Notes on the Theory of Games and the Social Sciences

By Royall Brandis *

I

Although many applications of mathematics to the social sciences have been attempted and some have certainly cast a helpful light on problems in the area, a fundamental difficulty remains. It is a commonplace that mathematics may be thought of as a language, but it may be more accurate to think of it as a form of logic. Many mathematical investigations of the last half-century have been devoted to tracing the logical basis and the logical implications of mathematical axioms and theorems.

The great difficulty for the social sciences is that mathematical development has always been along the lines of uncovering the logic of nature and of natural (i.e. physical) relationships rather than the relationships between human beings, the subject matter of the social sciences. Some mathematical discoveries have been the result of investigations into nature, for example, the beginnings of geometry in the Egyptian attempt to solve the problem of land measurement. Here, there is no question of the close connection between the physical problem and the mathematical tools developed to solve it.

A more puzzling case is that of mathematical discoveries which were spun out of the logical investigations of pure mathematicians without any reference to the problems of nature. In a surprising number of cases the mathematics developed in this fashion has later been used successfully to explain hitherto unexplainable natural processes and relationships. An excellent example of the latter is the use of Riemannian, non-Euclidian geometry to describe the physical universe (Einstein). Such applications of mathematics raise questions which this paper will not explore: Did the inventive mathematician possess the germ of the phy-

* Associate Professor of Economics, University of Illinois.

3–1956

Notes on the Theory of Games and the Social Sciences

By Royall Brandis *

I

Although many applications of mathematics to the social sciences have been attempted and some have certainly cast a helpful light on problems in the area, a fundamental difficulty remains. It is a commonplace that mathematics may be thought of as a language, but it may be more accurate to think of it as a form of logic. Many mathematical investigations of the last half-century have been devoted to tracing the logical basis and the logical implications of mathematical axioms and theorems.

The great difficulty for the social sciences is that mathematical development has always been along the lines of uncovering the logic of nature and of natural (i.e. physical) relationships rather than the relationships between human beings, the subject matter of the social sciences. Some mathematical discoveries have been the result of investigations into nature, for example, the beginnings of geometry in the Egyptian attempt to solve the problem of land measurement. Here, there is no question of the close connection between the physical problem and the mathematical tools developed to solve it.

A more puzzling case is that of mathematical discoveries which were spun out of the logical investigations of pure mathematicians without any reference to the problems of nature. In a surprising number of cases the mathematics developed in this fashion has later been used successfully to explain hitherto unexplainable natural processes and relationships. An excellent example of the latter is the use of Riemannian, non-Euclidian geometry to describe the physical universe (Einstein). Such applications of mathematics raise questions which this paper will not explore: Did the inventive mathematician possess the germ of the phy-

* Associate Professor of Economics, University of Illinois.

3–1956

sical theory, perhaps subconsciously, when he developed the mathematics? Did the physical scientist who made use of the mathematics press his concept of nature into a mold to which the available mathematical analysis would apply? Are the logical operations of the human mind, or at least of the minds of pure mathematicians, in some way governed by the construction of the physical world?

What all this seems to suggest is that we cannot be sure whether the failure of mathematics to make much contribution to the social sciences is due to the nature of mathematics or to the nature of the subject matter of social science. Certainly we cannot assume that the logic of the physical world is applicable to the logic of human social relations unless we assume that the latter are but one aspect of the former or that both fit into some higher framework which we do not understand nor have any very reliable evidence of.

II

Viewed in the light of the above remarks, the Theory of Games holds a unique position in the history of mathematics. For nowhere outside of the human social world do we find anything analogous to a game. We do not mean by this that probabilities rather than deterministic concepts are not a part of nature. Indeed, modern physical theory is based upon a probability structure, but these probabilities are of a very elementary type. *They are not the probabilities which arise from "A" asking himself the question: considering "B"'s reaction to each of the possible lines of action open to me, what is my optimal move?* This question, an ordinary one in man's relations with other men, has no counterpart in nature. It may be that poker is the indisputable evidence that man is, in a fundamental way, different from other living things and unlike the non-living universe.

