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Absolute stability properties of the explicit linear 3-step formulae

by

Zahari Zlatev

and

Ole Østerby

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Computer Science Department
AARHUS UNIVERSITY
Ny Munkegade – DK 8000 Aarhus C – DENMARK
Telephone: 06 – 12 83 55

Abstract

A three-parameter family of explicit linear 3-step formulae is derived. The conditions which ensure zero-stability of the formulae in the family are formulated. The absolute stability properties of the zero-stable formulae in the family are investigated both for $p = 3$ and $p = 2$ where p is the order of the formulae under consideration. Some numerical experiments are carried out in order to illustrate that formulae with good absolute stability properties can efficiently be used in the numerical solution of the problems in which the absolute stability properties are dominant.

Contents

1. Statement of the problem

Consider the initial value problem for first order ordinary differential equations defined by

(1.1) $y' = f(x,y)\ , \quad x \in [a,b]\ , \quad y \in D$

and

(1.2) $y(a) = yo$

where $[a,b]$ is an interval in $\grave{R}$ and D is a domain in $\grave{R}^s$ with $s \in \grave{N}$.

Assume that the true solution $y(x)$ of the problem (1.1) - (1.2) exists and is unique for $x \in [a,b]$ (it is well-known that this is so when f is continuous and Lipschitzian with regard to the second argument). Let the sequence

(1.3) $y_1, y_2, \ldots, y_N$

be a numerical solution of (1.1) - (1.2), i.e. each y_i is calculated so that

(1.4) $y_i - y(x_i) = r_i\ , \quad i = 1(1)N, \quad x_i \in [a,b]$

where r_i $(i = 1(1)N)$ are "small" in some sense, i.e. y_i can be considered as "good" approximations of $y(x_i)$.

In this paper it is assumed that : (i) the grid is equidistant and (ii) explicit linear multistep formulae are used to obtain (1.3) ; however, in the last section some brief remarks on the use of variable grid are given.

The assumption that the grid is equidistant implies that

(1.5) $x_i = x_o + ih\ , \quad x_o = a\ , \quad h = (b-a)/N\ , \quad i = 0(1)N.$

The general explicit k-step formulae is defined by (see Lambert [6], chapter 2)

$$(1.6) \qquad y_{n+k} + \sum_{i=0}^{k-1} \alpha_i \, y_{n+i} = h \sum_{i=0}^{k-1} \beta_i \, f_{n+i}$$

where

$$(1.7) \qquad f_{n+i} = f(x_{n+i} \, , \, y_{n+i})$$

and it is assumed that k starting vectors are available before the first use of (1.6).

By the use of formulae of type (1.6) the computational work per step in the calculation of (1.3) with a given stepsize h is minimized because: (i) only one function evaluation per step is needed and (ii) no problems with solving algebraic systems occur. Therefore the use of formulae of type (1.6) is desirable. Unfortunately, these methods have very small regions of absolute stability. This fact sometimes causes difficulties unless h is very small. The main purpose of this paper is to derive some special formulae (1.6) which have large absolute stability regions and which can be used in practical computations.

Denote by p the order of (1.6) (see [6], p. 23) and by S the absolute stability region of (1.6). Then the requirement $p = k$ will restrict severely the search for formulae (1.6) with large absolute stability regions S as seen from the following theorem.

Theorem 1.1 (Jeltsch & Nevanlinna, [5], p. 29).

> Let (1.6) be an explicit linear k-step method with $p = k$.
> If $[-a, 0] \subset S$ then $a \leq 2$ with equality only for $k = 1$.

□

Therefore the requirement $p = k$ must be relaxed when absolute stability regions which contain intervals greater than $[-2, 0]$ are wanted. It is well-known that for $p = k-1$ formulae (1.6) with absolute stability regions which contain intervals $[-a, 0]$ with an arbitrary $a > 0$ can be constructed. For $k = 2$ examples for one-parameter families of formulae of order $p = 1$ with long absolute stability regions are given in Mannshardt [7], van der Houwen [4], Zlatev & Thomsen [16, 17].

Practical implementations of such formulae are described in [16, 10]. A very general result for formulae (1.6) with $p = k-1$ is proved by Jeltsch and Nevanlinna [5].

Theorem 1.2 (Jeltsch & Nevanlinna, [5], p. 30).

> Let $0 \leq \alpha < \pi/2$, $R > 0$ and $k > 1$ be given. Then there exist explicit linear k-step methods of order $p = k-1$ such that they are absolutely stable in the set
> $\Omega_{\alpha, R} = \{\mu \in \mathbb{C} \, / \, |\mu| < R \wedge |\arg(-\mu)| \leq \alpha\}$.

