ON TYPE DEFINITIONS
WITH PARAMETERS

by

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On Type Definitions With Parameters

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Abstract

This paper analyzes some of the consequences of allowing the
definition of parameterized data types in programming languages. A
typical use of such types is:

\[
\text{type queue} \ (x) = \text{struct} \ (x, \ \text{ref} \ (\text{queue} \ (x))),
\]

\[
\text{intqueue} = \text{queue} \ (\text{int}).
\]

It is shown that the addition of parameters permits the definition of new
types not definable without parameters. In particular, the types definable
with parameters are closely related to the deterministic context-free
languages, whereas the author has previously shown that the types de-
finable without parameters are characterized by the regular (i.e. finite
state) languages.

An important consequence of this fact is that the type equivalence
problem, which is easily solvable in the absence of parameters, becomes
equivalent to the (currently open) equivalence problem for deterministic
pushdown automata.

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semantics.

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1. INTRODUCTION

Several authors [4, 6, 8, 12] have pointed out the usefulness of parameterized data types in a highly typed language such as ALGOL 68 [18] or Pascal [9]. For example, a general queueing facility might be defined by

\begin{align*}
(1.1) & \quad \text{type queue}(x) = \text{struct} (x, \text{ref queue} (x)) \text{ and later used by including definitions such as} \\
(1.2) & \quad \text{type intlist = queue (int)} \\
(1.3) & \quad \text{type waiting line = queue (person)}
\end{align*}

(\text{where person is some programmer defined data type). We shall adopt the terminology of [8, 12] and call an object like queue a modal.}

It should come as no great surprise that allowing modals creates new difficulties for the implementor. For example, if a parameter to a procedure can be declared to be of type \text{queue} (x), then the code generated may depend heavily on the size of objects of type x, and even if x is passed as a parameter (of type \text{type}) a good deal of run-time checking may be needed. Lindsey [12] suggests allowing only variables of type \text{ref queue} (x), and Gehani [4] discusses restrictions under which several copies of the troublesome procedure may be generated.

All these problems arise from attempts to use modals in variable declarations. The purpose of this paper is to point out a more subtle problem that remains even if modals are never used in variable declarations, but only as a tool for defining (ordinary) types as in (1.1-3.). The problem in question is to decide when two definitions are equivalent. As Král [10] points out, the algorithm for equivalence of modes in ALGOL 68 is similar to the equivalence algorithm for finite automata. As we showed in [16], this similarity is due to the fact that
modes defined by ALGOL 68-like mode declarations are regular, in the sense that they are characterized by regular sets of strings. With the introduction of parameters, this no longer holds true; the definable types are now characterized by arbitrarily complex deterministic context-free languages. The result of this is that the equivalence problem becomes of the same level of difficulty as the equivalence problem for deterministic languages. Since the latter problem is at present a famous open problem, we see that the addition of parameters adds an essential new complication.

1.1 Equivalence of Modes

The foregoing discussion presumes that there is an obvious canonical notion of "equivalence of types". This is not true. One school holds that any two distinct definitions define different types. Of course in this case the equivalence problem is trivial. We will take a point of view closer to that of ALGOL 68, one which has been supported in [11] and [16]. Briefly, we say that two types are equivalent if all values of one type are "the same shape as" values of the other type. To be somewhat more specific, we assume that there are atomic values partitioned into disjoint atomic type classes, and structuring operators which construct values from component values. Two values are the same type if they are both of the same atomic type or if they are constructed by the same operator from components which are, respectively, of the same types. This defines (recursively) an equivalence relation: "to be the same type as", and types are nothing more than equivalence classes. The important point is that the type of a variable $x$ should tell the following about the value of $x$: either the type of $x$ is atomic; then the value of $x$ is atomic and of that atomic type or the type of $x$ is $f(t_1, \ldots, t_k)$ where $f$ is a structuring
operator and \( t_1, \ldots, t_k \) are types; then \( k \) selection operations are applicable to the value of \( x \) and the result of applying the \( i \)th is a value of type \( t_i \). By induction, then, the type of a variable is sufficient to determine whether any given finite sequence of selection operations is applicable to \( x \), and if so, what the type of the result will be. This means that the type of information may be summarized as a (possibly infinite) tree.

1.2 Example

Figure 1a gives an example of a type definition which defines a modal of one argument and applies it to a constant type to obtain a new type. Figure 1b "unrolls" the definition by repeatedly replacing occurrences of the variable \( v_0 \) by its definition. The result of applying \( v_0 \) to \( \text{int} \) is shown in Figure 1c. As we shall see later, this type could not be defined without parameters.

The contents of the remainder of this paper are as follows: In section 2, we formally present a syntax for a type definition facility with parameters so as to have a concrete example to illustrate our ideas. In section 3, we endow the facility with a formal semantics based on ideas from \([11, 14, 15, 16]\) and elsewhere. The reader who is satisfied that the trees of figures 1b and 1c accurately reflect the "meaning" of the definition is figure 1a may skip this section, at least on first reading. He should, however, look at those paragraphs labeled "notation". Section 4 contains the central results of this paper (4.2.4 and 4.3.7). We show that types defined in our example language fragment are "deterministic context-free" and hence that the equivalence problem for types is of equal difficulty as the (open) equivalence problem for deterministic languages. Section 5 contains a discussion
of these results and their implications for language design.

2. A TYPE DEFINITION SYSTEM

Our example language fragment will be a stripped-down version of the type declaration facilities of ALGOL 68 [18] or Pascal [9] minimally augmented with the ability to specify a partially defined type as in (1, 1) or figure 1a. We assume that the following lexical classes are defined:

- \( \Omega \), a set of modal constants. Each \( \sigma \in \Omega \) has a rank (number of arguments) which is a non-negative integer. Modal constants of rank 0 may be called type constants and correspond to atomic types such as \texttt{int} or \texttt{real}. Modal constants of rank > 0 are the built-in type constructors of the language, such as \texttt{struct} and \texttt{ref} of ALGOL 68.

- \( V \), a set of modal variables. These also have ranks associated with them. A variable of rank 0 may be thought of as a type variable. Thus ordinary (non-parameterized) types are a special case of modals. For example, in figure 1a, \( v_0, v_1 \in V \) and rank \((v_0) = 1, \text{ rank } (v_1) = 0.\)

- \( X \), a set of parameter symbols. These are all of rank 0. We now specify the syntax of a modal definition using BNF and some context-sensitive restrictions.

\[
\text{<definition>} ::= \text{<declaration>} | \text{<definition<?>><declaration>}
\]

\[
\text{<declaration>} ::= \text{<modal variable>} \text{<formal parameters>} = \text{<modal expression>}
\]
<formal parameters> ::= <empty>|(<parameter list>)
<parameter list> ::= <parameter symbol>|<parameter list>,
    <parameter symbol>
<modal expression> ::= <modal constant> <actual parameters>
    | <modal variable> <actual parameters>
    | <parameter symbol>
<actual parameters> ::= <empty> |(<expression list>)
<expression list> ::= <modal expression>
    | <expression list>, <modal expression>

Restrictions

(1) There is exactly one <declaration> for each <modal variable>
    appearing in the <definition>.

