain other variables, such as customer goodwill, market share, sales level, time, personal prestige among his peers, and very likely, many more.

The symbolic representation of the utility function for the system is a fundemental task of the operations researcher. Sub-system optimization based upon utility functions of a single variable (e. g. profit) has been widely considered in the literature. However, without a theoretical treatment of the subject of utility functions for multi-goal systems, large system optimization is beyound our reach. Once we are to consider such systems, and to consider decisions which are of such significance that the resulting outcome may cause the system to move far from its existing state, then we must have a theoretical basis for developing analytic expressions for the system's utility. In other words, if operations research is to be able to consider total- versus sub-optimization, if it is to aid the decision-maker when he is faced with significant problems, then the operations researcher must have a well developed theory for developing utility functions for multi-goal systems. We hope that this paper will make some contribution to this important problem area.

One of the most widely referred to methods for analytically considering multi-goal systems is the conversion of all the performance measures into one measure, usually into units of money[1]). We shall refer to this method as Method A. For example, with Method A, so-called "transformations" are assumed to exist between the units used to describe customer goodwill, market share, sales level, time, etc., and the money units, thus converting all measures to their equivalent monetary values, while perhaps including some of the measures as constraints due to the difficulty of obtaining a proper transformation, or due to a notion that they are not important to the solution of the problem. If for each alternative course of action under consideration a joint probability density function for outcomes is developed, and if a utility function for money is developed, then the expected utility of each course of action can be obtained, and the alternative which maximizes the expected utility can be recommended as the optimal course of action.

There exists another widely used procedure for evaluating alternative actions when the performance measures of the possible outcomes are expresed in different units. This procedure is based upon the assumption

[1]) See Ackoff, R. L., *Scientific Method*, John Wiley and Sons, Inc., New York, 1962.

that, for each performance measure, a function describes the contribution of that measure to the overall utility of the system, and this function is independent of the levels of all other measures[2]). We shall refer to this method as Method B. The simplest example of such a method is the use of constant weights to represent the relative contribution of each measure; this is the form of linear value functions such as are used in linear programming problems. If such a utility function, based upon the assumption of independence or seperability, is developed and if a marginal probability density function for each outcome is developed, the expected utility for each alternative course of action can be obtained, and the alternative maximizing the expected utility can then be recommended as the optimal course of action.

It is hoped that the above comments on multiple-goal utility functions should suffice to introduce the subject matter of this article. In the remainder of the article, we have attempted to:

I. Analytically consider the calculation of expected utility and to introduce some notions which permit a precise discussion of the nature of the decision-making problem in a multi-goal system.

II. Examine the assumptions which underly Methods A and B, assumptions which the operations researcher might not always be willing to make were he aware of their implications. The mathematical proofs of the theorems presented are omitted from the paper due to considerations of space, and the desire to avoid confusing the non-mathematical reader.

III. Present a brief discussion of how, based upon the theorems presented, the operations researcher and the decision-maker, working together, may develop appropriate utility functions for operational decision-making in the "real world".

I. *The nature of the decision-making problem in a multi-goal system:*

We introduce the notion of a vector $(x_1, x_2, \ldots, x_m)$ as representing the state of a system, where each element of the vector represents some measure of a particular state variable, x_i, $i = 1, 2, \ldots, m$. We assume that to each state there can be associated a unique value, or utility,

[2]) This assumption of independence or separability is frequently used by Fishburn, P. C., in *Decision and Value Theory*, to be published in spring 1964, by John Wiley and Sons, Inc., New York.

$v = u(x_1, x_2, \ldots, x_m)$ and that a change in any of the state variables can result in a change in utility to the "individual" whose $m+1$, dimensional utility space, $(x_1, x_2, \ldots, x_m, v)$, is in question. "Individual" here might refer to a single person, or perhaps a group of people involved in the decision, such as the managers of the various sub-systems belonging to the overall system to be optimized.

We assume that when a decision-maker must choose between several available courses of action, and a set of state variables is considered as representing the outcome of a course of action, he will choose that action which maximizes his expected utility. That is, if the j^{th} course of action, c_j, yields an outcome state $(x_1, x_2, \ldots, x_m)$ with probability $dF_j(x_1, x_2, \ldots, x_m)$, the decision-maker acts so as to maximize

$$\int\int_{R_m} \cdots \int u(x_1, x_2, \ldots, x_m)\, dF_j(x_1, x_2, \ldots, x_m) \text{ over all } j = 1, 2, \ldots n;$$

where R_m is m-dimensional Euclidean space.

We will begin by considering the simple case of only two relevant state variables, and extend the results to m variables at a later stage. We introduce the following notions which occur frequently throughout the development.