□

However, note that the formulae used in the proof of this theorem seem to be difficult for practical implementation when R is large and α is closed to $\pi/2$. Therefore the requirement that R is arbitrarily large and α is arbitrarily close to $\pi/2$ will be relaxed but a new requirement about the size of the error constant will be imposed. More precisely, a three-parameter family of formulae will be derived from (1.6). The conditions, which ensure zero-stability of a considerably large subclass of formulae in the three-parameter family, will be formulated. A practical search for zero-stable formulae with large (or at least long) absolute stability regions will be carried out. Some examples, which demonstrate that the formulae found can be used efficiently in the solution of some problems, will be given. Finally, brief remarks on the use of formulae in a variable stepsize variable formulae methods are made.

2. A three-parameter family

Consider (1.6) with $k = 3$. Assume that :

$$(2.1) \qquad \alpha_0 = -1 - \alpha_2 - \alpha_1 \, ,$$

$$(2.2) \qquad \beta_1 = 4.5 + 2\alpha_2 + 0.5\alpha_1 - 2\beta_2 \, ,$$

$$(2.3) \qquad \beta_0 = -1.5 + 0.5\alpha_1 + \beta_2 \, .$$

Substitute α_0 , β_1 and β_2 in (1.6). The result is :

(2.4) $$y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} - (1 + \alpha_2 + \alpha_1) y_n$$
$$= h[\beta_2 f_{n+2} + (4.5 + 2\alpha_2 + 0.5\alpha_1 - 2\beta_2) f_{n+1}$$
$$+ (-1.5 + 0.5\alpha_1 + \beta_2) f_n]$$

It is readily seen that each triple $(\alpha_2, \alpha_1, \beta_2) \in \grave{R}^3$ determines an explicit 3-step formulae (2.4) which order is at least 2. In other words (2.4) determines a three-parameter family of explicit linear 3-step methods of order $p \in \{2, 3, 4, 5\}$.

Consider the stronger requirement : $p \in \{3, 4, 5\}$. Consider a triple $\{\alpha_2, \alpha_1, \beta_2\}$ and the formulae (2.4) generated by the triple chosen. This formula is of order p with $p \in \{3, 4, 5\}$ if the elements of the triple satisfy

(2.5) $$27 + 4\alpha_2 - \alpha_1 - 12\beta_2 = 0.$$

Since this equation is linear and since the triple was arbitrary, this means that for each pair $(\alpha_2, \alpha_1) \in \grave{R}^2$ a unique value of β_2 can be found so that (2.4) has order p at least equal to three. This special value of parameter β_2 will always be denoted by β_2^* and it is clear that

(2.6) $$\beta_2^* = 2.25 + \alpha_2/3 - \alpha_1/12.$$

Two examples which are well-known and commonly used in practice (especially in the predictor/corrector schemes) are given below.

Example 2.1 The Adams-Bashforth formula of order two is found by the following choice of the parameters

(2.7) $$\alpha_2 = -1 \ , \ \alpha_1 = 0 \ , \ \beta_2 = 1.5.$$

□

Example 2.2 The triple

(2.8) $$\alpha_2 = -1 \ , \ \alpha_1 = 0 \ , \ \beta_2^* = 23/12,$$

defines the Adams-Bahsforth formula of order three.

□

3. Zero-stability conditions

Consider the ρ-polynomial associated by the left-hand side of formula (1.6) :

(3.1) $$\rho(z) = z^k + \sum_{i=0}^{k-1} \alpha_i z^{k-i} .$$

The following definition corresponds to that given by Stetter [13] on p. 206.

Definition 3.1 A linear k-step method is said to be strongly zero-stable if (3.1) has no root outside the closed unit disk and 1 is the only root on the unit circle.

□

Consider the three-parameter family (2.4) again. The restrictions which must be put on the parameters of the family in order to obtain strongly zero-stable formulae are given by the following theorem.

Theorem 3.1 The formula obtained by an arbitrary triple $(\alpha_2, \alpha_1, \beta_2) \in \grave{R}^3$ from (2.4) is strongly zero-stable if the pair (α_2, α_1) is such that

(3.2) $$\alpha_2 < -\alpha_1 ,$$
(3.3) $$\alpha_1 > -1$$

and

(3.4) $$\alpha_1 > -3 - 2\alpha_2$$

are satisfied.