III

A simple approach to an understanding of the Theory of Games is by way of consideration of a coin-matching contest between two persons. Let us call them "A" and "B" and establish the rules of the game that "A" shall win when the two coins match and "B" shall win when they do not match.

Suppose "A" decides on the following strategy: to hold heads twice in succession, then tails once, then heads twice in succession, *ad infinitum.* "B", by observation, can discover this strategy after a few trials and, thereafter, win each time as long as "A" keeps to this strategy. No matter how elaborate a strategy "A" may work out, he always leaves "B" the opportunity of discovering the strategy and profiting from his discovery. How might "A" play so as to prevent this possible loss? The answer, of course, is that "A" should make his choices completely random, say by flipping the coin in the air and allowing it to lie as it falls. Note that, if "A" does this, there is no strategy which "B" can devise which will prevent "A" from breaking even in the game.

This example may be used to illustrate another aspect of the Theory of Games. Suppose the *rules* of the game require "A" and "B" to flip their coins into the air and let them lie as they fall. We would then have a *game of pure chance* and about such games the theory of games has nothing to say. For the theory of games is concerned only with *games of strategy*, that is, games in which intelligence and skill play a part.

We may remark again how appropriate the Theory of Games appears to be for the social sciences. Games of chance have, of course, been studied mathematically at least since Cardan (1501–1576) and the probabilty mathematics arising from these studies has been very useful in the natural sciences. Certain social phenomena have also been analyzed in this fashion, the most widespread example, of course, being insurance in all its forms. But the major problems of social science can seldom be profitably analyzed through the mathematics of games of chances because these problems often arise in a context in which human intelligence plays a significant role–a role that cannot be submerged in the Law of Large Numbers.

IV

The heart of the Theory of Games lies in what is called the "pay-off matrix" which, with the rules of the game, determines the optimal strategies of the players, gives the reward (positive or negative) for each player for each line of acition open to him and takes into consideration the lines of action open to this opponent. An example will clarify this point. Assume the following to be all the possible results of each play of a game, the plays being made in ignorance of the opponent's corresponding play:

(1) If "A" chooses red and "B" chooses red, "B" pays "A" \$5.
(2) If "A" chooses red and "B" chooses black, "B" pays "A" –\$2 (that is, "B" wins \$2 from "A").
(3) If "A" chooses black and "B" chooses red, "B" pays "A" \$10.
(4) If "A" chooses black and "B" chooses black, "B" pays "A" \$1.

The pay-off matrix would be: (in terms of "A"'s winnings)

		"B" red	"B" black
"A"	red	+ 5	− 2
	black	+10	+ 1

In this game, the simplest that can be constructed, the strategies chosen are immediately obvious. "A" and "B" will both chose black on every play. We generalize the reason for this in terms of the pay-off matrix as follows: Since for "A" every reward in the black *row* is greater than the corresponding reward in the red *row*, "A"'s optimal play is black. Since for "B" every reward in the black *column* is greater than the corresponding reward in the red column, "B"'s optimal play is also black. "B" will, therefore, lose $1 to "A" on each play of the game, but this is the best he can do, *taking into account the strategies available to "A" as well as those available to "B"*.

V

It might be useful to set up a real life "game" of this type. Let "A" be Eisenhower and "B", Stevenson, and the game be seeking the U.S. presidency in the 1956 election. The lines of action open to each are ignoring or discussing the President's health in the election campaign. Red is ignoring Eisenhower's physical condition, black is discussing it fully in the campaign. The pay-off matrix, above, can be interpreted as a measure of vote gain or loss on this campaign issue. The numbers in the pay-off matrix do not refer to actual votes, but to relative influences on the voting decisions of the electorate. The pay-off matrix is now interpreted as follows:

(1) If both Eisenhower and Stevenson ignore the issue of the President's health, Eisenhower will gain +5 (relative to Stevenson) because the voters will feel the issue to be unimportant if it is ignored in the campaign.

(2) If Eisenhower ignores the issue of his health and Stevenson discusses it, Eisenhower will gain −2 (that is, the positive gain will be Stevenson's) because voters will feel that Eisenhower cannot afford to meet Stevenson's discussion of the health issue because the President's health is not good.

