A MATHEMATICAL DESCRIPTION OF AN INFIELD-OUTFIELD SYSTEM

KJELD RASMUSSEN

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A simple mathematical model description of an infield-outfield system with cropping and grazing and including a ion-transfer mechanism is presented. The model illustrates important characteristics of the system such as its great flexibility and its ring-shaped land-use pattern.

Kjeld Rasmussen, cand.scient. with research scholarship at Geographical Institute, University of Copenhagen, Haraldsgade 68, DK-2100, Copenhagen Ø.

A mathematical description of an infield-outfield system by Kjeld Rasmussen.

1. INTRODUCTION

In Geografisk Tidsskrift, vol 77, 1978, Sofus Christiansen introduced a systematic view of infield-outfield systems based upon their basic ecological mechanisms. One of the most interesting and widely distributed of the variants discussed is the type called C, where cropping in the infield is combined with grazing in the outfield with daily movements of ruminants between the outfield and infield stables and pens. Since the droppings are concentrated on the infield, this system involves a ion transfer mechanism that makes it possible to obtain far higher yields from the crop production in the infield than otherwise feasible; thus the infield and the outfield are coupled both through their competition for work and through an areal interdependence, the yields of the infield being determined by its area and the total amount of manure at disposal, which again is determined by the animal production and thus by the area of the outfield. As noted by Sofus Christiansen this system has a large potential for development, because an increase in the amount of fertilizer at disposal (e.g. through the application of mineral fertilizers) can readily be utilized through an expansion of the infield.

This paper will present a mathematical model describing the functioning of this particular system. Even though this model-description does not, in principle, introduce anything not present in the verbal description, it is believed that something may be gained:

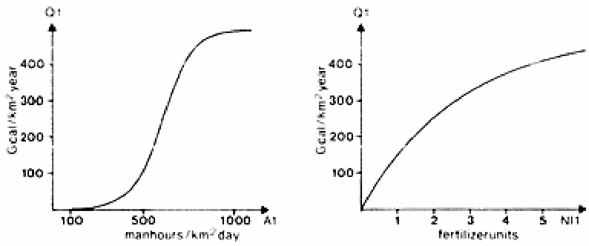

Fig. 1a. The relationsship between applied work and area-productivity for the infield. NI1 is held constant (= 2). Other parameters assume the standard values mentioned later in the text.
Fig. 1b. The relationsship between nutrient-ion concentration and area-productivity for the infield. Al is held constant (= 600). Other parameters assume standard values.
Fig. 1a. Arealproduktiviteten i indmarken som funktion af arbejdsindsatsen. NI1 er holdt konstant (= 2), mens andre parametre antager de i teksten omtalte standardværdier.
Fig. 1b. Arealproduktiviteten i indmarken som funktion af næringsionkoncentrationen. Al er holdt konstant (= 600), mens andre parametre antager standardværdier.

1) The model reveals clearly the main mechanisms at work and their relative importance. This facilitates discussion and criticism.
2) Employing computerized calculations any implications of a number of interacting mechanisms can be found without ambiguity.
3) The system's response to changes in numerical values of parameters may be analyzed.
4) If modified to fit an existing system, the model's predictions may be compared with measured values and thus be verified or falsified.

2. THE MODEL

2.1 The area-procutivity

The infield-outfield system is situated within a circular area with radius R2 on a homogenous plane. The infield is likewise circular with radius R1. The population is living in a village at the center of the circles.

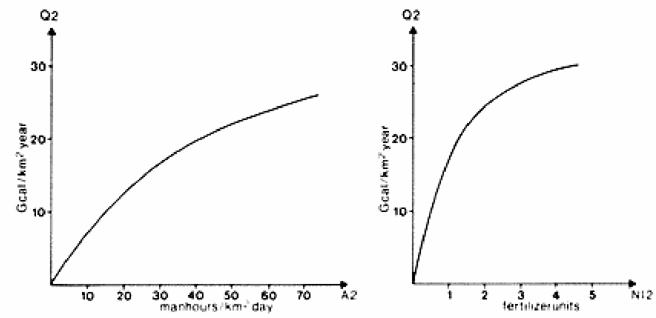

Fig. 2a. The outfield equivalent to fig. 1a. NI2 = 0.8.
Fig. 2b. The outfield equivalent to fig. 1b. A2 = 25.
Fig. 2a. Som fig. 1a, blot for udmarken. NI2 = 0,8.
Fig. 2b. Som fig. 1b, blot for udmarken. A2 = 25.