<u>Proof</u> Consider the ρ-polynomial associated by any method from the three-parameter family (2. 4)

(3. 5) $$\rho(z) = z^3 + \alpha_2 z^2 + \alpha_1 z - (1 + \alpha_2 + \alpha_1) .$$

Since $z = 1$ is a root of (3. 5), it is clear that

(3. 6) $$\rho(z) = (z-1) P(z)$$

where

(3. 7) $$P(z) = z^2 + (1 + \alpha_2) z + (1 + \alpha_2 + \alpha_1).$$

Consider

(3. 8) $$Q(z) = (1 + \alpha_2 + \alpha_1) z^2 + (1 + \alpha_2) z + 1$$

and

(3. 9) $$P(z) = \frac{1}{z} [Q(0) P(z) - P(0) Q(z)]$$
$$= (\alpha_2 + \alpha_1) [(\alpha_2 + \alpha_1 + 2) z + (1 + \alpha_2)].$$

By a theorem originally proved by Schur [11] (see also [6] and [8]) the roots of P(z) satisfy $|z_i| < 1$ (i = 1, 2) when

(3. 10) $$|P(0)| < |Q(0)|$$

and

(3. 11) $$|\bar{z}| < 1$$

where $\bar{z}$ is the root of (3. 9).

From (3. 10) it is readily seen that

(3. 12) $$|1 + \alpha_2 + \alpha_1| < 1$$

and therefore (3. 2) and

(3. 13) $$\alpha_2 + \alpha_1 + 2 > 0$$

hold.

From (3. 9) and (3. 13) it follows that (3. 11) is satisfied when

(3. 14) $$|1 + \alpha_2| < \alpha_2 + \alpha_1 + 2.$$

The inequalities (3. 3) and (3. 4) can easily be derived from (3. 14). Moreover (3. 3) and (3. 4) imply (3. 13). Thus the theorem is proved.

□

<u>Remark 3. 1</u> The region of strong zero-stability for the three-parameter family (2. 4) is the inside of the triangle ABC given in Fig. 3. 1.

<u>Remark 3. 2</u> The requirement for strong zero-stability can be relaxed so that (3. 1) is allowed to have simple roots different from 1 on the unit circle (see Dahlquist [1, 2], Henrici [3] and Lambert [6]). The formulae so found are called zero-stable (see Lambert [6], p. 33). It is easily seen that on the inside of AB the formulae of family (2. 4) are zero-stable but not on the end-points A (double root 1) and B (double root -1). The same is true for the inside of BC but not for the end-points B and C (1 is a root of multiplicity 3). On AC the formulae are zero-unstable. Indeed, on this side

(3. 15) $$3 + 2\alpha_2 + 1 = 0$$

which is equivalent to

(3. 16) $$\rho'(1) = 0.$$

The last equality shows that $z = 1$ is a double root for (3. 5).

The result obtained by Theorem 3.1 can easily be generalized. Consider (1.6) again. Let $k > 3$ and $p \in \{k-1, k, k+1, \ldots, 2k-1\}$. Assume also that

(3.17) $\alpha_{k+i} = 0 \qquad (i = 4(1)k)$

and

(3.18) $\alpha_{k-3} = 1 - \alpha_{k-1} - \alpha_{k-2}$.

Then the following theorem can be proved as Theorem 3.1 (observing that k-3 roots of the ρ-polynomial are equal to zero when (3.17) is satisfied and one of the other roots is equal to 1).

Theorem 3.2 The explicit linear k-step methods for which (3.17) - (3.18) hold are strongly zero-stable if the non-zero coefficients in (3.1) satisfy

(3.19) $\alpha_{k-1} < \alpha_{k-2}$,

(3.20) $\alpha_{k-2} > -1$

and

(3.21) $\alpha_{k-2} > -3 - 2\alpha_{k-1}$.

□

Finally, it should be mentioned that the following result can easily be obtained from Theorem 3.1.

Theorem 3.3 Let α_2 and α_1 satisfy (3.2) - (3.4). Then the order p of the method is at most 3 (i.e. $p \in \{2, 3\}$).