(2) If \( \sigma \in \Omega \cup \nabla \) and rank \( (\sigma) = 0 \), then each <formal parameters>
    or <actual parameters> following \( \sigma \) is <empty>; if rank \( (\sigma) = n \), then
    each <formal parameters> or <actual parameters> following \( \sigma \) has
    exactly \( n \) members.

(3) Each <parameter symbol> appearing in a <declaration>
    must appear in the <formal parameters> of that <declaration>. The
    <parameter symbol>s in any <parameter list> are all distinct.

2.1 Notation

To simplify notation, we will assume for the remainder of this
paper that we are talking about one fixed definition \( \mathcal{D} \), that the set
of modal variables is \( \mathcal{V} = \{v_0, \ldots, v_{n-1}\} \) for some \( n \), and that \( \mathcal{D} \)
has the form:
\[ v_0(x_0, \ldots, x_{\text{rank}(v_0)-1}) = e_0 \]

\[
\vdots
\]

\[ v_{n-1}(x_0, \ldots, x_{\text{rank}(v_{n-1})-1}) = e_{n-1} \]

where each \( e_i \) is an expression. We also adopt a convention due to B. Rosen \(^{[13]}\) and omit mention of ranks of symbols when they are evident from the context or irrelevant. For example, \( v_0(x_0, \ldots, x_{\text{rank}(v_0)-1}) \) is written as \( v_0(x_0, \ldots, x_{-1}) \). In such a context, "-1" is pronounced "last".

3. SEMANTICS

Once we have constructed \( T \), the set of types, it should be clear that the meaning of a modal \( t \) of rank \( n \) is a function \( \text{fun}(t) \): \( T^n \to T \). If each modal constant \( \sigma \) is given a standard meaning \( \text{fun}(\sigma) \) and a meaning \( \text{fun}(v_j) \) is assigned to each modal variable \( v_j \), then each expression \( e_i \) represents a function \( \text{fun}(e_i): T^k \to T \) in an obvious way. (Here \( k \) is any integer larger than any \( j \) such that \( x_i \) appears in \( e_i \).) Since \( \text{fun}(e_i) \) depends on the assignments of \( \text{fun}(v_j) \) to \( v_j \), \( e_i \) may also be thought of as a functional from \( D_0 \times D_1 \times \ldots \times D_{-1} \) to \( D_j \), where \( D_j \) is the set of functions from \( T^{\text{rank}(v_j)} \) to \( T \). In this way, a definition may be read as a set of simultaneous equations asserting the equality of various functions over \( T \). Under certain circumstances, the equations have a unique solution. (See \(^{[16]}\) for more about when the solution is unique. A sufficient condition is that no \( e_i \) is of the form \( v_j(\ldots) \) or \( v_j' \).)
The set we use for $T$ is the same as the one we constructed in [16]. See that paper, [11], and section 1 of this paper for motivation for this choice of model. We briefly review the construction of $T$ and some of its properties.

3.1 Trees and Cpo's

3.1.1 Definition

Let $\Sigma$ be any ranked set (a set together with a function rank: $\Sigma \to \mathbb{N}$). A tree over $\Sigma$ is a partial function $t: \mathbb{N}^* \to \Sigma$ with domain $\text{dom}(t) \subseteq \mathbb{N}^*$ satisfying:

(3.1.1) If $\alpha n \in \text{dom}(t)$ for some $\alpha \in \mathbb{N}^*$ and $n \in \mathbb{N}$, then $n < \text{rank}(t(\alpha))$ and $\beta \in \text{dom}(t)$ for all prefixes $\beta$ of $\alpha$ (including $\alpha$).

3.1.2 Notation

Let $T[\Sigma]$ denote the set of trees over $\Sigma$. Let $T = T[\Omega]$. We will write $t[\alpha]$ rather than $t(\alpha)$ for the value of $t$ at the string $\alpha \in \mathbb{N}^*$. Let $L$ denote the tree with empty domain, and let $t[\alpha] = \bot$ mean that $\alpha \notin \text{dom}(t)$. If $\sigma \in \Sigma$, then we also use $\sigma$ to denote the function $\sigma: T[\Sigma]^{\text{rank}(\sigma)} \to T[\Sigma]$ where

$\dagger$ $\mathbb{N}$ denotes the set of non-negative integers. $\mathbb{N}^*$ is the set of finite sequences of elements of $\mathbb{N}$ including $\epsilon$, the sequence of length 0.
\[
\sigma(t_0, \ldots, t_{-1})[\alpha] = \begin{cases} 
\sigma & \text{if } \alpha = \varepsilon \\
t_i[\beta] & \text{if } \alpha = i\beta \text{ and } i < \text{rank}(\sigma) \\
\text{undefined otherwise.}
\end{cases}
\]

It is easy to verify that this satisfies (3.1.1). \qed

This gives us a notation for all finite trees as the following remark shows.

3.1.3 Remark

Let \( F[\Sigma] \) denote the set of finite trees over \( \Sigma \) (trees with finite domain). Then \( F[\Sigma] \) is the least set satisfying:

\[
\bot \in F[\Sigma]
\]

(3.1.2) If \( \sigma \in \Sigma \) and \( t_0, \ldots, t_{-1} \in F[\Sigma] \), then

\[
\sigma(t_0, \ldots, t_{-1}) \in F[\Sigma].
\]

Moreover, each \( t \in F[\Sigma] \) can be written in the form of (3.1.2) in a unique way. \qed

This result allows us to define functions on \( F[\Sigma] \) by induction on the complexity of the tree.

3.1.4 Example

The tree depicted

\[
\begin{array}{c}
\text{informatively} \\
\text{informally}
\end{array}
\]

\[
\begin{array}{c}
t = b \\
a \\
c \\
a \\
\end{array}
\]

\[
\begin{array}{c}
t[\varepsilon] = a \\
t[01] = e \\
t[0] = b \\
t[21] = b \\
t[1] = c \\
t[22] = c \\
t[2] = a \\
t[\alpha] = \bot \text{ otherwise}
\end{array}
\]

and may be denoted by \( t = a(b(e), c, a(b(\bot), c, \bot)) \).
3.1.5 Theorem

(i) The relation $\leq$ on $T[\Sigma]$ defined by

$$(3.1.3) \ t_1 \leq t_2 \text{ iff for all } \alpha \in \mathbb{N}, \ t_1[\alpha] = \bot \text{ or } t_1[\alpha] = t_2[\alpha]$$

is a partial order (reflexive, transitive, and anti-symmetric).

(ii) $\bot \leq t$ for all $t \in T$

(iii) If $t_0 \leq t_1 \leq \ldots$ is a countable sequence of trees in $T[\Sigma]$, then there is a unique tree, denoted $\text{lub}\{t_i\}$ such that $t_i \leq \text{lub}\{t_i\}$ for all $i$ and if $t_i \leq t$ for all $i$, then $\text{lub}\{t_i\} \leq t$.

