Bootstrapping the Primitive Recursive Functions by 47 Colors

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Bootstrapping the Primitive Recursive Functions by 47 Colours

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Abstract

I construct a concrete colouring of the 3 element subsets of \( \mathbb{N} \). It has the property that each homogeneous set \( \{s_0, s_1, s_2, \ldots, s_r\}, r \geq s_0 - 1 \) has its elements spread so much apart that \( F_\omega(s_i) < s_{i+1} \) for \( i = 1, 2, \ldots, r - 1 \). It uses only 47 colours. This is more economical than the approximately 160000 colours used in [1].

1 Introduction and preliminaries

In the famous paper [2] L.Harrington and J.Paris showed that a certain finitary version \( \text{PH} \) of Ramseys Theorem is true, but unprovable in the celebrated system of Peanos Arithmetic. This is an example of Gödels incompleteness theorem. However, unlike Gödels consistency statement \( \text{PH} \) has generally been accepted to be a natural statement from Arithmetic. In

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Ketonen and Solovay gave a careful analysis of the underlying growth-rate of PH. As a first step in this analysis it was shown that for each increasing primitive recursive function \( f \) there exists \( n \) and a colouring of the 3 element subsets of \( \{ n, n+1, n+2, \ldots, f(n) \} \) such that there are no homogeneous sets \( \{ s_0, s_1, s_2, \ldots, s_r \} \) with \( r \geq s_0-1 \). The real point is that the number of colours can always be chosen to be less than a number fixed in advance. Ketonen and Solovay defined various algebras and took a series of products, in order to obtain the required colouring. An examination of their proof shows that they used approximately \( 1 \times 10^5 \) colours. However they clearly did not try to be economical. Actually in the work of Ketonen and Solovay the important point is that the number is finite. In this paper I construct a concrete colouring which uses only 47 colours.

Recall that the first functions in the Wainer hierarchy [3] are defined by

\[
F_0(n) := n + 1, \quad F_k^{j}(n) := F_k(n), \quad F_k^{j+1}(x) := F_k^{j}(F_k(n)), \quad F_k^{j+1}(n) := F_k^{j}(n), \quad F_\omega(n) := F_n(n).
\]

The function \( F_\omega \) is the first function in this hierarchy which growth faster than each primitive recursive function.

Let \( S^{[k]} \) denote the collection of \( k \) element subsets of \( S \). We use the convention that the elements in displayed in sets \( S = \{ s_0, s_1, \ldots, s_r \} \subseteq \mathbb{N} \) are listed after size (i.e. \( s_0 < s_1 < \ldots s_r \)). Let \( g : \mathbb{N}^{[k]} \to C \). We say that \( S \subseteq \mathbb{N} \) is homogeneous (for \( g \)) if \( u \geq k+1 \) and \( g \) takes a constant value on \( S^{[k]} \). The elements in \( C \) are called colours. If \( g_1 : \mathbb{N}^{[k]} \to C_1, g_2 : \mathbb{N}^{[k]} \to C_2, \ldots, g_u : \mathbb{N}^{[k]} \to C_u \) we define the product colouring \( g := g_1 \times g_2 \times \ldots \times g_u \) as the product map \( g : \mathbb{N}^{[k]} \to C_1 \times C_2 \times \ldots \times C_u \). Notice that \( S \) is homogeneous for \( g \) if and only if \( S \) is homogeneous for all the maps \( g_1, \ldots, g_u \).

## 2 Definition of the colouring

Let \( j(x, y) \) be the smallest \( j \) such that \( y \leq F_j(x) \). Consider the following 7 open propositions:

\[
\psi_1(\{ x_0, x_1 \}) := x_1 \leq F_\omega(x_0)
\]
\[ \psi_2(x_0, x_1) := j(x_0, x_1) > x_0 \]
\[ \psi_3(x_0, x_1) := j(x_0, x_1) \geq \left\lfloor \frac{x_0}{2} \right\rfloor \]
\[ \psi_4(x_0, x_1, x_2) := j(x_0, x_1) \neq j(x_0, x_2) \]
\[ \psi_5(x_0, x_1) := x_1 < F^{x_0-1}_{j(x_0, x_1)}(x_0) \text{ where } j := j(x_0, x_1). \]
\[ \psi_6(x_0, x_1, x_2) := j(x_0, x_1) > j(x_1, x_2) \]
\[ \psi_7(x_0, x_1) := j(x_0, x_1) \geq 2. \]

Now we define 7 auxiliary colourings \( h_1, h_2, \ldots, h_7 \) as follows. The colouring \( h_i : \mathbb{N}^2 \rightarrow \{0, 1\}; \ i = 1, 2, 3, 5, 7 \) takes the value 1 exactly when \( \psi_i \) holds.