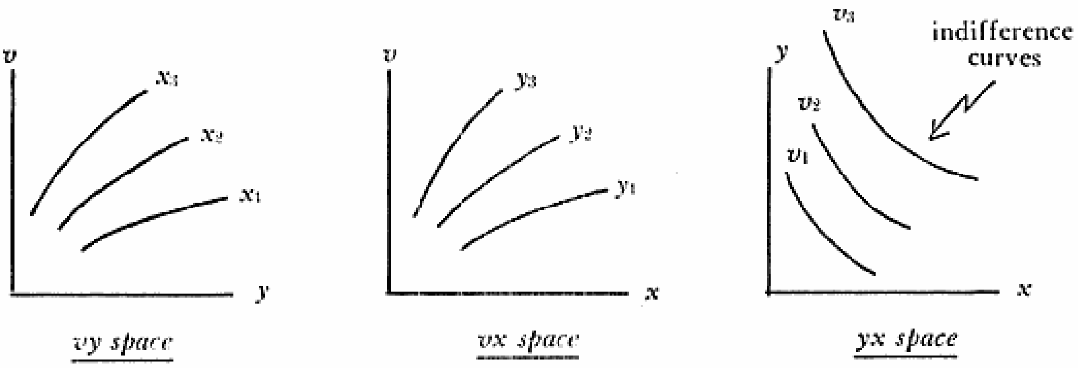
1. (x_0, y_0) represents the initial state of the system.

2. The courses of action available are $c_1, c_2, \ldots, c_j, \ldots, c_n$.

3. To the j^{th} course of action there is associated a joint probability density function $dF_j(x, y)$ which gives the probability that a course of action c_j will have the result that the state changes from (x_0, y_0) to (x, y). That is, $dF_j(x, y) = Pr[(x,y) \mid (x_0,y_0),c_j]$.

4. To each state (x, y) there can be assigned a relative value or utility, $v = u(x, y)$. To define the scale of v, we assign some arbitrary value to the present state (x_0, y_0) and some other arbitrary value to another state, say (x_1, y_1).

5. We will consider the following three spaces:



6. It is assumed that for any of these surfaces, we can develop the unique inverse functions of $v = u(x, y)$

(i) $y = Y(x, v)$ (ii) $x = X(y, v)$.

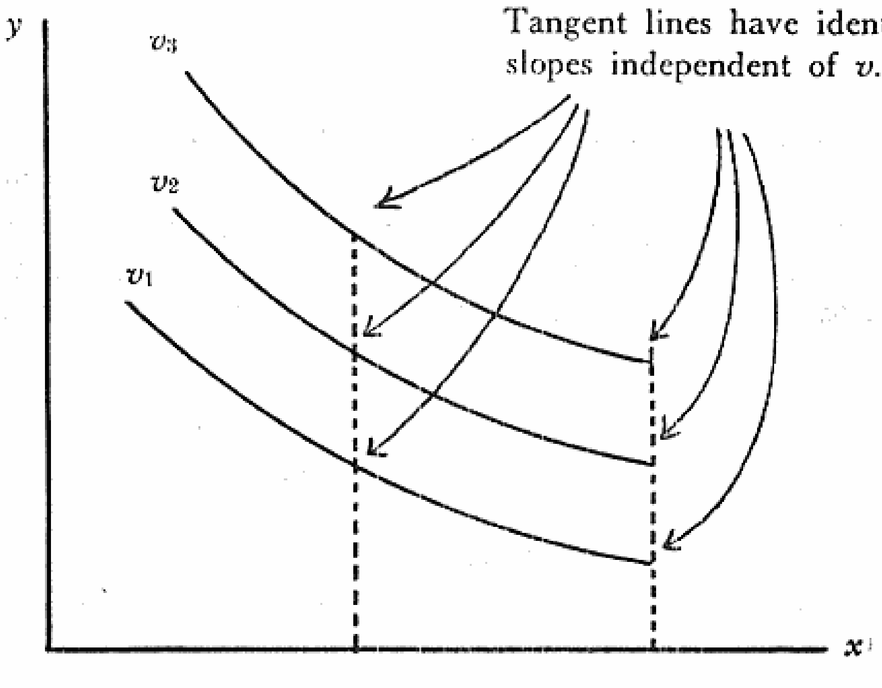
7. We define the term Vertically Parallel in the following way:

a. In *vy* space: $V.P._{vy} \rightarrow \dfrac{\partial v}{\partial y} = h_1(y)$

b. In *vx* space: $V.P._{vx} \rightarrow \dfrac{\partial v}{\partial x} = h_2(x)$

c. In *yx* space: $V.P._{yx} \rightarrow \dfrac{\partial y}{\partial x} = h_3(x)$