□

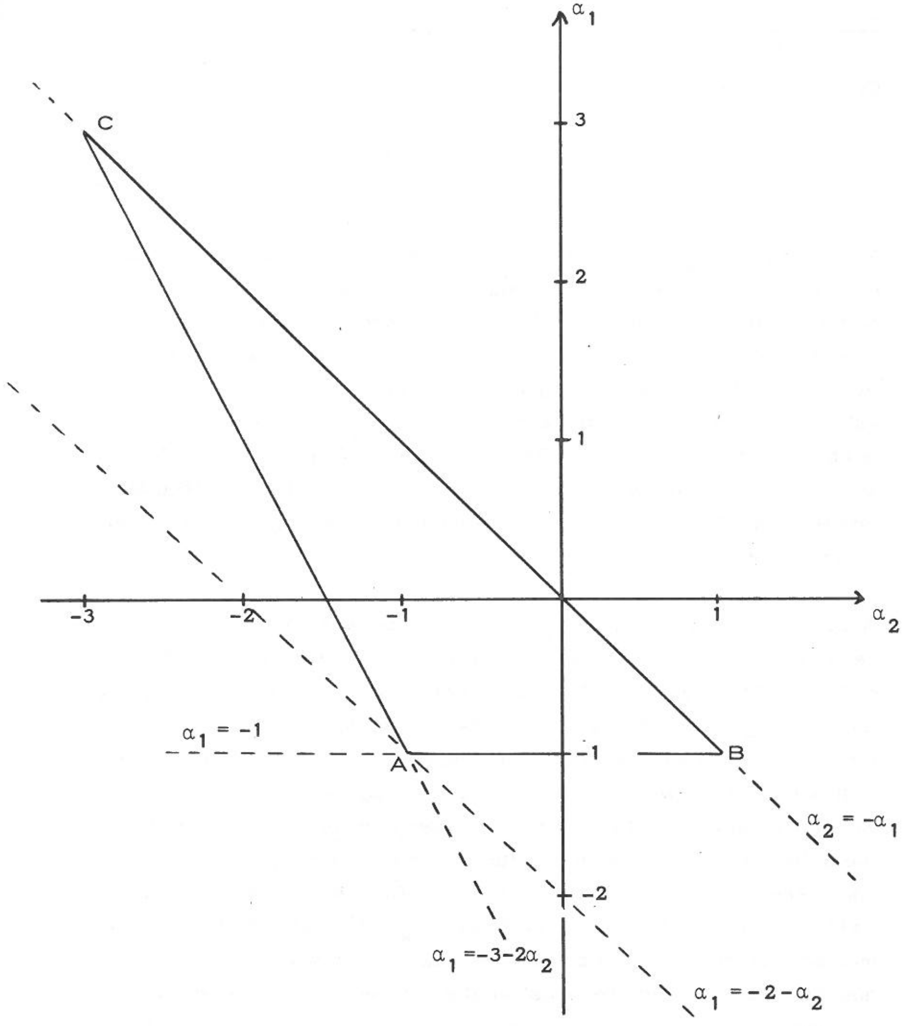

Fig. 3.1

Zero-stability region for the three-parameter family (2.4) - each pair (α_2, α_1) in the inside of the triangle ABC gives a strongly zero-stable formulae.

4. Absolute stability properties of the formulae with p = k

Consider the set

$$(4.1)\quad A = \{(\alpha_2,\alpha_1)/\alpha_2 = -2.95(0.05)0.95,\ \alpha_1 = -\alpha_2 - 0.05(-0.05)\max(-2.95-2\alpha_2, -0.95)\}.$$

For each pair $(\alpha_2,\alpha_1) \in A$ the corresponding parameter β_2^* has been computed by (2.6). Denote the length of the absolute stability interval on the negative part of the real axis by h_1 and the length of the absolute stability interval on the positive part of the imaginary axis by h_2. Both h_1 and h_2 are non-negative functions of α_2 and α_1. The values of these functions for each pair $(\alpha_2, \alpha_1) \in A$ have been computed by an algorithm based on Schur's criterion (see [6], pp. 77-79) and plotted using the standard plot-subroutines available at RECAU (the Regional Computing Centre at Aarhus University). The plots are given in Fig. 4.1 and Fig. 4.2.

According to Theorem 1.1 $h_1 < 2$ in this case. From Fig. 4.1 it is seen that the largest values of h_1 can be found in a neighbourhood of p. C (see also Fig. 3.1). The numerical experiments show that $h_1 \to 2$ for $\alpha_2 \to -3_+ \wedge \alpha_1 \to 3_-$ (see the plots on Fig. 4.3; the package developed by Sand and Østerby [9] has been used to draw all plots given in this paper). However, the methods found by values of α_2 and α_1 which are very close to -3 and 3 have large error constants (here the definition of the error constant given by Henrici [3] on p. 223 is considered; Lambert [6] uses another definition, which is the same as Henrici's for $\rho'(1) = \beta_0 + \beta_1 + \dots + \beta_k = 1$). Nevertheless, some methods can successfully be used in practical computations when the absolute stability requirements for the problem under consideration are dominant over the accuracy requirements (i.e. if smaller value of the stepsize h should be used this is in order to ensure stable computations rather than to obtain more accurate results ; see Shampine [12] and Zlatev and Thomsen [18]).