The colouring \( h_j : \mathbb{N}^3 \rightarrow \{0, 1\}; \ j = 4, 6 \) takes the value 1 exactly when \( \psi_j \) holds.

**Lemma:** Suppose that \( S = \{s_0, s_1, \ldots, s_r\} \subseteq \mathbb{N} \) contains at least \( s_0 \) elements, \( s_0 \geq 5 \) and \( S \) is homogeneous for the colourings \( h_1, h_2, \ldots, h_7 \). Then \( F_\omega(s_i) < s_{i+1} \) for \( i = 1, 2, \ldots, r - 1 \).

**Proof:**

1. If \( h_1 \equiv 0 \) on \( S^2 \) then \( F_\omega(s_i) < s_{i+1} \) for \( i = 0, 1, 2, \ldots, r - 1 \). This is what we want to show.
2. So assume that \( h_1 \equiv 1 \) on \( S^2 \). According to the definition \( F_\omega(x) := F_x(x) \). So \( s_{i+1} \leq F_\omega(s_i) = F_s(s_i), \ i = 0, 1, 2, \ldots, r - 1. \)
3. For \( i = 0 \) this gives \( s_1 \leq F_\omega(s_0) \).
4. According to the definition \( j(s_0, s_1) \leq s_0 \).
5. This shows that \( h_2 \equiv 0 \) on \( S^2 \).
6. In particular \( j(s_0, s_1), j(s_0, s_2), \ldots, j(s_0, s_r) \leq s_0. \)
7. Now \( h_3 \equiv 0 \) or \( h_3 \equiv 1 \) on \( S^2 \) by (5) we know that \( j(s_0, s_1), j(s_0, s_2), \ldots, j(s_0, s_r) \) takes at most \( \left\lfloor \frac{s_0}{2} \right\rfloor + 1 \) different values.
8. Now \( h_4 \equiv 0 \) on \( S^3 \), because otherwise \( j(s_0, s_1), j(s_0, s_2), \ldots, j(s_0, s_r) \) would all take different values. This is impossible because \( r \geq s_0 - 1 > \left\lfloor \frac{s_0}{2} \right\rfloor + 1 \) and \( s_0 \geq 5. \)
9. But if \( h_4 \equiv 0 \) on \( S^3 \), then \( j(s_0, s_1) = j(s_0, s_2) = \ldots = j(s_0, s_r) \). Let \( j_0 \) denote this value.
(9) The value \( j_0 \) cannot be 0, because then according to the definition of \( j(s_0, s_r) \) we would have \( s_0 + 4 \leq s_r \leq F_0(s_0) = s_0 + 1. \)

(10) According to (9) \( j_0 > 0 \). By the definition of \( j_0 \) we have \( F_{x_0 - 1}(s_0) < s_i \leq F_{j_0}(s_0) \) when \( i = 0, 1, \ldots, r. \)

(11) Now \( h_0 \) cannot take the value 1 on \( S^{[3]} \). To see this suppose that \( h_0 \equiv 1 \) on \( S^{[3]} \). Then \( s_0 \geq j(s_0, s_1) > j(s_1, s_2) > \ldots > j(s_{r-1}, s_r) \) and especially \( j(s_0, s_1) > 2. \) Then by the definition of \( h_7 \) this would have the consequence that \( j(s_{r-1}, s_r) > 2. \) But this is a contradiction because: \( j(s_0, s_1) \geq j(s_{r-1}, s_r) + r - 1, \) so \( j(s_0, s_1) \geq r + 1 > s_0 \geq j(s_0, s_1). \)

(12) So \( h_0 \equiv 0 \) on \( S^{[3]} \). In particular \( j_0 = j(s_0, s_1) \leq j(s_1, s_2) \leq \ldots \leq j(s_{r-1}, s_r). \)