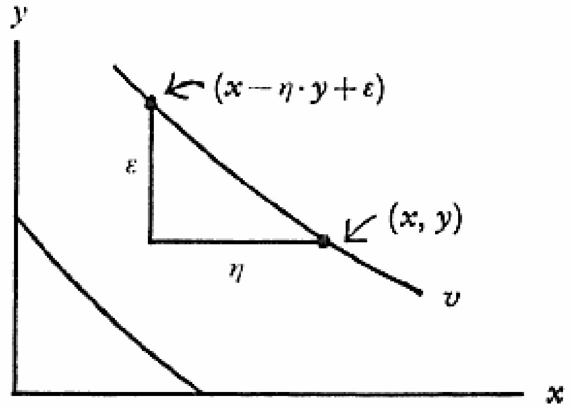
That is, two parametric curves are *V.P.* if their slopes are equal. For example, in *yx* space the indifference curves are vertically parallel if they are as shown below:



Horizontal parallelism is an identical notion in the horizontal direction. That is,

$$V.P._{vy} \Rightarrow H.P._{yv}$$
$$V.P._{vx} \Rightarrow H.P._{xv}$$
$$V.P._{yx} \Rightarrow H.P._{xy}$$

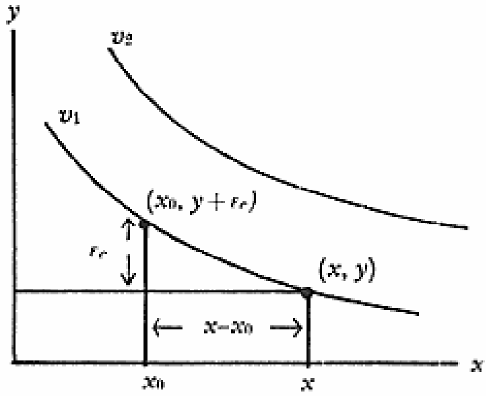
8. We define a trade-off function ε as follows:

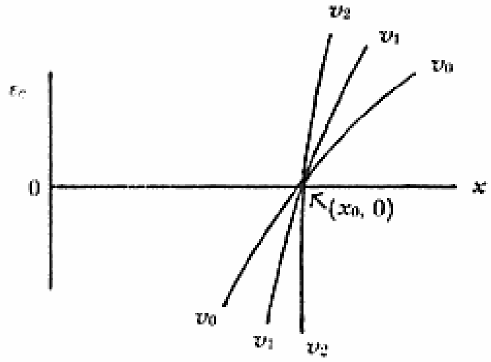


Given any η, define $\varepsilon = Y(x-\eta, v) - Y(x, v) = f(\eta, x, v)$ where η and ε can be either positive or negative. Clearly if $\eta = 0$, then $\varepsilon = 0$ and $f(0, x, v) = 0$.

Associated with the notion of the trade-off function ε is the notion of sliding along a particular indifference curve from a state (x, y) to a new state $(x-\eta, y+\varepsilon)$. The trade-off function then tells us how many additional units of y the decision-maker requires in order to remain on a particular indifference curve if he is to give up η units of x.

We define a special form of trade-off function, the Complete Trade-Off Functions, as $\varepsilon_c = f(x-x_0, x, v) = g(x_0, x, v)$. This function specifies the amount ε_c of y which the decision-maker requires to keep him on the same indifference curve if we take $\eta = x-x_0$, that is, if we are to slide from the state (x, y) to the state $(x_0, y+\varepsilon_c)$. Clearly when $x = x_0$, $g(x_0, x, v) = 0$.





9. We define a "utility line" to be a particular cut through the utility surface, passing through the initial state (x_0, y_0). The utility line $u(x_0, y)$ represents the intersection of the utility surface $u(x, y)$ and the plane $x = x_0$. Similarly, the utility line $u(x, y_0)$ represents the line formed by the interesection of the utility surface $u(x, y)$ with the plane $y = y_0$. It should be noted that a utility line is what Ackoff refers to as a "value function"[3]).

II. *Calculation of Expected Utility, Method A ($V.P._{yx_i}$ Approximation):*

If the functional form of the utility surface were known over the entire range of possible outcome states, (x, y), the computation and selection of the optimal course of action would be routine. To obtain this surface $v = u(x, y)$, the decision-maker would be required to specify a relative value to each possible outcome state. Ideally, the operations researcher might attempt to help the decision-maker to construct such a surface through a series of questions constructed so as to elicit his value structure. For example, he might help the decision-maker to generate his indifference curves by judiciously choosing intervals of x and y and then forming iso-value curves via some sort of best fit procedure.

However, the construction of the utility surface would require the decision-maker to place himself in many hypothetical outcome states with which he may have had little or no past experience. The conception of such states could be quite difficult for him. It would seem to be a logical statement that the further an outcome state (x, y) is from his initial state (x_0, y_0), the less certain would he be of its relative value; within a certain domain of outcome states near (x_0, y_0), we might expect his estimates to reflect his true relative values fairly well.