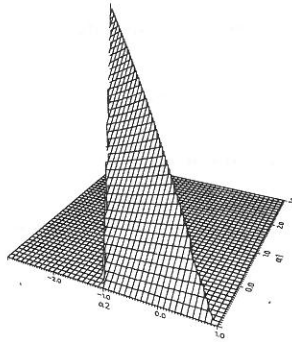

Fig. 4.1

The length h_1 of the absolute stability interval on the negative part of the real axis as function of α_2 and α_1. In this plot $h_1 = 0$ when $(\alpha_2, \alpha_1) \notin A$.

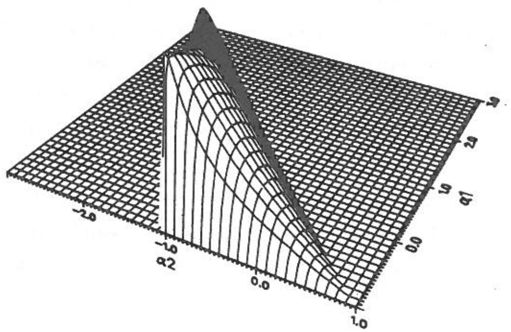

Fig. 4.2

The length h_2 of the absolute stability interval on the positive part of the imaginary axis as function of α_2 and α_1. In this plot $h_2 = 0$ when $(\alpha_2, \alpha_1) \notin A$.

The following example illustrates that the application of some methods of family (2.4) with good absolute stability properties may be more profitable than the corresponding Adams formula.

Example 4.1 (Shampine [12]) Solve the problem defined by

$$(4.2)\qquad y_1' = 0.1\,y_1 - 199.9\,y_2\,,\qquad y_1(0) = 2.0\,,$$

$$(4.3)\qquad y_2' = \qquad - 200.0\,y_2\,,\qquad y_2(0) = 1.0\,,$$

for $x \in [0, 6]$.

□

The solution of this problem contains a small interval $[0, c]$ where the accuracy requirements are dominant in the efforts to obtain an acceptable numerical solution. On the interval $[c, 6]$ the stability requirements are dominant. The number c depends on accuracy required (if the required accuracy is lower, then c is smaller).

The integration on $[0, c]$ has been carried out with Euler's formula and with $h = 0.0001$. A method selected by $\alpha_2 = 2.35$, $\alpha_1 = 2.05$ and $\beta_2 = \beta_2^*$ has been used in the integration on $[c, 6]$ with $h = 0.0075$. The whole integration has been performed with an error of magnitude $O(10^{-5})$ for $c = 0.045$. The attempt to use the third order Adams-Bashforth on $[0.045, 6]$ with $h = 0.0075$ has not been successful; the computations have been unstable. This method can be used to solve this problem only if h is reduced to 0.0025 (i.e. three times).

Note too that if the accuracy required is of magnitude $O(10^{-2})$, then the Euler formula should be used only on $[0, 0.015]$, while $h = 0.0075$ can be used on $[0.015, 6]$; i.e. c is equal to 0.015 in this case.

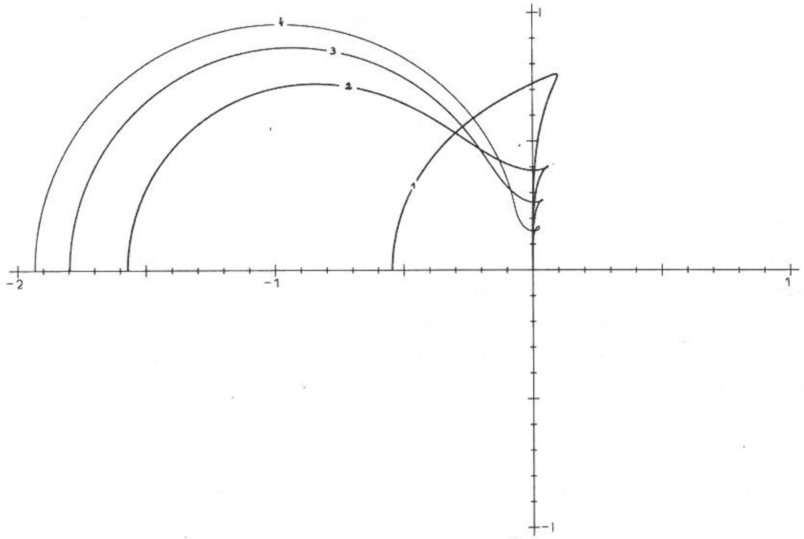

Fig. 4.3

Stability regions for some formulae in the three-parameter family. The parameters used are as follows :