(3) If Eisenhower discusses his health while Stevenson ignores the

issue, Eisenhower will gain +10 because voters will discount the question of the President's physical fitness if his opponent does not appear able to meet Eisenhower's arguments.

(4) If both Eisenhower and Stevenson discuss the issue of physical fitness, Eisenhower will gain +1, because his willingness to discuss his side of the matter in opposition to Stevenson's arguments will lead some voters to conclude that he is indeed, able to carry the burdens of the presidency.

Assuming that both Eisenhower and Stevenson know the pay-off matrix, the play of each is clear. Both will discuss the President's health as a campaign issue. This is the optimal play for each, *taking into account the lines of action open to the other.**

* For an example of the use of the Theory of Games in analysis of a Danish political contest see: Gustav Leunbach, "Landstingsvalget som et strategisk spil," *National-økonomisk Tidsskrift*, 1950, pr. 201 ff. I am indebted to my colleague, Professor Hans Brems, for this reference.

VI

Suppose we now take a more complicated pay-off matrix–say that of the coin-matching game described in III. The pay-off matrix would be:

		"B" Heads	Tails
"A"	Heads	+ 1	− 1
	Tails	− 1	+ 1

There is, for this game, no optimal solution that requires "A" or "B" to make the same play each time. As a matter of fact, as we saw, if either player should play in a consistent fashion, the other could then win on every play. The optimal strategy for "A" is, therefore, to play heads with a probability of ½ and tails with a probability of ½, that is to say, his playing should be perfectly random. This notion of assigning probabilities to each of the various possible plays is the key to understanding much of the theory of games.

Notice the application if we change our pay-off matrix slightly as follows:

		"B" Heads	Tails
"A"	Heads	+ 1	− 1
	Tails	− 1	+ 2

In other words "A" not only wins but gets a bonus of +1 whenever both "A" and "B" hold tails. But this does not mean that "A" should hold tails on every play of the game, for B" could then win by holding heads on every play. On the other hand, "B" should not adopt the strategy of holding heads on every play (assuming "A"'s strategy now to be unknown to "B"), for "A" could then win each time by holding heads also.

What, then, would be the optimal strategy for "A"? Suppose "A" plays heads with frequency x and plays tails with frequency 1 – x, and suppose "B" plays heads with frequency y and plays tails with frequency 1 – y. Then the mathematical expectation of "A" in winning is:

$$E(x, y) = 1\,xy - 1x\,(1-y) - 1\,(1-x)\,x\,y + 2\,(1-x)\,(1-y)$$

Through elementary algebra we have:

$$\begin{aligned} E(x, y) &= 5\,xy - 3x - 3y + 2 \\ &= 5\,(xy - 3/5x - 3/5y) + 2 \\ &= 5\,(x - 3/5)\,(y - 3/5) + 1/5 \end{aligned}$$

It can be easily seen that, if $x = 3/5$, the value of the first term, above, becomes 0 no matter what may be the value of y. "A"'s expectation of winning then becomes +1/5. Therefore, the optimal strategy for "A" is to choose heads with a probability of 3/5 for he may then guarantee that he will win, on the average, 1/5 on each play. In other words, if there are a large number of plays, "A" should choose heads three out of five times. In this particular game, "B" should also choose heads with a frequency of three out of five for by doing so he can hold "A"'s winnings on each play to 1/5 which is the best he can hope to do since the pay-off matrix is unfavorable to him.

Intuitively, we would feel that "A" should choose tails more often than heads since his winnings on a "tails-tails" result carries a bonus. But, on further thought, it can be seen that if he plays tails with a higher frequency than two out of five, "B", by increasing the frequency with which he plays heads, can reduce "A"'s winnings below 1/5.

It should not be supposed that "A" and "B" always find their optimal strategies to be the same frequencies for each. With a slightly different pay-off matrix, for example:

		"B" Heads	"B" Tails
"A"	Heads	+ 1	− 2
	Tails	− 1	+ 1

"A" should play heads with a frequency of two out of five while "B" should play heads with a frequency of three out of five. The value of this game, incidentally, is $-1/5$ indicating, as may be expected, that the optimal strategies leave "B" the winner.