3) Ackoff, R. L. *Scientific Method*, John Wiley and Sons, Inc., New York 1962, p. 42.

The difficulty is that, as yet, there does not exist a proven methodology for generating a utility space, even within a well defined region of the initial state, (x_0, y_0). However, extensive work has been done in the area of generating "utility lines" for operational problems; see Ackoff, *Scientific Method*[4]). We shall show that under certain conditions the utility lines can be used to generate the entire utility surface. In cases where the utility lines cannot be used to generate the entire utility surface, we may be able to use them to generate a surface, which, through subsequent modifications by the decision-maker, will lead to an approximation to the actual surface.

Before outlining the procedures for developing the utility surface from the utility lines, it is necessary to develop a series of theorems:

Theorem I. A necessary and sufficient condition for the complete trade-off function $\varepsilon_c = g(x_0, x, v)$ to be of the form $\varepsilon_c = h(x_0, x)$, independent of v, is:

$$u(x, y) = u(x_0, y + h(x_0, x))$$

where $u(x_0, y)$ is the utility line for outcome y.

Corollary I.

$$\varepsilon_c = h(x_0, x) \Rightarrow \begin{cases} (1) & V.P._{\cdot yx} \\ (2) & H.P._{\cdot vy} \end{cases}$$

Theorem II. If $u(x, y)$ is such that the indifference curves are vertically parallel, then there exists a unique transform between the units of the state variable x and the units of the state variable y, independent of y and given by

$$\varepsilon_c = h(x_0, x)$$

or, equivalently,

$$V.P._{\cdot yx} \Rightarrow \varepsilon_c = h(x_0, x)$$

Interpretation of Theorems I and II:

If we are willing to make the assumption of vertically parallel indifference curves, then we can transform any outcome state (x, y) to a state $(x_0, y + \varepsilon_c)$ having the same value (i. e., lying on the same indif-

[4]) ibid. Chapter 3.

ference curve) but in a plane such that we know the utility line $u(x_0, y)$. We can then generate the entire utility space, which will be of the form

$$u(x, y) = u(x_0, y + h(x_0, x)).$$

Conversely, if we postulate that we can transform the units of state variables x into the units of state variable y by a transform function $\varepsilon_c = h(x_0, x)$, we are implying that the utility surface is of a special form, namely having the properties of $V.P._{yx}$ and $H.P._{xy}$.

The implications of the existence of such a transform, which does not depend on the state variable y, is that the state variable x is important to the decision-maker only because it can be converted directly into units of state variable y. If the importance of x is dependent on the level of y, or if, as will be shown later, the decision-maker does implicitly or explicitly value x with respect to other state variables in the system, then the indifference curves cannot be $V.P._{yx}$ and the unique transform of x into units of y cannot be made.

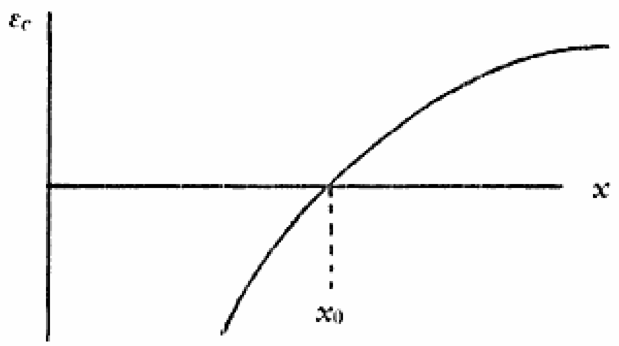
Illustration:

Suppose the decision-maker is the president of a firm and we define

$$x = \text{market share (in percent)}$$

$$y = \text{profit rate (in kroner per unit time).}$$

Then, the existence of a complete transform function ε_c (same for all



levels of y) implies that the market share is of importance to the decision-maker only in so far as it affects profit rate. In many cases this could quite possibly be true, particularly in a region not too far removed from the initial state, (x_0, y_0). If, however, the value the decision-maker assigns to an increase in market share is dependent upon the level of the profit rate, or if he also values an increase in market share because it enhances his prestige and/or self-satisfaction, or if it bears some relationship to any other component of what we might abstractly consider as his entire value space, then his indifference curves will not be parallel, and no such transformation will exist. This is not to say that there is no relation between percent of market and profit rate. In fact, if sufficient data is available such a relationship can often be established statistically (see for example Ackoff, *Scientific Method*, pp. 78–79)[5]). The importance of the foregoing theorems is that such a relationship is equivalent to ε_c only when the indifference curves are vertically parallel.