(1) $\alpha_2 = -1.0000$, $\alpha_1 = 0.0000$, $\beta_2^* = 1.9167$

(2) $\alpha_2 = -2.3500$, $\alpha_1 = 2.0500$, $\beta_2^* = 1.2958$

(3) $\alpha_2 = -2.7000$, $\alpha_1 = 2.5500$, $\beta_2^* = 1.1375$

(4) $\alpha_2 = -2.9000$, $\alpha_1 = 2.8500$, $\beta_2^* = 1.0458$

Let us consider now the Fig. 4.2. It is seen that $h_2 \le 1$ for all methods of the family with $\beta_2 = \beta_2^*$. This can be considered as a numerical illistration (for k = 3) of the following theorem proved by Jeltsch and Nevanlinna [5].

Theorem 4.1 (Jeltsch and Nevanlinna [5], pp. 34-35)

Let $r \in [0,1]$ and $k \in \{2,3,4\}$ be given. Then there exist explicit linear k-step method of order p = k with

(4.4) $\qquad (-r, r) \in S.$ □

Let α_2 be fixed and close to -1. Then $h_2 \to 1_-$ for $\alpha_1 \to -3-2\alpha_2$ and $(\alpha_2, \alpha_1) \in A$. Note that in this case $h_1 \to 0$. This should be expected; see Theorem 5.1 in Jeltsch and Nevanlinna [5], p. 32. It is not necessary to formulate this theorem here (too many auxiliary definitions and statements are needed). Roughly speaking, the theorem states that for the explicit linear k-step methods either $[-i, i] \notin S$ or $[-i, i] = S$; see more details in [5].

The above considerations are illustrated in Fig. 4.4 where $\alpha_2 = -1.05$ has been fixed and the absolute stability regions for the third order formulae obtained with $\alpha_1 = 0.4$, $\alpha_1 = -0.1$, $\alpha_1 = -0.55$ and $\alpha_1 = -0.9$ are plotted.

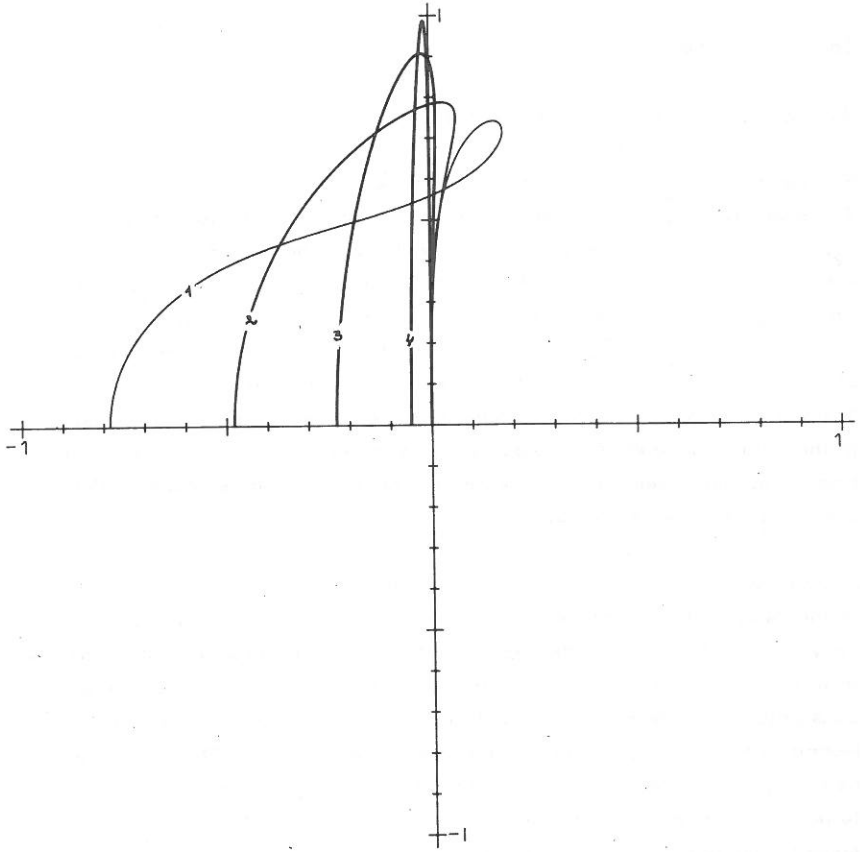

Fig. 4.4

The absolute stability regions for some formulae in the three parameter family. The parameters used are $\alpha_2 = -1.05$, $\beta_2 = \beta_2^*$ and