The two involved production systems, the plant production for human consumption in the infield and the meat and/or milk production based on grazing in the outfield, have the area-productivity functions, Q1 and Q2. These are supposed each to be function of two variables only: the invested amount of work per unit area per unit of time, A1 resp. A2, and concentration of nutrient-ions in the soil, NI1, resp. NI2. In fig. 1a and 2a are shown the supposed shapes of the functions relating area-productivities and invested work in infield and outfield for constant values of the nutrient-ion-concentration. Fig. 1b and 2b illustrates the supposed relations between the nutrient-ion-concentrations and the area-productivities in infield and outfield for constant values of invested work.

The curves shown in figs. 1 and 2 can be approximated by the functions:

$$Q1 = \frac{U1(1-e^{-NI1 \cdot T11})}{1+B \cdot e^{-A1 \cdot T12}} \qquad (1)$$

$$Q2 = U2(1-e^{-NI2 \cdot T21})(1-e^{-A2 \cdot T22}) \qquad (2)$$

U1 and U2 represent the highest possible area productivities in in- and outfield.

T11, T12, T21, T22 and B are constants that determine the »stretching« of the curves in the direction of the l-axis.

2.2 The distance effect

In order to calculate the work inputs, A1 and A2, in the two production systems a number of parameters must be specified:

1) The fraction, c, of the total available worktime (including time spent on transport) which is used in infield.
2) The areas (and thus the radia) of in- and outfield, AREA1 and AREA2, where

$$AREA1 = \pi R1^2 \quad \text{and} \quad AREA2 = \pi(R2^2-R1^2).$$

3) The travelling velocity, v.
4) The daily workhours, TD, including journeying to and from work.
5) The number of people in the workforce, N.

The expressions for mean effective applied work per unit area (in hours/(km²day)) can be obtained by subtracting the time spent travelling from the total daily workhours and finding the mean value by means of an integration:

$$A1 = \frac{1}{AREA1} \cdot c \cdot N \cdot \left(TD - \frac{1}{AREA1}\int_0^{R1} \frac{2R}{v} \cdot 2\pi R dR\right) = \frac{c \cdot N \cdot \left(TD - \frac{4R1}{3v}\right)}{AREA1} \qquad (3)$$

$$A2 = \frac{1}{AREA2} \cdot (1-c) \cdot N \cdot \left(TD - \frac{1}{AREA2}\int_{R1}^{R2} \frac{2R}{v} \cdot 2\pi R dR\right) = \frac{(1-c) \cdot N \cdot \left(TD - \frac{4(R2^3-R1^3)}{3 \cdot v \cdot (R2^2-R1^2)}\right)}{AREA2} \qquad (4)$$

The values of the expressions

$$TD - \frac{4R1}{3v} \quad \text{and} \quad TD - \frac{4(R2^3-R1^3)}{3 \cdot v \cdot (R2^2-R1^2)}$$

as functions of R1 are shown in fig. 3.

2.3 The effect of nutrient-ion transfer

The method of calculating the nutrient-ion concentrations NI1 and NI2 is sketched in fig. 4.

Since the leaching of nutrients by rain-water is strongly dependent on their concentration, the nutrient-ion subsystem has a built-in buffer mechanism keeping the concentrations within a reasonable range. This is the equivalent of saying that the concentrations NI1 and NI2 have a functional dependence on the net ion balances NB1 and NB2, defined as the yearly nutrient-ion input from fertilizing less the yearly loss of ions by harvesting and grazing as shown in fig. 5. The units used in this subsystem are arbitrary »fertilizer-units«, not specific physical units. When analyzing a real system these units must be transformed into units relevant in that particular system. If, for instance, the limiting factor for the crop production is K-ions, then the »fertilizer-units« can be changed into units of concentration of K-ions.