VII

Simple inspection and intuition would lead one to suppose that as the number of persons playing a game increased, the number of lines of action open to each person increased, and the rules of the game became more complicated, the mathematics would become more difficult. This is the case. Indeed, the rapid increase in mathematical complexity as we move from the simplest game to slightly more complicated ones has undoubtedly discouraged both mathematicians and social scientists from pursuing the Theory of Games very far. This is the elusive, frustrating element in studying the Theory of Games. One sees the possibility of framing so many social science questions in game theory terms if only the mathematics did not become hopelessly complex. By "hopelessly" I do not mean the difficulty which any of us may have in studying a subject for the first time. I refer to the difficulty which mathematicians themselves face in devising methods of logically handling the relationships and in evaluating solutions. Progress, but very slow progress, is being made in this field perhaps in part because of the small number of mathematicians working in the field compared with other branches of mathematics.

The principal application of the Theory of Games until now has been in economics and can be dated precisely from the publication of *The Theory of Games and Economic Behavior* by Von Neuman (the originator of the mathematical theory) and Morgenstern (an economist) in 1944. Certain problems in economic theory seem almost to demand this type of treatment, for example, the analysis of duopoly, the situation in which there are only two sellers in a market each of whom must, of course, take into consideration the reaction of the other to any price and production policy. The price and production strategy for a duopolist which would allow him to extract the maximum profit *considering the possible lines of action open to his opponent* is obviousby not a simple maximization problem which might be handled by the calculus. The rational duopolist will attempt to obtain the "value of the game" rather than to follow a policy which would give him a larger profit based on a particular reaction from his opponent but would bankrupt him if his opponent seized upon the proper counter-strategy.

No doubt one reason that most of the applications of the Theory of Games have been in economics is that the pay-off matrix can be given real meaning in terms of profit and loss or of cost and revenue. In other fields of social science pure numbers may have to be assigned to the various possible results of the game by the investigator on weak empirical evidence. The example earlier in this paper of the Eisenhower-Stevenson strategies in the 1956 presidential election is a case in point. Here, however, we might have been able to use actual votes in the pay-off matrix.

In general, in political science where votes (in conventions, committees, legislative bodies, popular elections) are often the measure of success or failure, the Theory of Games may be worth considerable exploration by political scientists. Even where strict mathematical analysis within the framework of the theory would do violence to reality, initial conception of the problem in game theory terms may afford insights that might otherwise be missed. A few personal words may be suggestive on this point.

This writer has been working for some time, with little success, on a theory of a (hot or cold) war economy. Present economic theory has little to offer on this score since it is based on the pre-conception that the aim of economic activity is the maximization of individual satisfactions from limited rescources rather than the maximization of the security of a nation-state.

One thing becomes evident at the outset. The efficacy of any action directed towards maximization of national security can only be judged in the light of the reactions it provokes among the actual or avowed enemy and among (presently) neutral nations. For example, you do not increase your security by building an aircraft carrier if by so doing you provoke the enemy into building two carriers or if your action drives an island neutral into the camp of the enemy.

I have concluded, bearing in mind the limits of the present mathematical development of the theory of games, that a fruitful line of approach would be to conceive of a three-person game (much more complicated mathematically than a two-person game) in which the players are the subject nation and her allies, the neutrals, and the enemy nation and her allies. The problem then becomes one of constructing the pay-off matrix and establishing the rules of the game in such a way that they reflect the real world.

A possibility that arises immediately is that of coalitions between subject nation and neutrals or between enemy nation and neutrals. What price could the subject nation afford to pay to induce the neutrals to enter a coalition? This possibility of coalition in any multi-person game (except the two-person game) is one reason the mathematics becomes much more complicated in the three-person game than the two-person game (assuming coalitions are allowed by the rules of the game as they frequently are in the real world).