(1) $\alpha_1 = 0.4$

(2) $\alpha_1 = -0,1$

(3) $\alpha_1 = -0.55$

(4) $\alpha_1 = -0.9$

5. Absolute stability properties of the formulae with p = k-1

Consider the set

$$(5.1)\quad B = \{(\alpha_2, \alpha_1)/\alpha_2 = -2.8(0.2)0.8;\ \alpha_1 = -\alpha_2 - 0.1(-0.2)\max(-2.8-2\alpha_2, -1.0)\}$$

For each pair $(\alpha_2, \alpha_1) \in B$ formula (2.6) has been used to compute β_2^*. The quantities h_1 and h_2 (defined at the beginning of section 4) depend on α_2, α_1 and β_2 in the case where p = k-1 = 2. For each pair $(\alpha_2, \alpha_1) \in B$ a direct search for the value $\bar{\beta}_2$, such that $h_1(\alpha_2, \alpha_1, \bar{\beta}_2) \geq h_1(\alpha_2, \alpha_1, \beta_2)$ where β_2 is in a neighbourhood of β_2^*, has been carried out. Denote $\bar{h}_1(\alpha_2, \alpha_1) = h_1(\alpha_2, \alpha_1, \bar{\beta}_2)$. It is clear that $\bar{h}_1$ is a non-negative function of α_2 and α_1. The corresponding function $\bar{h}_2$ is defined in a similar way. The values of $\bar{h}_1$ and $\bar{h}_2$ have been computed for all pairs $(\alpha_2, \alpha_1) \in B$. In the neighbourhood of some points (as e.g. (-3, 3) and (-1, 0)) more refined grids have been used. The results obtained by these computations can be summarized as follows.

According to Theorem 1.2 $\bar{h}_1$ can be larger than 2 in this case. This is the case in the neighbourhood of the point $(\alpha_2, \alpha_1) = (-3, 3)$. In fact, very large values of $\bar{h}_1$ can be found for $(\alpha_2, \alpha_1) \to (-3, 3)$. Unfortunately, in this case $\rho'(1) = \beta_2 + \beta_1 + \beta_0$ tends to zero. This shows that the error constants of the formulae so found (again the definition proposed by Henrici [3], p. 223, is considered) become very large. Therefore the pair (α_2, α_1) should not be chosen too close to (-3, 3) if the method is to be used in practical computations. The absolute stability regions of some formulae are given in Fig. 5.1.

The usefulness of the formulae in the numerical integration of some problems is illustrated by the following example.

Example 5.1 As Example 4.1 in section 4 with $x \in [0, 200]$. □

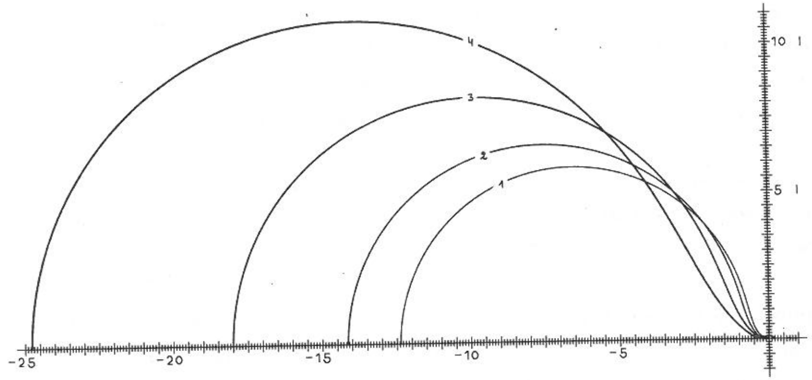

Fig. 5.1

The stability regions for some formulae in the three-parameter family. The parameters are as follows

(1) $\alpha_2 = -2.98$, $\alpha_1 = 2.9610$, $\beta_2 = 0.17$

(2) $\alpha_2 = -2.98$, $\alpha_1 = 2.9609$, $\beta_2 = 0.15$

(3) $\alpha_2 = -2.98$, $\alpha_1 = 2.9608$, $\beta_2 = 0.12$

(4) $\alpha_2 = -2.98$, $\alpha_1 = 2.9607$, $\beta_2 = 0.09$

The formula generated by $\alpha_2 = -2.98$, $\alpha_1 = 2.961$ and $\beta_2 = 0.17$ has been used on the interval $[c, 200]$ with a stepsize $h = 0.05$, while the Euler formula has been used (as in Example 4. 1) on the interval $[0, c]$ with a stepsize $h = 0.0001$. Note that in this case c should have a larger value (compared with the value obtained in Example 4. 1) because both the formula is not so accurate (the order is $p = 2$) and a much larger value of h has to be used. With $c = 4$ the largest error observed on the whole interval $[0, 200]$ was $0(10^{-3})$. The Adams formula of order 2 could be used with maximal stepsize $h = 0.005$ for this problem. It should be noted that the first phase of the integration could be performed much more efficiently (e.g. using the classical 4'th order Runge-Kutta method or the method used in section 4. 1). However, our purpose here is only to show that large stepsizes could be used. The problem of efficiency of the computation is briefly discussed in section 6.