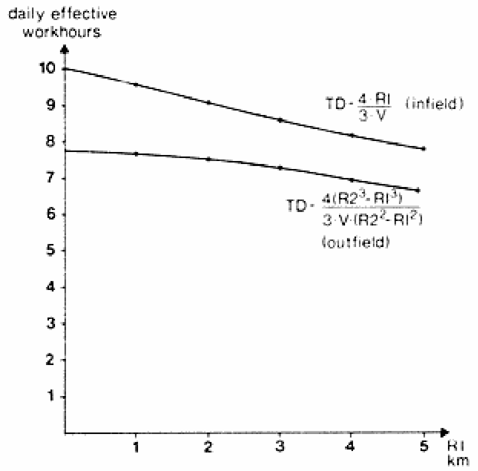

Fig. 3. Mean daily effective workhours for the infield and the outfield as a function of infield radius. Other parameters assume standard values.
Fig. 3. Den gennemsnitlige daglige effektive arbejdstid i indmark og udmark som funktion af indmarkens radius. Andre parametre antager standardværdier.

The curve shown in fig. 5 can be approximated by the following mathematical expressions:

$$NI1 = A11\left(\frac{\pi}{2} + \arctan(NB1 + A12)\right) \qquad (5)$$

$$NI2 = A21\left(\frac{\pi}{2} + \arctan(NB2 + A22)\right) \qquad (6)$$

Arctan is the inverse tangens-function, i.e. the tangens-function tilted 90°.
A11, A12, A21 and A22 are constants determining the »streching« of the curves along the 2-axis and the position along the 1-axis. The net balances are calculated in the following way:

$$NB1 = MF + s \cdot \frac{AREA2}{AREA1} \cdot Y2 \cdot Q2 - Y1 \cdot Q1 \qquad (7)$$

$$NB2 = -s \cdot Y2 \cdot Q2 \qquad (8)$$

MF is the yearly input per unit area of nutrient-ions form other sources such as mineral fertilizer or mulching.

s is the fraction of animal droppings utilized as fertilizer in the infield.

Y1 and Y2 are factors converting area-productivities Q1 and Q2 into the equivalent amount of nutrient-ions per unit area.

It can be seen from the equations (1), (2), (5), (6), (7) and (8), that the nutrient-ion concentrations are functions of the net balances, which again depend on area-productivities, which are functions of the nutrient-ion concentrations. Thus we deal with a coupled system of equations, and since they are not linear, there exists no general analytical method of solution. A numerical solution is possible, however, when using a digital computer.

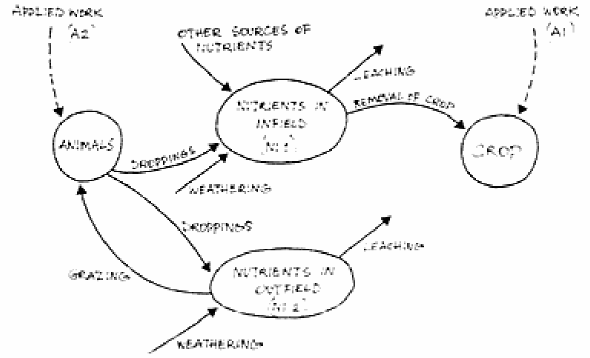

Fig. 4. A sketch of the nutrient-ion subsystem. Solid lines denote ion flows.
Fig. 4. Skitse af næringsion-subsystemet. Optrukne pile angiver ionstrømme.

The various functions illustrated in figs. 1, 2 and 5 and expressed in the equations (1), (2), (5), (6), (7) and (8) are tentative type curves. In an analysis of real systems the corresponding functions may look somewhat different, and functions based on measurement should be applied.

2.4 Optimizing

It is believed that the structure and functioning of an infield-outfield system is the outcome of some kind of optimizing procedure, performed by the people that operate the system. The available workforce and area are allocated to the two production systems in order to obtain the best possible result in relation to the objectives of the people operating the system. These objectives may be hard to reveal and even harder to express mathematically, since they may be time-dependent. The present work is restricted to time-independent objectives, which are thought to have a considerable relevance in systems, where physical features do not constitute a barrier against an expansion of the infield.