It appears to me that the Theory of Games might be a rewarding approach to the study of the family by psychologist or sociologist. The notion that the members of a family are trying to "maximize some single-valued function"–the father, authority; the child, independence, etc.–does not appear as fruitful a concept as that which could be gained from the Theory of Games. Here we would view family life as a game and the participants (at least the rational ones) as attempting to gain the "value of the game" rather than adopting "all or nothing" strategies. Every parent who has passed up an opportunity to punish a child for a particular misdeed grasps this notion of family relationships as a game with a set of (fairly well-understood) rules not subject to a simple maximization approach. And the parent whose strategy is based on a simple maximization strategy (of authority, say) often finds the child quickly discovers this strategy and may win the game, even though the pay-off matrix be unfavorable to him, by choosing the proper counter-strategy (which may include a coalition with the other parent).

VIII

A few concluding remarks may be useful. I have done no more than hint at the mathematics of the Theory of Games. This paper is intended only to be suggestive of the possibilities which the Theory of Games may hold for the social sciences. Two difficulties appear to be the most important.

The first difficulty lies in framing rules of the game and constructing a pay-off matrix which accurately reflects the real world situation. This difficulty is, of course, always present when mathematical analysis of real world phenomena is attempted. This difficulty is not, therefore, peculiar to the Theory of Games. In fact, if the logic of the Theory of Games is more apropos to the analysis of human social behavior than the logic of other forms of mathematics, the construction of reasonably accurate models should be more nearly possible with the Theory of Games than with other types of mathematical analysis of social science problems.

The second difficulty lies in the mathematics itself. It is easy to conceive of a social science problem which could be set in the Theory of Games, but whose analysis would break down because no mathematical method is available which would yield the optimal strategy and the "value of the game". Hence, if the Theory of Games is to be used, the problem must be forced into a mold of a game which is simple enough to yield results by known mathematical methods. Yet this mold may be too restrictive–that is, it may do such violence to the real world situation that the results are valueless. But progress in the mathematics of the Theory of Games is continually being made so that the theory becomes increasingly useful provided, of course, that the social scientist masters the mathematics or can work with a mathematician who has done so.

IX Bibliography

1. McDonald, J., *Strategy in Poker, Business, and War*, New York, W. W. Norton & Co., 1950, This is a good way to begin the study of the Theory of Games. A. remarkable job of simplification of a complex study.
2. McKinsey, J., *Introduction to the Theory of Games*, New York, McGraw-Hill, 1952. A textbook in the mathematics of the theory. A complete bibliography is appended to this work.
3. Von Neuman, J., and O. Morgenstern, *Theory of Games and Economic Behavior*, 2nd edition, Princeton, Princeton University Press, 1947. The classic work in the field.
4. Williams, J. D., *The Complete Strategyst*, McGraw-Hill, New York, 1954. The best introduction by now.

The second difficulty lies in the mathematics itself. It is easy to conceive of a social science problem which could be set in the Theory of Games, but whose analysis would break down because no mathematical method is available which would yield the optimal strategy and the "value of the game". Hence, if the Theory of Games is to be used, the problem must be forced into a mold of a game which is simple enough to yield results by known mathematical methods. Yet this mold may be too restrictive–that is, it may do such violence to the real world situation that the results are valueless. But progress in the mathematics of the Theory of Games is continually being made so that the theory becomes increasingly useful provided, of course, that the social scientist masters the mathematics or can work with a mathematician who has done so.

IX Bibliography

1. McDonald, J., *Strategy in Poker, Business, and War*, New York, W. W. Norton & Co., 1950, This is a good way to begin the study of the Theory of Games. A. remarkable job of simplification of a complex study.
2. McKinsey, J., *Introduction to the Theory of Games*, New York, McGraw-Hill, 1952. A textbook in the mathematics of the theory. A complete bibliography is appended to this work.
3. Von Neuman, J., and O. Morgenstern, *Theory of Games and Economic Behavior*, 2nd edition, Princeton, Princeton University Press, 1947. The classic work in the field.
4. Williams, J. D., *The Complete Strategyst*, McGraw-Hill, New York, 1954. The best introduction by now.