Let us consider now the size of $\bar{h}_2$. Again the largest values of $\bar{h}_2$ are in the neighbourhood of $(-1, -1)$ and again $\bar{h}_2 < 1$. Some absolute stability regions are given in Fig. 5. 2.

The problems for which the formulae should have absolute stability regions with large values of h_2 (i.e. the problems whose Jacobians have some eigenvalues which are large in absolute value and close to the imaginary axis) require normally a greater degree of accuracy also. Therefore the explicit linear multistep methods are not suitable for such problems. If the use of explicit methods is desirable, then some Runge-Kutta method (e.g. of order 4) will be preferable in comparison with the explicit linear multistep methods (the scaled absolute stability intervals on the imaginary axis will be comparable, while the accuracy properties of the Runge-Kutta formula will be much better).

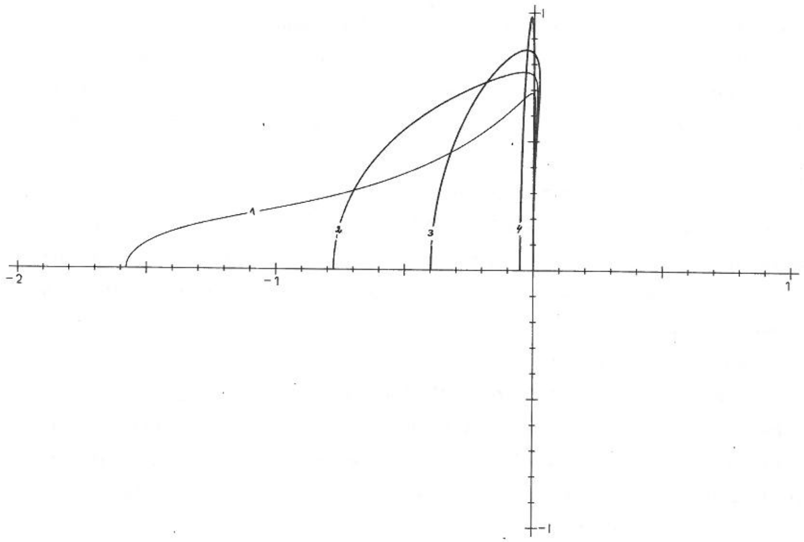

Fig. 5. 2

Stability regions for some formulae in the three-parameter family. The parameters used are as follows

(1)	$\alpha_2 = -1.0$,	$\alpha_1 = 0.5$,	$\beta_2 = 1.4750$
(2)	$\alpha_2 = -1.0$,	$\alpha_1 = 0.1$,	$\beta_2 = 1.7083$
(3)	$\alpha_2 = -1.0$,	$\alpha_1 = -0.3$,	$\beta_2 = 1.8792$
(4)	$\alpha_2 = -1.0$,	$\alpha_1 = -0.9$,	$\beta_2 = 1.9792$

6. On the practical use of formulae with extended absolute stability regions

Typical for many practical problems is the fact that at the beginning of the integration the accuracy requirements are dominant, while after that the absolute stability requirements are much more important. This normally means that the numerical integration cannot be started with a large stepsize h in this situation unless the formula is both very accurate and very stable. If the additional requirement that the formula is both explicit and linear multistep is imposed, then the formula will not simultaneously be very stable and very accurate. This shows that the formulae with extended absolute stability regions should be combined with some other formulae (which are more accurate) in a VSVFM (variable stepsize variable formula method; see e.g. [15] and [18]). The more accurate formulae should be used in the starting phase of the numerical integration. When the absolute stability requirements become dominant, an automatic switch to the formulae with extended absolute stability regions has to be performed. In the case where predictor-corrector schemes are used, these ideas have already been used by Zlatev and Thomsen [19] in a VSVFM based on Adams predictor-corrector schemes (which are the accurate formulae) and on some predictor-corrector schemes derived in [14, 18] (which have better absolute stability properties than the corresponding Adams schemes). Similar ideas can be used for the methods found in this paper.

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