(13) According to (12) \( F_{j_0 - 1}(s_i) \leq F_{j(s_i, s_{i+1})}(s_i) \). The definition of the function \( j \) shows that \( F_{j(s_i, s_{i+1}) - 1}(s_i) < s_i + 1. \) Combining this shows that \( F_{j_0 - 1}(s_i) < s_i + 1. \)

(14) According to (13) \( s_i > F_{x_0 - 1}(s_{r-1}) > F_{x_0 - 1}(F_{x_0 - 1}(s_{r-2})) > \ldots > F_{x_0 - 1}(s_0). \)

(15) Now \( r \geq s_0 - 1 \) so by (14) \( s_i > F_{x_0 - 1}(s_0) \) so \( h_5(\{s_0, s_r\}) = 0. \)

(16) So \( h_5 \equiv 0 \) on \( S^{[3]}, \) and then \( s_{i+1} > F_{j(s_i, s_{i+1})}(s_i), i = 0, 1, 2, \ldots, r - 1. \)

(17) Now \( s_{i-1} \geq s_0 + 1 \) so according to (12) \( j(s_i, s_{i+1}) \geq j_0, \) and thus \( F_{j(s_i, s_{i+1}) - 1}(s_i) \geq F_{j_0 - 1}(s_0). \)

(18) This shows that \( s_r > F_{x_0 - 1}(s_{r-1}) > \ldots > F_{x_0 - 1}(s_0) \).

(19) Now \( r \cdot (s_0 - 1) > s_0 + 1 (s_0 \geq 5) \) so \( s_r > F_{x_0 - 1}(s_0) = F_{x_0}(s_0). \) This shows that \( j(s_0, s_r) > j_0 \) which violates (8) \( j(s_0, s_r) = j_0. \)

(20) The contradiction in (19) shows that the assumption in (2) is impossible. Thus \( h_1 \equiv 0 \) and we are back to (1).

**Lemma:** There is a colouring \( U : \mathbb{N}^{[3]} \rightarrow \{1, 2, \ldots, 44\} \) using 44 different colours such that if \( S \) is homogeneous for \( h \) then \( S \) is simultaneously homogeneous for the maps \( h_1, h_2, \ldots, h_7 \)

**Proof:** Now \( 1 + 5 \cdot 2 = 11 \) so by [1] there exists a colouring \( U_1 : \mathbb{N}^{[3]} \rightarrow \{1, 2, \ldots, 11\} \) such that if \( S \) is homogeneous for \( U_1 \) then \( S \) is simultaneously...
homogeneous for $h_1, h_2, h_3, h_5$ and $h_7$. Now let $U : \mathbb{N}^3 \to \{1, 2, \ldots, 11\} \times \{0, 1\} \times \{0, 1\}$ be the product of $U_1, h_4$ and $h_6$. It uses 44 colours. \hfill \Box

**Theorem:** There is a colouring $W : \mathbb{N}^3 \to \{1, 2, \ldots, 47\}$ such that if $S := \{s_0, \ldots, s_n\}$ is homogeneous for $W$ then $F_\omega(s_i) < s_{i+1}$.

**Proof:** Define $W$ as $U$ except that $W(\{s_0, s_1, s_2\})$ gets colour $45$ if $s_0 < 5$ and $s_1 \geq 5$ or $s_0, s_1, s_2 < 5$ and $s_2 = 4$, and colour $46$ if $s_0, s_1 < 5$ and $s_2 \geq 5$, and colour $47$ if $s_0, s_1, s_2 < 5$ and $s_2 \neq 4$. It is straightforward to show that any set $S := \{s_0, s_1, s_2, s_3\}$ which is homogeneous for $W$ must have $s_0 \geq 5$.

### 3 Final remarks and open questions

There is no reason to believe that 47 is a natural constant. Actually by a slight change in the problem I can show that 12 colours suffice. This suggests that the following question might be critical:

**Problem 1:** Is it possible to use only 12 colours?

One can also ask for the asymptotic answer. Here I think the critical question could be whether:

**Problem 2:** Is it possible to use only 3 colours?

To my knowledge the 47 colours used in this paper provides the best known lower bound to both of these questions.

### References


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