Proof

(i) and (ii) are trivial. For (iii), $\text{lub}\{t_i\} = t$ where $t[\alpha] = \sigma$ if there is some $\sigma \neq \bot$ and some $i$ such that $t_i[\alpha] = \sigma$. (There is at most one such $\sigma$.)

3.1.6 Definition

A set, together with an element $\bot$ and a relation $\leq$ satisfying the conditions of theorem 3.1.5 is called a cpo (chain-complete poset).

3.1.7 Theorem

Let $p_n: T[\Sigma] \rightarrow F[\Sigma]$ by $p_n(t)[\alpha] = t[\alpha]$ if length $(\alpha) < n$

$\bot$ otherwise.

Given a tree $t \in T[\Sigma]$, let $\Psi(t)$ be the sequence $<t_0, t_1, \ldots>$ of members of $F[\Sigma]$ defined by $t_i = p_i(t)$. Then

(i) $t_i \leq t_{i+1}$ for all $i$

(ii) $t_i = p_i(t_j)$ for all $j \geq i$

(iii) $t = \text{lub}\{t_i\}$

(iv) $\Psi$ is an order isomorphism between $T[\Sigma]$ and the set of sequences over $F[\Sigma]$ satisfying (i) and (ii) and ordered componentwise. 

$\square$
3.1.8 Remark

By the above theorem, we could take 3.1.3 as the definition of $F[\Sigma]$ and define $T[\Sigma]$ to be the set of sequences satisfying (i) and (ii) of 3.1.7. This is the approach taken in [15] and [16]. We prefer the method here (which was inspired by [5]) since it seems more intuitively accessible.

3.1.9 Definition

If $A$ and $B$ are cpo's then $f: A \to B$ is continuous if $f(\bot) = \bot$, $x \leq y$ implies $f(x) \leq f(y)$, and $x_0 \leq x_1 \leq \ldots$ implies $\text{lub}\{t(x_i)\} = f(\text{lub}\{x_i\})$. $[A \to B]$ denotes the set of continuous functions from $A$ to $B$, ordered point-wise, that is $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in A$. $A \times B$ denotes the cartesian product of $A$ and $B$ ordered by $<x_1, y_1> \leq <x_2, y_2>$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$.

3.1.10 Proof

(i) $[A \to B]$ and $A \times B$ are cpo's.

(ii) If $f \in [A \to B]$ and $g \in [B \to C]$ then $g \circ f \in [A \to C]$.

(iii) If $f \in [A \to B]$ and $g \in [A \to C]$ then $f \times g \in [A \to B \times C]$ where $(f \times g)(x) = <f(x), g(x)>$.

(iv) $\pi_j \in [A_0 \times \ldots \times A_{-1} \to A_j]$ where $\pi_j(x_0, \ldots, x_{-1}) = x_j$.

(v) If $A$ is any set, then $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ is a cpo, where $\bot = \emptyset$, $B \leq C$ iff $B \subseteq C$, and $\text{lub}\{B_i\} = \bigcup B_i$. If $f: A \to \mathcal{P}(B)$ is any function, then $\hat{f} \in [\mathcal{P}(A) \to \mathcal{P}(B)]$ where $\hat{f}(C) = \bigcup \{f(x) \mid x \in C\}$. If $A$ is countable, then every $g \in [\mathcal{P}(A) \to \mathcal{P}(B)]$ is $\hat{f}$ for some $f: A \to \mathcal{P}(B)$.
3.1.11 Remarks

(1) If $S$ is any set, then $\hat{S} = S \cup \{\bot\}$ is a cpo, where $\bot$ is a new symbol and $x \leq y$ iff $x = y$ or $x = \bot$.

(2) $T[\Sigma]$ is a sub-cpo of $[\hat{\mathbb{N}}^* \to \hat{\Sigma}]$.

(3) If $A$ and $B$ are alphabets, then any substitution (as defined in [7, p. 124] or [1, p. 146]) $h: \mathcal{P}(A^*) \to \mathcal{P}(B^*)$ is continuous by 3.1.1(v). □

3.2 Substitution Operators

We stated above that under the assignment of functions to modal variables, each expression $e_i$ denotes a function "in an obvious way". We now make this more precise.

Recall from 3.1.2 that each $\sigma \in \Omega$ also denotes a function $\sigma: T^{\operatorname{rank}(\sigma)} \to T$.

3.2.1 Lemma

$\sigma$ is continuous. □

3.2.2 Definition

Let $e = <e_0, \ldots, e_{-1}>$ be a sequence of expressions - i.e. members of $F[\Omega \cup \nu \cup X]$ and let $f = <f_0, \ldots, f_{-1}>$ be a sequence of functions where $f_i: T^{\operatorname{rank}(\nu_i)} \to T$. Then $e \leftarrow f$ is the function defined by induction on the complexity of $\bar{e}$ as follows:

$\bar{e} \leftarrow f = (e_0 \leftarrow f) \times \ldots \times (e_{-1} \leftarrow f)$

$\sigma(e_0, \ldots, e_{-1}) \leftarrow f = \sigma \circ (\bar{e} \leftarrow f)$

$\nu_i(e_0, \ldots, e_{-1}) \leftarrow f = f_i \circ (\bar{e} \leftarrow f)$

$(x_i \leftarrow f) = \pi_i$

(By 3.1.3 this defines $\bar{e} \leftarrow f$ completely.) □
The reader may find this definition easier to understand if he notices that $(f \cdot (g_1 \times g_2 \times \ldots \times g_{-1}))(x) = f(g_0(x), \ldots, g_{-1}(x))$.

3.2.3 Lemma

(i) If each $f_i$ is continuous then $\bar{e} \cdot \bar{f}$ is continuous.

(ii) The operator $\mathcal{F}$ defined by $\mathcal{F}(\bar{f}) = \bar{e} \cdot \bar{f}$ is continuous – i.e. $\mathcal{F} \in [D_0 \times D_1 \times \ldots \times D_{-1} \rightarrow D_0 \times D_1 \times \ldots \times D_{-1}]$

where $D_i = [\text{Trank}(v_i) \rightarrow T]$.

Proof

(i) follows from 3.1.10(ii, iii, and iv) and 3.2.1.

(ii) follows from the fact that the operators $\cdot$ and $\times$ are continuous.

3.2.4 Theorem (Tarski [17])

If $A$ is any cpo and $f \in [A \rightarrow A]$, then $f$ has a fixed point, an element $x \in A$ such that $f(x) = x$. In fact, $f$ has at least fixed point, denoted $Yf$, which may be computed by

$$f_0 = 1$$

$$f_{n+1} = f(f_n)$$

$$Yf = \text{lub}\{f_n\}$$

3.2.5 Corollary

For each definition $D$ as in (2.1.1), there is a sequence $F = \langle f_0, \ldots, f_{-1} \rangle$ such that $\bar{f} = \bar{e} \cdot \bar{f}$.