Interpretation of Parallelism in m-dimensional Value Space:

In general, we may consider a decision-maker as having a value space consisting of state variables $x_1, x_2, \ldots, x_m, y$ where y is used to signify some important measure, usually in units of money (e. g., profits, costs, etc.). We can extend Theorem I to the case of m dimensions as follows:

Theorem III. A necessary and sufficient condition for the existence of a set of complete trade-off functions (or transform functions) $\varepsilon_{jc} = h_j(x_{j0}, x_j)$, $j = 1, 2, \ldots, m$, which transform units of x_j into equivalent units of y, is:

$$v = u(x_1, \ldots, x_m, y) = u\left(x_{10}, \ldots, x_{m0}, y + \sum_{j=1}^{m} h_j(x_{j0}, x_j)\right)$$

where x_{j0} is the initial value of state variable x_j, $j = 1, 2, \ldots, m$, and $u(x_{10}, \ldots, x_{m0}, y)$ is the utility line for y developed for all other state variables held at their initial values.

Similarly, in analogy to the earlier corollary and theorem in three dimensional $(x, y, u(xy))$ space we state the following:

Corollary I.

$$\varepsilon_{jc} = h_j(x_{j0}, x_j), j = 1, \ldots, m \Rightarrow \begin{cases} (1) \quad V.P._{yx_j}, \; j = 1, \ldots, m \\ (2) \quad H.P._{xy} \end{cases}$$

Theorem IV.

$$V.P._{yx_i} \Rightarrow \varepsilon_{ic} = h_i(x_{i0}, x_i), \; i = 1, 2, \ldots, m$$

[5]) ibid.

Summary of Theorems I–IV:

We now have developed the necessary theoretical concepts to generate the entire $m+2$ dimensional $(x_1, \ldots, x_m, y, v)$ utility space under the assumption of $V.P._{yx_i}$ which enables us to transform each outcome state variables into equivalent units of state variable y (usually measured in units of money). The method of generation is simply to replace y in its utility line function by

$$y + \sum_{i=1}^{m} h_i(x_{i0}, x_i).$$

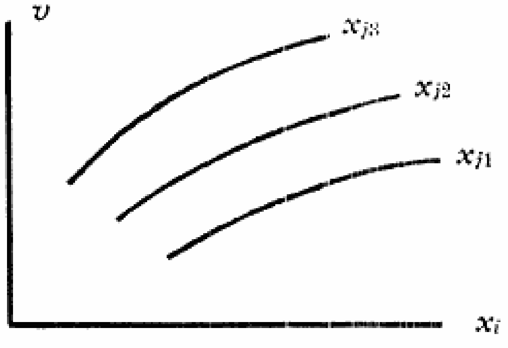
The expected utility of the j^{th} course of action is thus

$$E(v) = \iint_{R_{m+1}} \cdots \int u\left(x_{10}, \ldots, x_{m0}, y + \sum_{i=1}^{m} h_i(x_{i0}, x_i)\right) dF_j(x_1, \ldots, x_m, y).$$

We can now refer to Method A as $V.P._{yx_i}$ approximation.

Method B ($V.P._{vx_i}$ Approximation):

When, for one reason or another, $V.P._{yx_i}$ approximation does not give a good fit (i. e., the indifference curves are not parallel), a second type of approximation is to assume vertical parallelism in all vx_i spaces, $x_1, x_2, \ldots, x_m$.



$j = 1, 2, \ldots, m$

$i = 1, 2, \ldots, m$

$V.P._{vx_i} \Rightarrow \dfrac{\partial v}{\partial x_i} = c_i(x_i)$, where $c_i(x_i)$ is a function of x_i only. Since this must hold for all values of the state variables, it holds for the case when all other state variables take on the initial conditions.

Thus

$$c_i(x_i) = \frac{\partial}{\partial x_i} u(x_{10}, \ldots, x_i, \ldots, x_{m0})$$

where $u(x_{10}, \ldots, x_i, \ldots, x_{m0})$ is the utility line for x_i generated with the decision-maker's other state variables held constant at their initial values.

Hence the following theorem:

Theorem V. A necessary and sufficient condition for $V.P._{vx_i}$, $i = 1, 2, \ldots, m$, is that the utility surface be

$$u(x_1, \ldots, x_m) = v_0 + \sum_{i=1}^{m} M_i(x_i)$$

where $M_i(x_i)$ is the marginal utility of x_i and is given by

$$M_i(x_i) = u(x_{10}, \ldots, x_i, \ldots, x_{m0}) - v_0$$

$$i = 1, 2, \ldots, m; \quad v_0 = u(x_{10}, \ldots, x_{m0}).$$

Comments on Theorem V:

The $V.P._{vx_i}$ condition might be described as a "separability" or "independence" condition where the value contributed by each state variable depends in no way upon the levels of other state variables. It is difficult to conceive of such a relation holding over the entire utility space. However, it may give a good approximation within a decision domain near the initial state $(x_{10}, \ldots, x_{m0})$.