The simplest possible objective might be the maximizing of the total production for a given area and a given

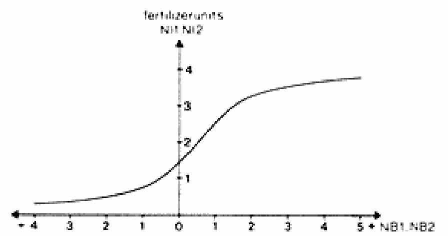

Fig. 5. The relationship between net balance, as defined in the equations (7) and (8) and ion concentration. The relationship is assumed to be similar for infield and outfield. All parameters assume standard values.
Fig. 5. Sammenhængen mellem nettobalancen, defineret i ligningerne (7) og (8), og ionkoncentrationen. Funktionen antages at have samme udseende for ind- og udmark. Andre parametre antager standardværdier.

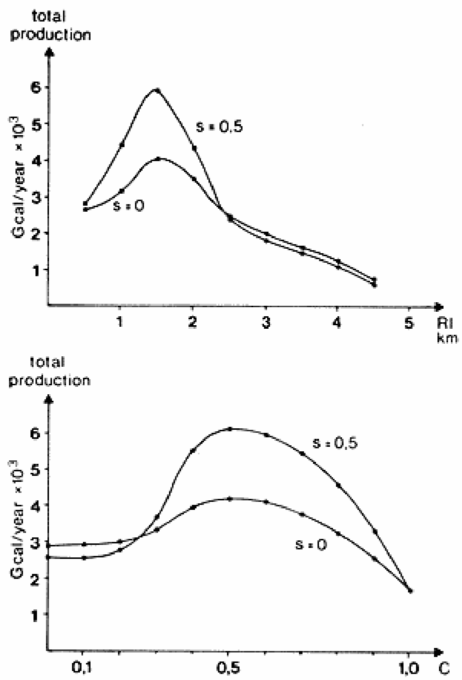

Fig. 6a. The relationship between total production and infield radius for a constant near-optimum value of the work allocation constant, c (= 0,6), in the case of a ion-transfer mechanism, s = 0,5, and without a ion-transfer mechanism, s = 0.
Fig. 6b. The relationship between total production and the work allocation constant, c, for a constant near-optimum value of infield radius, R1 (= 1,5 km), with (s = 0,5) and without (s = 0) a ion-transfer mechanism.
Fig. 6a. Sammenhængen mellem total produktion og indmarkens radius for en nær-optimum værdi af arbejdsallokeringskonstanten c (= 0,6), med (s = 0,5) og uden (s = 0) iontransport-mekanisme.
Fig. 6b. Sammenhængen mellem total produktion og arbejdsallokeringskonstanten, c, for en nær-optimum værdi af indmarkens radius, R1 (= 1,5 km), med (s = 0,5) og uden (s = 0) iontransport-mekanisme.

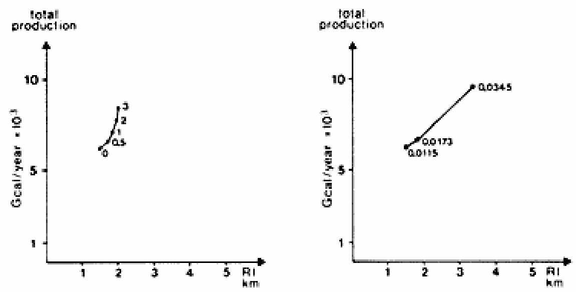

Fig. 7a. The effect on infield radius and total production with application of mineral fertilizer. The numbers on the curve denote the value of the parameter MF in that point.
Fig. 7b. The effect on infield radius and total production if introducing work-saving equipment in the crop production, in the model simulated by changing the value of T12. The numbers on the curve denote the value of T12 in that point.
Fig. 7a. Effekten af anvendelsen af mineralgødning i indmarken på indmarkens radius og den totale produktion. Tallene på kurven angiver værdien af parameteren MF i punktet.
Fig. 7b. Effekten af introduktion af arbejdsbesparende maskiner i planteproduktionen i indmarken på indmarkens radius og den totale produktion. Tallene på kurven angiver værdien af parameteren T12 i punktet.

total input of work. Another relevant objective might be a minimizing of the daily workhours, given the area and a total production (assumed to be in a reasonable agreement with the number of people and the area). These two approaches will be used to examine the effect of mineral fertilizer, of mechanization, and of increasing population pressure on the system.