This is the promised semantics for type definitions.

Summarizing, a mode definition $D$ is of the form $\{v_i(x_0, \ldots, x_{-1}) = e_i \mid i = 0, 1, 2, \ldots\}$. A solution to the sequence of equations is
a sequence \( \tilde{f} \) of continuous functions such that \( f_i(t_0, \ldots, t_{-1}) = (e_i + \tilde{f})(t_0, \ldots, t_{-1}) \) for all \( i \) and all \( t_0, \ldots, t_{-1} \).

3.2.6 Notation

We will say that the solution of \( D \) is the least fixed point \( Y \) computed as in theorem 3.2.4, and the function defined by \( D \) is the first component of this fixed point, \( \Pi_0(Y_F) \). Finally, and most important, two definitions are equivalent if they define the same function.

3.3 Functions and Trees

In 3.1.2 we used a symbol \( \sigma \in \Sigma \) to denote a function. More generally, many (not all) functions on \( T[\Sigma] \) can be represented by trees.

3.3.1 Definition

Let \( X_n = \{x_0, \ldots, x_{n-1}\} \) and let \( t \in T[\Omega \cup X_n] \). Then for all \( m \geq n \) and all \( \Sigma \supseteq \Omega \), \( t \) denotes a function \( \text{fun}(t) : T[\Sigma]^m \to T[\Sigma] \) where \( \text{fun}(t)(t_0, \ldots, t_{m-1}) \) is the result of replacing each occurrence of \( x_i \) by \( t \) in \( t_i \). More precisely, let \( s \) be the tree:

\[
\begin{align*}
\sigma[\alpha] &= t[\alpha] \text{ if } t[\alpha] \in \Omega \\
\sigma[\alpha\beta] &= t_i[\beta] \text{ if } t[\alpha] = x_i
\end{align*}
\]

It is straightforward to check that this defines a unique tree. Let \( \text{fun}(t)(t_0, \ldots, t_{m-1}) = s \).

Notice that this definition extends 3.1.2 in the sense that \( \sigma = \text{fun}(\sigma(x_0, \ldots, x_{-1})) \) (as a function). Next, we extend this notation of "replacing occurrences of the symbol \( \tau \) in \( t \) with the tree \( s \)" to the case in which \( \text{rank}(\tau) \neq 0 \) i.e. in which \( \tau \) can appear at an interior node.
3.3.2 Definition

Let \( \bar{e} = <e_0, \ldots, e_{-1}> \) and \( \bar{t} = <t_0, \ldots, t_{-1}> \) be finite sequences of trees where \( e_i \in F[\Omega \cup \mathcal{V} \cup \mathcal{X}] \) and \( t_i \in T[\Omega \cup \mathcal{V} \cup \mathcal{X}] \). Then \( \bar{e} \leftrightarrow \bar{t} \), the result of replacing \( v_i \) by \( t_i \) in \( \bar{e} \) is defined by induction on the complexity of \( \bar{e} \) as follows:

\[
\bar{e} \leftrightarrow \bar{t} = <e_0 \leftrightarrow \bar{t}, \ldots, e_{-1} \leftrightarrow \bar{t}>
\]

\[
\sigma(e_0, \ldots, e_{-1}) \leftrightarrow \bar{t} = \sigma(\bar{e} \leftrightarrow \bar{t})
\]

\[
v_i(e_0, \ldots, e_{-1}) \leftrightarrow \bar{t} = \text{fun}(t_i)(\bar{e} \leftrightarrow \bar{t})
\]

\[
x_i \leftrightarrow \bar{t} = x_i
\]

(See, for example figure 1b or 2a.)

3.3.3 Remark

The restriction that each \( e_i \) be finite is not essential. We could define \( \bar{e} \leftrightarrow \bar{t} \) directly (rather than inductively) for arbitrary \( \bar{e} \), but the definition would be much more complicated, and we are only interested in the case of finite \( e_i \).

In view of the similarity between definitions 3.2.2 and 3.3.2 it is not surprising that we have the following result.

3.3.4 Lemma

\[
\text{fun}(\bar{e} \leftrightarrow \bar{t}) = \bar{e} \leftrightarrow \text{fun}(\bar{t}).
\]

3.3.5 Remark

In the statement of 3.3.4 we implicitly extended the domain of \( \text{fun} \) from trees to tuples of trees by letting \( \text{fun}(<t_0, \ldots, t_{-1}>) = <\text{fun}(t_0), \ldots, \text{fun}(t_{-1})> \). We will continue to make such extensions without explicit mention.
3.3.6 Lemma

(i) fun: $T[\Omega \cup X_n] \to [T^n \to T]$ is continuous

(ii) fun is one-to-one

(iii) $\leftrightarrow$ as an operation on $T[\Omega \cup V \cup X]$ is associative

wherever it is defined. That is, $\bar{\varepsilon} \leftrightarrow (\bar{\varepsilon}' \leftrightarrow \bar{t}) = (\bar{\varepsilon} \leftrightarrow \bar{\varepsilon}') \leftrightarrow \bar{t}$.

3.3.7 Theorem

The minimal fixed point $\gamma \mathcal{F}$ of 3.2.3 can be described by trees.

More precisely, $\gamma \mathcal{F} = \text{fun}(\bar{t})$ where $\bar{t} = \text{lub}(\bar{t}^1)$ and $\bar{t}^1$ is defined by

$\bar{t}^0 = \langle \bot, \ldots, \bot \rangle$

$\bar{t}^{n+1} = \bar{\varepsilon} \leftrightarrow \bar{t}^n$

Proof

By 3.2.4, $\gamma \mathcal{F} = \text{lub}(\mathcal{F}^n(\bot))$ where $\bot$ is the nowhere defined function.

We will show by induction on $n$ that $\mathcal{F}^n(\bot) = \text{fun}(\bar{t}^n)$:

$\mathcal{F}^0(\bot) = \bot$ (where $\bot$ is the nowhere defined function)

$= \text{fun}(\bar{t}^0)$ (where $\bot$ is the nowhere defined tree)

$= \text{fun}(\bar{t})$

$\mathcal{F}^{n+1}(\bot) = \mathcal{F}(\mathcal{F}^n(\bot))$

$= \bar{\varepsilon} \leftrightarrow \mathcal{F}^n(\bot)$ by definition of $\mathcal{F}$

$= \bar{\varepsilon} \leftrightarrow \text{fun}(\bar{t}^n)$ by induction hypothesis

$= \text{fun}(\bar{\varepsilon} \leftrightarrow \bar{t}^n)$ by lemma 3.3.4

$= \text{fun}(\bar{t}^{n+1})$ by definition of $\bar{t}^n$

Hence $\gamma \mathcal{F} = \text{lub}(\mathcal{F}^n(\bot))$

$= \text{lub}\{\text{fun}(\bar{t}^n)\}$

$= \text{fun}(\text{lub}(\bar{t}^n))$ by 3.3.6 (i)