If we assume the V.P. $_{vx_i}$ condition then the calculation of expected utility is *greatly* simplified for we no longer require a joint density function as with $V.P._{yx_i}$ conditions, but need only the marginal probability density functions, $dF_i(x_i)$. That is, for a given course of action,

$$E(v) = v_0 + \int\int_{R_m} \cdots \int \sum_{i=1}^{m} M_i(x_i)\, dF_i(x_1, \ldots, x_m)$$

$$= v_0 + \sum_{i=1}^{m} \int_{R_i} M_i(x_i)\, dF_i(x_i).$$

Combination of $V.P._{yx_i}$ and $V.P._{vx_i}$ Approximations:

It is interesting to note the effect of assuming vertical parallelism in both yx space and vx space, or equivalently, assuming both the one-to-one transformation functions and the "independence" of state variables. It turns out that both assumptions combined imply a linear utility curve in y. Hence, we no longer have a problem in value theory.

In fact an even weaker condition, namely, $V.P._{vy}$, when combined with $V.P._{yx_i}$, $i = 1, \ldots, m$ implies $V.P._{vx_i}$, hence "independence". Therefore $V.P._{yx_i}$ and $V.P._{vy}$ combined imply a linear utility curve in y.

Theorem VI. The combined properties of $V.P._{yx_i}$, $i = 1, 2, \ldots, m$ and $V.P._{vy}$ are necessary and sufficient for a utility surface to be of the form $v = v_0 + k_1 (y - y_0) + k_1 \sum_{i=1}^{m} h_i (x_{i0}, x_i)$ where k_1 is a constant and $h_i (x_{i0}, x_i)$ is the complete trade-off function for x_i implied by $V.P._{vx_i}$.

Corollary I. The combined properties of $V.P._{yx_i}$, $i = 1, 2, \ldots, m$, and $V.P._{vy}$ are necessary and sufficient for "independence" as defined by $V.P._{vx_i}$, $i = 1, 2, \ldots, m$.

Corollary II. Under combined properties of Theorem VI, the utility v is just the sum of initial state value, v_0 and the marginal utilities of y and of x_i which in this case are identical with the transform functions ε_{ic}, $i = 1, 2, \ldots, m$.

Comments on Theorem V:

From the point of view of value theory, applying the assumptions of Theorem VI results in a degeneracy. Under the assumptions of this theorem, if we for the moment assume that y represents profit, then we maximize expected utility by maximizing expected profit, taking into account that we transform all other state variables into their equivalent profit. This is the same as saying that we assume that the decision-maker's utility line for profit is linear. It will be noted that in most applications of operations research, "intangibles", such as shortages in inventory problems, machine down-time in replacement, customer waiting time in queueing, etc., are transformed into equivalent costs and revenues, and then the "optimal" policy is chosen as that which maximizes expected profit. Thus, when they follow such a procedure, the operations researchers implicitly assume that the decision-maker's utility line for money is linear!

We can now formally state, in terms of value theory, the implicit assumptions we are making about the nature of the utility space when we follow these standard procedures; namely, we assume vertical parallelism of all indifference curves, vertical parallelism in all vx_i spaces, and horizontal and vertical parallelism (hence linearity) in vy space.

III. *A Suggested Procedure for the Practical Establishment of Utility Surfaces.*

The motivation behind this section is to show how, using the theorems presented earlier, the operations researcher may help the decision-maker to establish his appropriate utility surface. The general outline of the procedure is to consider whether:

> Method A (i. e., $V.P._{yx_j}$) is an appropriate assumption
>
> Method B or "independence" of utilities (i. e., $V.P._{vx_j}$) is an appropriate assumption.

In the discussion that follows, we restrict ourselves to the case where the decision-maker's value space is assumed to be adequately described by two state variables, x and y, and the corresponding surface $v = (x,y)$[6]).

By considering only two state variables, the presentation is simplified and we can take advantage of graphic demonstrations. Many of the results can be extended into a general n-dimensional state space but this is beyond the scope of this section.

The assumption is made that it is difficult for the decision-maker to describe his feeling about relative values, trade-offs, etc., when he is removed from his present state, described by x_0, y_0 and $v_0 = u(x_0, y_0)$. Therefore, we attempt to develop a procedure for eliciting information about the actual utility space by continually allowing the decision-maker to refer to at least one of x_0, y_0 or v_0.