3 RESULTS

Two computer programs have been designed, one assuming a maximizing of the total production, the other a minimizing of the daily workhours as the relevant objective. By means of a number of iterative procedures the system of equations is solved numerically. In the confined space of a paper like this a full documentation and test of the model cannot be presented. Only a few representative results obtained will therefore be shown.

In all the calculations the same basic set of numerical values of constants has been used. Any variations from this set will be indicated in the text. The following values have been employed: R2 = 5 km, v = 3 km/hr, U1 = 1000 Gcal/(km² year), U2 = 60 Gcal (km² year), MF = 0, A11 = A21 = 1,27, A12 = A22 = - 0,5, s = 0,5, Y1 = 0,004, Y2 = 0,02, T11 = 0,35, T12 = 0,0115, T21 = 0,85, T22 = 0,028 and B = 1000.

Most of the units have been omitted, but can be deducted from the rest. In the program maximizing the total production, the daily workhours (TD) are set at 10 hours. In the program minimizing TD, it is assumed that the total production in units of Gcal/year, approximately equivalent to the yearly energy intake of a human being, equals four times the workforce, N.

In fig. 6 is shown the dependence of total production on c and R1. It is clearly illustrated that there is a well defined combination of c and R1 which gives the highest total production. Likewise, fig. 6 shows the same functions for a system with no daily movements of animals from outfield to infield and therefore being without a ion transfer mechanism. In the model this situation is described putting s equal to zero. It is revealed that the total prodution is lower and the optimum less well developed than in the presence of a ion transfer mechanism.

The effect of applying increasing amounts of mineral fertilizer in the infield is shown in fig. 7 a, employing the program maximizing the total production. As assumed, the result is an increase in the total production and a slight increase in the radius of the infield. The introduction of work-saving equipment in the crop production has been simulated by changing the numerical value of T12, and as fig. 7b shows this will also increase the total production and the radius of the infield. The higher sensitivity of R1 to variations in degree of mechanization than to variations in input of mineral fertilizer is a result of the combination of numerical values used here, and is not a general property of the model.

Employing the program minimizing the daily work-hours, the effect of a variation of the population pressure has been investigated. In fig. 8 are shown all combinations of c and R1 that give less than 12 daily workhours for various values of N. With increasing population, the infield encroaches on the outfield, and the possible combinations of c and R1 lie within increasingly narrow intervals. When the population exceeds a certain limit (in this case about 8000) the system will of course collapse.

4. CONCLUSION

The very simple model described here seems to contain the essential mechanisms responsible for the important characteristics of a system with infield cropping and outfield grazing, since the expected behaviour can be simulated. Thus, in addition to giving a concise and explicit statement of the basic principles, the model opens up possibilities to test the sensitivity of the system to changes in numerical values of the involved parameters, and, perhaps the most important, it makes it possible, at least in principle, to test the validity of the concept of the system.

The main conclusions that may be drawn from the results shown in figs. 6, 7 and 8 are:

1) The ring-shaped land use pattern of this variant of infield-outfield systems is not primarily the result of a von Thünen transport effect, but rather of the ecological functioning of the system and in particular of the ion transfer mechanism.
2) The wide distribution and the development of the system can be partly explained by its ecological characteristics, in particular its easy adjustment to varying population densities.

A great number of changes could be introduced in order to make this primitive model more realistic, but it will be useless to attempt to list them. The most serious shortcomings exist perhaps in the nutrient ion sector, where a comprehensive treatment of concentrations and flows af various nutrients and their influence on crop and pasture growth is urgently needed.

Finally it must be noted that the presented model is a purely static equilibrium model. Many important problems, such as over- and undergrazing phenomena and adjustments to changes in externally influences parameters, are not equlibrium phenomena, and thus unfit for this approach. The transformation of the present model into a dynamical, optimizing model is a major task that lies far beyond the scope of this paper.