$= \text{fun}(\bar{t})$
3.3.8 **Corollary**

The function defined by definition $\mathcal{D}$ is $\text{fun}(\pi_0(\text{lub}[t^n]))$. □

3.3.9 **Notation**

Let $\mathcal{D}$ be a definition, and $\bar{t} = \text{lub}[t^n]$ be derived from $\mathcal{D}$ as in 3.3.7. Then $\bar{t}$ is said to be the **tuple of trees** defined by $\mathcal{D}$. The **tree defined by** $\mathcal{D}$ is the first component of this tuple, $\pi_0(\bar{t})$. □

3.3.10 **Remark**

Although a definition $\mathcal{D}$ technically defines a function, by virtue of 3.3.8 we can pretend that $\mathcal{D}$ defines a tree: If $f_0$ is the function defined by $\mathcal{D}$ and $t_0$ is the tree defined by $\mathcal{D}$ then $f_0 = \text{fun}(t_0)$. Since $\text{fun}$ is one-to-one, two definitions are equivalent iff they define the same tree. □

3.3.11 **Example**

Return to example 1.2. The meaning of figure 1b can now be made more precise. Notice that the tuples $t^n$ of 3.3.7 have the property that $t^n = e^n \leftarrow <\perp, \ldots, \perp>$ where $e^n = \overline{e} \leftarrow \overline{e} \leftarrow \ldots \leftarrow \overline{e}$ (n factors) or (by induction)

$$
e^0 = <v_0(x_0, \ldots, x_{-1}), \ldots, v_{-1}(x_0, \ldots, x_{-1})>
$$

$$
e^{n+1} = \overline{e} \leftarrow e^n = e^n \leftarrow \overline{e}$$

(by the associativity of $\leftarrow$).

The first 4 trees of figure 1b represent $e^0_0$, $e^1_0 = e_0$, $e^2_0 = e_0 \leftarrow \overline{e}$, and $e^3_0 = e_0 \leftarrow \overline{e} \leftarrow \overline{e}$. (Since $v_1$ does not appear in $e_0$, we need not show $e_{-1}$.) The last tree in 1b represents $t_0$, and the tree of 1c represents $\text{fun}(t_0)(\text{int}) = t_1$. □
4  TREES AND LANGUAGES

In this section, we establish an intimate connection between modals and deterministic context-free languages, and use this connection to prove the main results of the paper. The first step has already been taken by noting (in 3.3.10) that we can confine our attention to trees. The next step is to notice that the trees involved all have finite range.

4.1 Lemma

All trees mentioned in theorem 3.3.7 have finite range. (The range of \( t \) is \( \{ \sigma \mid t[\alpha] = \sigma \text{ for some } \alpha \in N^* \} \).

Proof

Clearly, if \( t \) is any of these trees, then \( t[\alpha] = \sigma \) only if \( \sigma \) appears somewhere in \( D \). But \( D \) contains a finite number of symbols.

Now let \( \Sigma \) be a (not necessarily finite) ranked set and let \( t \in T[\Sigma] \). Suppose the range of \( t \) is finite. Let \( n = \max \{ \text{rank}(\sigma) \mid \sigma \in \text{range}(t) \} \), and let \( [n] = \{0, 1, \ldots, n-1\} \).
For each \( \sigma \in \text{range}(t) \), \( t^{-1}[\sigma] \subseteq [n]^* \); that is, \( t^{-1}[\sigma] \) is a language over the finite alphabet \( [n] \). \( t^{-1}[\sigma] \) may be thought of as "the set of addresses of nodes where \( \sigma \) lives". This finite collection of languages completely characterizes \( t \). This observation is so important in what follows, that we state it as a theorem.
4.2 Theorem

If $t \in T[\Sigma]$ has a finite range, then there is a finite alphabet $[n] \subseteq \mathbb{N}$ and a finite collection of disjoint languages $L_\sigma \subseteq [n]^*$ such that

$$(4.1) \quad t[\alpha] = \sigma \iff \alpha \in L_\sigma$$

and $t[\alpha] = \bot \iff \alpha \notin \bigcup \{L_\sigma\}$. 

In [16] we showed that for ordinary type definitions (i.e. those without parameters), these languages are all regular sets, and, conversely, that given any finite collection of regular sets such that (4.1) defines a tree $t$, $t$ is definable by some definition. Notice, however, that the definable tree $t$ shown in 1c has the property that $t^{-1}[\text{int}] = \{ t^n0^{n+1} \mid n \geq 0 \}$ which is not a regular set [7].

4.1 Description Languages

In the previous section we showed how a tree can be described by a finite collection of languages. Here we introduce a device for describing a tree by a single language.

4.1.1 Definition

Let $\Sigma$ be a ranked set and let $\hat{\Sigma} = \Sigma \cup \{<\sigma, n> \mid \sigma \in \Sigma \text{ and } n < \text{rank}(\sigma)\}$.

Let $\psi : T[\Sigma] \to \mathcal{P}(\hat{\Sigma}^*)$ by

$\psi(t) = \{<\sigma_0, n_0>, \ldots, <\sigma_{k-1}, n_{k-1}, \sigma_k> \mid k \geq 0 \text{ and } t[n_0, \ldots, n_{j-1}] = \sigma_j \text{ for all } j \leq k\}.$

Note that if $\Sigma$ is infinite, then $\hat{\Sigma}$ is infinite, but if range($t$) is finite, then $\psi(t)$ is finite for some finite subset $A \subseteq \hat{\Sigma}$. $\psi(t)$ is called the description language of $t$. 
4. 1. 2 Lemma

\$ is continuous. \hfill \Box

4. 1. 3 Definition

For each \( \sigma \in \Sigma \), let \( \sigma \) be the finite substitution ([7, p. 124] or [1, p. 196]) defined by

\[
\begin{align*}
    h_\sigma(<\tau, i>) & = \{i\} \text{ for all } \tau \in \Sigma \\
    h_\sigma(\tau) & = \emptyset \text{ if } \tau \neq \sigma \\
    h_\sigma(\epsilon) & = \{\epsilon\}.
\end{align*}
\]

Since \( h_\sigma(\omega) \) contains at most one string for any string \( \omega \), we will regard \( h_\sigma \) as a partial function and write \( h_\sigma(\omega) = \alpha \) rather than \( h_\sigma(\omega) = \{\alpha\} \).

\hfill \Box

4. 1. 4 Lemma

\( t^{-1}[\sigma] = h_\sigma(\delta(t)) \).

\hfill \Box

4. 1. 5 Corollary

\( t_1 = t_2 \) iff \( \delta(t_1) = \delta(t_2) \).

\hfill \Box

We have already given two meanings to the operator \( + \). Now we give a third.