Depending on the nature of the problem at hand, the operations researcher may wish to check for $V.P._{yx}$ parallelism first or for $V.P._{vx}$ (or $V.P._{xy}$) parallelism first, as he may consider one or the other to be

[6]) To select relevant state variables, the partial derivates of v with respect to these variables must be different than zero. That is, if x, y and $u(x,y)$ are sufficient to describe the value spacer over some reasonable decision-domain, the decision-maker considers that

$$\frac{\partial u}{\partial z}(u, y, z) = 0$$

for any other state variable z.

more desirable. It is likely that *if* he is considering many state variables, he would wish to check first for $V.P._{vx}$ (or $V.P._{vy}$) parallelism as this form of parallelism would permit him to use marginal density functions for x and y when calculating expected utility of an action. On the other hand, he may wish to have yx parallelism when dealing with a state space described by only a few state variables.

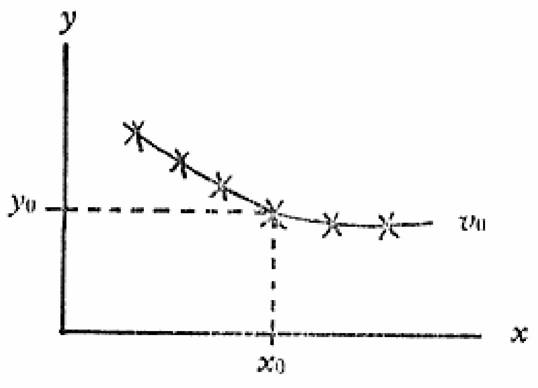
(A) Determining appropriateness of an assumption of parallelism in yx space:

It is clear from the theorems that if there exists $V.P._{yx}$ then there exists a unique complete trade-off function which can be used to translate units of x into units of y, independent of the actual magnitude of the state variable y (or y into units of x in the case of $V.P._{xy}$).

Thus, if such transformations are considered as proper, the value space describable by x, y, $u(x, y)$ can be equally described by y', $u(x_0, y')$, where $y' = y + h(x_0, x)$, which is far easier to generate than the former space. This becomes all the more true when the number of state variables increases. However, when the number of state variables becomes large, the assumption of "independence" is more desirable mathematically than that of yx parallelism; the latter requires the use of a joint density function associated with each course of action in order to determine expected utility of action, while the former requires only the marginal density functions.

The following steps are suggested to determine the appropriateness of the assumption of parallelism in yx space.

1. Generate the indifference curve v_0 which passes through (x_0, y_0). This can be done by presenting the decision-maker with a series



of different states of x and asking him to select the corresponding states y which would permit him to be indifferent between remaining at (x_0, y_0) or moving to (x, y), a state having the same value, v_0, to him.

2. Hold x at x_0 and generate $u(x_0, y)$ by the Case Method or some other suitable method. Assume a value for v_0, say v_0 equal 100, and assume a value for some other state (x_0, y_1) say v_1 equal 200. This is sufficient to uniquely define a scale of relative values for the utility line $u(x_0,y)$. See Figure 2 below.

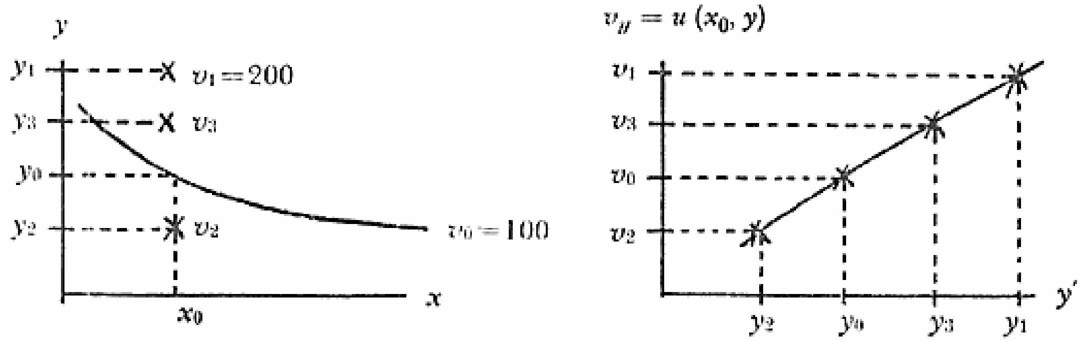


Figure 2

3. Holding y at y_0, determine the states (x, y_0) which correspond to the same relative value as the points (x_0, y_1), (x_0, y_2) . . . . . These points can be obtained by asking the decision-maker what value of x_1 he would require to be indifferent between being at the state (x_0, y_1) and the state (x_1, y_0), etc. This permits determination of $u(x_1, y_0)$, etc. See Figure 3 for the resulting points of equal value for $v_s = u(x, y_0)$.