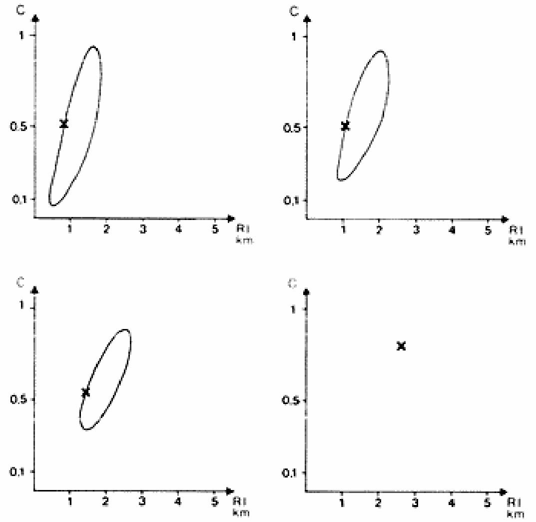

Fig. 8. The possible combination of c- and R1- values giving less daily workhours than twelve are situated within the areas delimited by the curves. The cross denotes the optimum, i.e. the combination that gives the minimum daily workhours. a) N = 500, b) N = 1000, c) = 1500, d) N = 2000.

Fig. 8. De kombinationer af c- og R1-værdier, der giver en daglig arbejdstid kortere end 12 timer, findes i arealerne begrænset af kurverne. Krydset angiver optimum, dvs. den kombination der giver den korteste daglige arbejdstid. a) N = 500, b) = 1000, c) N = 1500, d) N = 2000.

RESUME

Sofus Christiansen (1977) har på økologisk grundlag etableret en klassifikation af indmark-udmark systemer. I denne artikel præsenteres en matematisk modelbeskrivelse af én af de mest interessante varianter, af Sofus Christiansen betegnet C, der består i planteproduktion til menneskelig konsum i indmarken og mælke- og/eller kødproduktion baseret på græsning i udmarken. Den daglige flytning af dyr fra udmarken til stalde eller indhegninger i indmarken bevirker samtidig en flytning af næringsioner fra udmark til indmark, hvilket muliggør højere udbytter i indmarken.

Ud fra en række antagelser angående arealproduktivitetens afhængighed af arbejdsindsats og næringskoncentration, samt sidstnævntes afhængighed af til- og fraførsel af ioner ved gødning, høst og græsning, opstilles et ligningssystem, der tilnærmelsesvis beskriver systemets økologiske funktion. Idet det antages, at systemet er optimeret med henblik på opnåelse af størst mulig total produktion eller kortest mulig arbejdstid for en given produktion, er der udarbejdet to computerprogrammer, der finder optimale løsninger for systemets funktion.

Af resultaterne fremgår det, at ion-flytningsmekanismen er af afgørende betydning for eksistensen af et veldefineret ringformet udnyttelsesmønster i sådanne indmark-udmark systemer. Desuden påvises det, at systemet besidder en høj grad af fleksibilitet over for ændringer i befolkningstæthed, hvilket kan medvirke til at forklare dets holdbarhed og vide udbredelse.

REFERENCES

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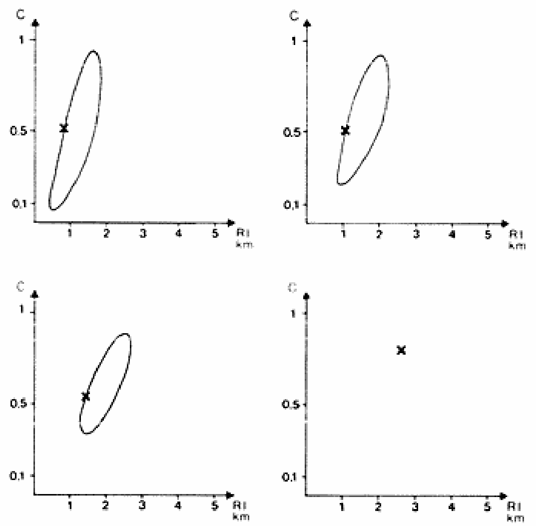

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