4. 1. 6 Definition

Let \( \hat{\Sigma} \) be as in 4. 1. 1 where \( \Sigma = \Omega \cup \mathcal{V} \cup \mathcal{X} \). Let \( M_0, \ldots, M_{-1} \subseteq \hat{\Sigma}^* \), where the number of \( M \)'s is \( ||\mathcal{V}||^+ \). Let \( \Psi \) be the substitution defined by

\[ + \text{ For any set } S, ||S|| \text{ is the cardinality of } S. \]
\[ \Psi(v_j) = M_j - \hat{\Sigma} \times \]
\[ = \{ \alpha \in M_j \mid \alpha \neq \beta x_i \text{ for any } \beta \text{ and } i \} \]
\[ \Psi(<v_j, \tau>) = M_j / \{ x_i \} = \{ \alpha \in \hat{\Sigma} \times | \alpha x_i \in M_j \} \]
\[ \Psi(\tau) = \{ \tau \} \text{ for any other } \tau \in \hat{\Sigma}. \]

Let \( \tilde{L}_0, \ldots, \tilde{L}_{-1} \in \hat{\Sigma} \times \). Then \( <\tilde{L}_0, \ldots, \tilde{L}_{-1}> \leftarrow <M_0, \ldots, M_{-1}> = \]
\[ = <\Psi(\tilde{L}_0), \ldots, \Psi(\tilde{L}_{-1})>. \]

The reason for "overloading" the symbol \( \leftarrow \) is the following:

4.1.7 Proposition

Let \( \tilde{t} \) and \( \tilde{e} \) be tuples of trees in \( T[\Omega \cup \nu \cup \chi] \). Then
\[ \delta(\tilde{t} \leftarrow \tilde{e}) = \delta(\tilde{t}) \leftarrow \delta(\tilde{e}). \]

4.1.8 Corollary

Let \( t \) be the tree defined by the definition \( D \). Then \( \delta(t) = \pi_0(\tilde{L}) \)

where
\[ L^0 = <\emptyset, \ldots, \emptyset> \]
\[ L^{n+1} = \delta(\emptyset) \leftarrow L^n \]

and \( \tilde{L} = \bigcup_{n=0}^{\infty} \tilde{L}^n \).

Proof

By 3.3.7, 4.1.2, and the fact that \( \delta(\bot) = \emptyset. \)

4.2 From Definitions to Pushdown Automata.

In this section we show how, given a definition \( D \) and a symbol \( \sigma \) appearing in \( D \) we can construct a deterministic pushdown automaton (DPDA) \( P \) such that \( L(P) = \tilde{L}^{-1}_0[\sigma] \). Our notation for DPDA's is exactly as in \([1]\) (q, v) except that we use \( \emptyset \) rather than \( e \) for the null string. The description languages introduced in section 4.1
are not used directly in the construction, but are used in the proof
of correctness of the construction. First we present the construc-
tion, then we give an example. [A detailed correctness proof is
omitted in this draft.]

4.2.1 Construction

Let $D$ be a definition as in (2.1.1) and let $\sigma$ be a symbol
appearing in $D$. Let $P = <Q, \Sigma, \Gamma, \delta, q_0, Z_0, F>$ where

$Q$ (the set of states) = $\{<i, \alpha> | \alpha \in \text{dom}(e_i)\}$

(A state is a node in some tree of $D$.)

$\Sigma$ (the input alphabet) = $\{0, \ldots, m-1\}$ where

$m = \max \{\text{rank}(\tau) | \tau \in \Omega \text{ appears in } D\}$

$\Gamma$ (the stack alphabet) = $\{<i, \alpha> \in Q | e_i[\alpha] \in \nu \} \cup Z_0$

(A stack symbol is a node labeled by a modal variable.)

$q_0$ (the initial state) = $<p, e>$ (the root of the first tree in $D$)

$F$ (the set of final states) = $\{<j, \alpha> \in Q | e_j[\alpha] = \sigma\}$

(A final state is a node labeled $\sigma$.)

and $\delta$ (the transition function) is defined as follows:

(i) If $e[i] \in \Omega$, then $\delta(<i, \alpha>, j, Z) = (<i, \alpha j>, Z) \forall Z \in \Gamma$

(ii) If $e[i] = \nu_k$, then $\delta(<i, \alpha>, \varepsilon, Z) = (<k, \varepsilon>, <i, \alpha>, Z) \forall Z \in \Gamma$

(iii) If $e[i] = x_k$, then $\delta(<i, \alpha>, \varepsilon, <j, \beta>) = (<j, \beta k>, \varepsilon)$

4.2.2 Example

Let $\sigma, \tau, \rho \in \Omega$ and let $\text{rank}(\sigma) = 3$, $\text{rank}(\tau) = \text{rank}(\rho) = 0$.

Let $D$ be the definition $v_0(x_0, x_1) = \sigma(v_0(\sigma(\tau, x_0, \rho), \sigma(\tau, \rho, x_1)), x_0, x_1)$.

Let $t$ be the tree defined by $D$. The derivation of $t$ is illustrated
in figure 2a. Figures 2b and 2c illustrate the action of $P$ on input

$00110$. $L(P) = t^{-1}[\tau] = \{0^m1^n0 | m \geq n \geq 1\} \cup \{0^m2^n0 | m \geq n \geq 1\}$

(Notice that $L(P)$ is not an LL language [1]).
4.2.3 Theorem

Let $P$ be the DPDA constructed in 4.2.1. Then $L(P) = t^{-1}[\sigma]$ where $t$ is the tree defined by $D$.

4.2.4 Corollary

If the DPDA equivalence problem is decidable, then the type equivalence problem is decidable.

Proof

Given two definitions $D$ and $D'$, $D$ is equivalent to $D'$ iff $t_0 = t'_0$ where $t_0$ and $t'_0$ are the trees defined by $D$ and $D'$, by 3.3.1. For each symbol $\sigma$ appearing in $D$ or $D'$, construct DPDA's $P_0$ and $P'_0$ according to 4.2.1. Then $t_0 = t'_0$ iff $L(P_\sigma) = t^{-1}_0[\sigma] = t'^{-1}_0[\sigma] = L(P'_\sigma)$ for each $\sigma$.

4.3 From Pushdown Automata to Definitions

By theorem 4.2.3, if $t$ is a definable tree, then there is a finite set of deterministic languages $\{L_\sigma\}$ which completely characterize $t$ in the sense of (4.1). Unfortunately, the converse is not true. Even if $t$ has finite range and $t^{-1}[\sigma]$ is deterministic context-free for all $\sigma$, it may not be the case that $t$ is definable. Fortunately, the converse of 4.2.4 does not need the full converse of 4.2.3; a weaker result suffices. Given any deterministic context-free language $L$ (satisfying certain properties), there is a definable tree $t$ built up from the symbols $\sigma$ and $\tau$ such that $t^{-1}[\tau] = L$ and $t^{-1}[\sigma]$ is completely determined by $L$. The first step is to reduce an arbitrary DPDA to an easily manageable form. Fortunately, this has already been done in [1].