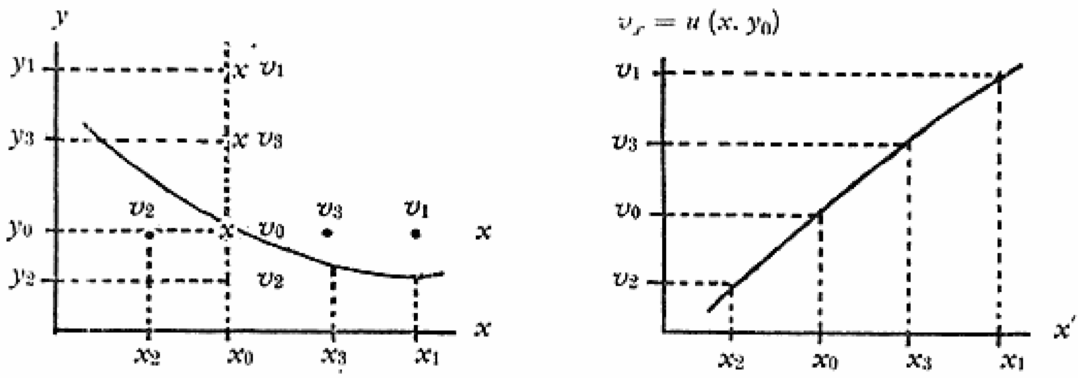


Figure 3

4. Check for parallelism in yx space. This can be done by sliding the indifference curve for v_0 upward and downward and seeing how well it passes through iso-value points. If it passes through these points reasonably well, we can assume $V.P._{yx}$. (Similarly we check for $H.P._{yx} \Rightarrow V.P._{xy}$). In order to determine the "goodness of fit" various

procedures might be used, including the "eyeball" method, or more sophisticated tests.

5. If yx parallelism exists, this implies that there exists a unique complete trade-off function and all that is required for decision-making which is consistent with the objective of maximizing expected utility exists in the form of the complete trade-off function ε_c, the joint probability distribution associated with a course of action, and the utility line $v = u(x_0, y)$.

(B) Determine the appropriateness of the independence assumption:

1. From (A) we have developed both $u(x_0, y)$ and $u(x, y_0)$, the so-called utility lines. Furthermore, these curves are appropriately scaled with respect to each other via steps (A–2, 3). We now develop an analytic expression for each of these curves, perhaps in terms of polynomials, etc.

2. Assuming the property of independence we then derive the mathematical form of the indifference curve with v equal v_0. That is, we solve analytical or numerically for the values of x and y which will be such that $v_x + v_y$ equal v_0.

3. Compare this plot of y versus x with that graph of y versus x obtained in step A–1 (see Figure 1). It will be recalled that that plot of the indifference curve v_0 was obtained via a direct questioning of the decision-maker.

4. If, based upon some statistical or other form of test procedure, it is concluded that the agreement between these curves is sufficient to allow the assumption of independence (i. e., $V.P._{r.c_j}$), then, given the marginal distribution of x and y for action c_N we can immediately calculate the expected utility of choosing the course of action as

$$E(v) = \int_{R_x} v_x \, dF_N(x) + \int_{R_y} v_y \, dG_N(y)$$

where R_x and R_y represent the decision domain for x and y, and where $dF_N(x)$ *and* $dG_N(y)$ represent the marginal density functions of x and y respectively with the N^{th} course of action, c_N, and where v_x and v_y are the utility lines established from steps A–2 and A–3.

(C) If neither assumption of parallelism nor independence is acceptable:

If neither of the assumptions appear at first to be acceptable by the test criteria, then the decision-maker and the operations researcher must generate the value space $v = u(x, y)$, either by placing the decision-maker in a large number of hypothetical situations (i. e., states) and then attempting to assign relative values to each such state, or else by modifying the utility surface which would be generated under either assumption, until it is acceptable to the decision-maker. The first of these alternatives will be a very difficult procedure, even in the case of only two relevant state variables, and almost impossible with more than two such state variables. (Clearly then, before the two assumptions are both rejected, rejection must be based upon some economic considerations of the cost of error, cost of additional investigation, etc. It has not been our goal to consider the test procedures used in accepting or rejecting the assumptions, but this is a significant problem if the above approaches are to be followed).

The second of the two alternatives, that of modifying the utility surfaces which would be generated under either the assumption of yx parallelism or of independence, is therefore a much more appealing alternative. Certainly the fact that the decision-maker and the operations resarcher could have originally rejected the notions of parallelism or independence, supplies information which can be used to modify the surface so as to more closely represent the actual, but as yet unknown, utility surface.

Furthermore, it may be found that one or the other of these assumptions hold in part of the decision domain and thus permit the description of this part of the surface with information already available. This could then provide a lead into determining how the remainder of the surface should be changed so as to more accurately represent the decision-maker's true utility surface.