4.3.1 Definition (quoted from [1, p. 691])

A DPDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is in normal form if it has
all the following properties:

1. \( P \) is loop-free. Thus, on each input, \( P \) can make only a bounded number of moves.

2. \( F \) has a single member, \( q_f \), and if \((q_0, w, Z_0) \vdash (q_f, \epsilon, \gamma)\), then \( \gamma = Z_0 \). That is, if \( P \) accepts an input string, then \( P \) is in the final state \( q_f \) and the pushdown list consists of the start symbol alone.

3. \( Q \) can be written as \( Q = Q_s \cup Q_w \cup Q_e \cup \{ q_f \} \), where \( Q_s \), \( Q_w \), and \( Q_e \) are disjoint sets, called the scan, write, and erase states, respectively; \( q_f \) is in none of these three sets. The states have the following properties:
   
   a. If \( q \) is in \( Q_s \), then for each \( a \in \Sigma \), there is some state \( p_a \) such that \( \delta(q, a, Z) = (p_a, Z) \) for all \( Z \). Thus, if \( P \) is in a scan state, the next move is to scan the input symbol. In addition, this move is always independent of the symbol on top of the pushdown list.
   
   b. If \( q \) is in \( Q_w \), then \( \delta(q, \epsilon, Z) = (p, YZ) \) for some \( p \) and \( Y \) and for all \( Z \). A write state always prints a new symbol on top of the pushdown list, and the move is independent of the current input symbol and the symbol on top of the pushdown list.
   
   c. If \( q \) is in \( Q_e \), then for each \( Z \in \Gamma \), there is some state \( p_Z \) such that \( \delta(q, \epsilon, Z) = (p_Z, \epsilon) \). An erase state always removes the topmost symbol from the pushdown list without scanning a new input symbol.
   
   d. \( \delta(q, a, Z) = \emptyset \) for all \( a \in \Sigma \cup \{ \epsilon \} \) and \( Z \in \Gamma \). No moves are possible in the final state.

4. If \((q, w, Z) \vdash (p, \epsilon, Z)\), then \( w \neq \epsilon \). That is, a sequence of moves which (possibly) enlarges the stack and returns to the same level cannot occur on \( \epsilon \) input. A sequence of moves \((q, w, Z) \vdash (p, \epsilon, Z)\) will be called a traverse. Note that the possibility \( \Rightarrow \) impossibility of a
traverse for given \( q, p, \) and \( w \) is independent of \( Z \), the symbol on top of the pushdown list,

In short, a scan state reads the next input symbol, a write state prints a new symbol on the stack, and an erase state examines the top stack symbol, erasing it. Only scan states may shift the input head. \( \square \)

4.3.2 Theorem (quoted from [1, p. 691])

If \( L \subseteq \Sigma^* \) is a deterministic language, and \( \xi \) is not in \( \Sigma \), then \( L\xi \) is \( L(P) \) for some DPDA \( P \) in normal form. \( \square \)

We will assume henceforth that \( \Sigma = \{0, 1, \ldots, n-1\} \) for some \( n \).

4.3.3 Definition

Let \( P \) be a DPDA in normal form. Then the function \( \delta \) may be characterized by three functions: the scan \( \text{gto} \) function \( f \), the push-pop function \( g \), and the push-goto function \( h \) defined as follows:

\[
\begin{align*}
\delta(q_i, n, Z) &= (f(q_i, n), Z) \\
g : Q_s \times Q_e &\to Q \text{ where for all } Z \in \Gamma, \ q_i \in Q_s, \ n \in \Sigma \\
\delta(q_i, e, Z) &= (h(q_i), \gamma, Z) \text{ for some } \gamma \in \Gamma \text{ depending only on } q_i \text{ and } Z \text{ in } Q_e \\
\delta(p_j, e, \gamma) &= (g(q_i, p_j), e)
\end{align*}
\]

(Notice that \( h(q_i) \notin Q_e \) since otherwise there would be a traverse \( (q_i, e, Z) \longrightarrow (h(q_i), e, \gamma, Z) \longrightarrow (g(q_i, p_j), e, Z) \) contradicting point (4) of 4.3.1. \( \square \)

Now we can describe the construction of a type definition from a normal form DPDA \( P \).

4.3.4 Construction

Let \( P \) be a DPDA in normal form. Without loss of generality,
we may assume that $q_0 \in Q_e$. Let $\Omega = \{ \sigma, \tau \}$ where $\text{rank}(\sigma) = ||\Sigma||$ and $\text{rank}(\tau) = 0$. Let $V = \{ v_0, \ldots, v_{-1} \}$ where $||V|| = ||Q - Q_e||$ and $\text{rank}(v_i) = ||Q_e||$ for each $i$. Let $\vec{x}$ denote $<x_0, \ldots, x_{||Q_e||-1}>$. Arbitrarily order $Q_e$ and $Q - Q_e$ so that $Q_e = \{ p_0, \ldots, p_{-1} \}$, $Q - Q_e = \{ q_0, \ldots, q_{-1} \}$ and $q_0$ is the initial state.

For each $q_i$, let $k(q_i)$ be the expression (tree)

$"v_i(\vec{x})" \in T[\Omega \cup V \cup X]$ and for each $p_i$, let $k(p_i)$ be the expression $"x_i" \in T[\Omega \cup V \cup X]$.

Let $D$ be the definition

$$\begin{align*}
v_0(\vec{x}) &= e_0 \\
&\vdots \\
v_{-1}(\vec{x}) &= e_{-1}
\end{align*}$$

where

- if $q_i \in Q_s$ then $e_i = \sigma(k(f(q_i, 0)), \ldots, k(f(q_i, -1)))$
- if $q_i \in Q_w$ then $e_i = v_j(k(g(q_i, p_0), \ldots, k(g(q_i, p_{-1})))$ where $q_j = h(g_i)$
- if $q_i = q_f$ then $e_i = \tau$.

4.3.5 Example

Let $P$ be the normal form DPDA in figure 3a. $L(P) = \{ 0^n1^n1^n | n \geq 0 \}$. Then the functions used and the definition constructed according to 4.3.4 are illustrated in figure 3b.

4.3.6 Theorem

Let $P$ be a DPDA in normal form and let $D$ be the definition constructed from $P$ by 4.3.4. Let $t_0$ be the tree defined by $D$. Then $t_0[\alpha] = \begin{cases} 
\tau & \text{if } \alpha \notin L(P) \\
\bot & \text{if } \alpha \notin L(P) \text{ but } \alpha = \beta\gamma \text{ for some } \beta \in L(P) \\
\sigma & \text{otherwise}
\end{cases}$
4.3.7 Corollary

If the type equivalence problem is decidable, then the DPDA equivalence problem is decidable.

Proof

Let $L, L' \subseteq \Sigma^*$. By 4.3.2, we can construct DPDA's $P$ and $P'$ such that $L(P) = L \in (\Sigma \cup \{n\})^*$ and $L(P') = L' \in (\Sigma \cup \{n\})^*$. Let $D$ and $D'$ be definitions defining trees $t_0$ and $t'_0$, constructed according to 4.3.4. Then

$D$ is equivalent to $D'$

iff $t_0 = t'_0$ \hspace{1cm} (by 3.3.10)

iff $L(P) = L(P')$ \hspace{1cm} (by 4.3.6)

iff $L = L'$ \hspace{1cm} $\square$

5. SUMMARY AND CONCLUSIONS

We have presented a modest extension to the usual type definition facilities found in current programming languages. We have given a precise mathematical semantics to this extended facility using a so-called "lattice theoretic" or "Scott-like" model. Using this model, we were able to show an intimate connection between definable types and deterministic context-free languages. In particular, we have shown that the equivalence problem for definable types is of equal difficulty with the equivalence problem for deterministic languages. Since the latter problem remains open in the face of several
years of concentrated efforts of researchers to close it, we must either abandon attempts to decide equivalence of types or restrict the range of types that may be defined.

What are the practical implications of this result? As we mentioned in § 1.1, we can avoid the problem entirely by refusing to consider separately declared types equivalent, or to consider them equivalent only if the declarations are of "essentially the same form" in some sense. On the other hand, it may be that even the modest extension we presented here is unnecessarily strong. For example, definitions (1.1), (1.2), and (1.3), which were presented as motivation for modals, define regular types and thus could be defined without the use of parameters. In this case, the parameters serve merely as a convenience, allowing various aspects of the type \texttt{intlist} to be specified separately. This one could require that all types defined be regular (in the sense that $t^{-1}[\sigma]$ is a regular set for all $\sigma$). In view of the fact that it is decidable whether a given deterministic context-free set is regular [7, p. 230], it should not be hard to show that there is an algorithm to enforce this restriction. It remains to be seen whether non-regular types are of any practical use.

In fact, a somewhat stronger restriction, which is sufficient to ensure that all types declared are regular, is to change the syntax so that actual parameters to modal variables must be parameter symbols or modal constants of rank 0. This still allows definitions such as (1.1) but would be easier to enforce than the restriction of the previous paragraph. More experience with modals is necessary before we can state whether such restrictions prohibit any truly useful type definitions.
Acknowledgements

The results reported in this paper were obtained with the invaluable help and encouragement of Alan Demers. The construction of 4.3.4 was inspired by the proof of theorem 4.2.4 in [3]. We feel that the content of § 4 is essentially the same as that of [2], although the parallel is not exact, and we obtained our results before we were aware of [2]. Portions of this research were completed while the author was a student at Cornell University, Ithaca, N.Y.
References


\[ \nu_0 (\chi_0) = \text{struct} (\chi_0, \nu_0 (\text{ref} (\chi_0))) \]

\[ \nu_1 = \nu_0 (\text{int}) \]
Figure 2a
Figure 2b
States of P

<table>
<thead>
<tr>
<th>State</th>
<th>Stack</th>
<th>Remaining Input</th>
<th>Move</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$Z_0$</td>
<td>00 1 1 0</td>
<td>i</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$Z_0$</td>
<td>0 1 1 0</td>
<td>ii</td>
</tr>
<tr>
<td>$q_0$</td>
<td>$q_1Z_0$</td>
<td>0 1 1 0</td>
<td>i</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_1Z_0$</td>
<td>1 1 0</td>
<td>ii</td>
</tr>
<tr>
<td>$q_0$</td>
<td>$q_1q_1Z_0$</td>
<td>1 1 0</td>
<td>i</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_1q_1Z_0$</td>
<td>1 0</td>
<td>iii</td>
</tr>
<tr>
<td>$q_4$</td>
<td>$q_1Z_0$</td>
<td>1 0</td>
<td>i</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q_1Z_0$</td>
<td>0</td>
<td>iii</td>
</tr>
<tr>
<td>$q_4$</td>
<td>$Z_0$</td>
<td>0</td>
<td>i</td>
</tr>
<tr>
<td>$q_6$</td>
<td>$Z_0$</td>
<td>E</td>
<td>accept</td>
</tr>
</tbody>
</table>

Figure 2c
Action of P on input 00110
\[ Q_w = \{ q_0, q_1 \} \]
\[ Q_\delta = \{ q_2, q_3, q_4 \} \]
\[ Q_e = \{ p_0, p_1 \} \]
\[ F = \{ q_5 \} \]
\[ \Sigma = \{ 0, 1 \} \]
\[ \Gamma = \{ a, b \} \]

**Figure 3a**

A pushdown automaton in normal form.

<table>
<thead>
<tr>
<th>( f )</th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>( q_2 )</td>
<td>9_1</td>
<td>9_3</td>
</tr>
<tr>
<td>( q_3 )</td>
<td>( p_0 )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>( q_4 )</td>
<td>9_4</td>
<td>9_6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( g )</th>
<th>( p_0 )</th>
<th>( p_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 9_0 )</td>
<td>9_4</td>
<td>9_5</td>
</tr>
<tr>
<td>( 9_1 )</td>
<td>( 9_3 )</td>
<td>( 9_4 )</td>
</tr>
<tr>
<td>( 9_5 )</td>
<td>( 9_2 )</td>
<td></td>
</tr>
</tbody>
</table>

\[ k \circ f \]

<table>
<thead>
<tr>
<th>( k \circ f )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_2 )</td>
<td>( v_1(x_0, x_1) )</td>
<td>( v_2(x_0, x_1) )</td>
</tr>
<tr>
<td>( q_3 )</td>
<td>( x_0 )</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>( q_4 )</td>
<td>( v_4(x_0, x_1) )</td>
<td>( v_5(x_0, x_1) )</td>
</tr>
</tbody>
</table>

\[ k \circ g \]

<table>
<thead>
<tr>
<th>( k \circ g )</th>
<th>( p_0 )</th>
<th>( p_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 9_0 )</td>
<td>( v_3(x_0, x_1) )</td>
<td>( v_5(x_0, x_1) )</td>
</tr>
<tr>
<td>( 9_1 )</td>
<td>( v_4(x_0, x_1) )</td>
<td>( v_5(x_0, x_1) )</td>
</tr>
</tbody>
</table>

\[ v_0(x_0, x_1) = v_2(v_4(x_0, x_1), v_5(x_0, x_1)) \]
\[ v_1(x_0, x_1) = v_2(v_3(x_0, x_1), v_5(x_0, x_1)) \]
\[ v_2(x_0, x_1) = \sigma(v_1(x_0, x_1), v_3(x_0, x_1)) \]
\[ v_3(x_0, x_1) = \sigma(x_0, x_1) \]
\[ v_4(x_0, x_1) = \sigma(v_4(x_0, x_1), v_5(x_0, x_1)) \]
\[ v_5(x_0, x_1) = \tau \]

**Figure 3b**

The type definition derived from 3a.
Let $t_y = \ldots$

$t_o^{-1} [2] = \{0^n10^n1 | n \geq 0\}$

Figure 3c