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A Correction to "A Modal Logic for a Subclass of Event Structures"

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0. Introduction

The proof of the completeness result presented in the report DAIMI PB-220 ("A Modal Logic for a Subclass of Event Structures" by K. Lodaya and P.S. Thiagarajan) contains a flaw. Our aim here is to rectify this flaw. To save space, we shall assume that the reader has a copy of DAIMI PB-220. The flaw occurs in the proof of Lemma 4.19. Hence the material we present here is to *replace* the material in Section 4 starting with Lemma 4.19.

1. The Correction

Before proceeding to present a correct proof of Lemma 4.19, it will be convenient to rework the definition of an n-agent event structure. Let $ES = (E, \leq, \#)$ be an n-agent event structure. Then it is easy to observe that $\#$ is a *derived* relation. Stated differently, once $\leq$ and $E_1, E_2, \ldots, E_n$ are fixed, then $\#$ can be extracted in a unique fashion provided the property asserted below is satisfied. And this property is always satisfied by an n-agent event structure.

Proposition 4.19

Let $ES = (E, \leq, \#)$ be an n-agent event structure. Then

$$(\forall e \in E)(\forall i)(\forall e_1, e_2 \in E_i)[e_1 \leq e \text{ and } e_2 \leq e \Rightarrow e_1 \leq e_2 \text{ or } e_2 \leq e_1]$$

Proof: Suppose $e \in E$ and $e_1, e_2 \in E_i$ such that $e_1 \leq e$ and $e_2 \leq e$. If $e_1 \not\leq e_2$ and $e_2 \not\leq e_1$ then $e_1 \# e_2$ by the definition of an n-agent event structure. But then $e_1 \leq e$ and $e_2 \leq e$ would together imply $e \# e$ which is a contradiction because $\#$ is supposed to be irreflexive. □

Proposition 4.20

Let $ES' = (E_1, E_2, \ldots, E_n, \leq)$ be a structure such that the following conditions are satisfied.

(i) $i \neq j \Rightarrow E_i \cap E_j = \emptyset$

(ii) $(E, \leq)$ is a poset where $E = \bigcup_{i=1}^{n} E_i$

(iii) $(\forall e \in E)(\forall i)(\forall e_1, e_2 \in E_i)[e_1 \leq e$ and $e_2 \leq e \Rightarrow e_1 \leq e_2$ or $e_2 \leq e_1]$

Then $ES = (E_1, E_2, \ldots, E_n, \leq, \#)$ is an n-agent event structure where $\#$ is the least subset of $E \times E$ given by

(i) $(\forall i)(\forall e_1, e_2 \in E_i)[e_1 \not\leq e_2$ and $e_2 \not\leq e_1 \Rightarrow e_1 \# e_2]$

(ii) $e_1, e_2 \in E_i$ and $e_1 \# e_2$ and $e_1 \leq e_1'$ and $e_2 \leq e_2'$ implies $e_1' \# e_2'$.

Proof: $\#$ is symmetric by definition. $\#$ is irreflexive by part (iii) of the conditions satisfied by the structure of ES'. Since $\leq$ is reflexive and transitive $e_1 \# e_2 \leq e_3$ would at once imply that $e_1 \# e_3$. It is now routine to complete the proof. □

Due to Proposition 4.19 and Proposition 4.20 we can work with the following alternative definition of an n-agent event structure.

Definition 4.21

An n-agent event structure is a structure $ES = (E_1, E_2, \ldots, E_n, \leq)$ such that

(i) $i \neq j \Rightarrow E_i \cap E_j = \emptyset$

(ii) $(E, \leq)$ is a poset where $E = \bigcup_{i=1}^{n} E_i$

(iii) $(\forall e \in E)(\forall i)(\forall e_1, e_2 \in E_i)[e_1 \leq e$ and $e_2 \leq e \Rightarrow e_1 \leq e_2$ or $e_2 \leq e_1]$

The conflict relation # then becomes a derived notion as outlined in the statement of Prop. 4.20. It is this alternative definition we will work with from now on. Note that the material presented in DAIMI PB-220 concerning chronicles (i.e. starting from Def. 4.14) goes through without any modifications w.r.t. the alternative formulation of an n-agent event structure presented in Def. 4.21.

In what follows we will use a weakened notion of a chronicle structure, hence Def. 4.17 in DAIMI PB-220 is to be read as follows.

Definition 4.17 (new)

A chronicle structure is a pair (ES, T), where $ES = (E, \leq)$ is an n-agent event structure and T is a coherent chronicle on ES. □

As in the above definition we will often say that $(E, \leq)$ is an n-agent event structure instead of saying that $(E_1, E_2, \ldots, E_n, \leq)$ is an n-agent event structure.

We can now briefly explain the flaw in the proof of Lemma 4.19 in DAIMI PB-220. For case 1 and case 2 in the proof the resulting structure ES' will, in general *not* satisfy part (iii) of Def. 4.21! As a result, $\#'$ might not be irreflexive.

First let us observe an elementary fact about posets.

Proposition 4.22

Let $PO = (X, \sqsubseteq)$ be a poset and $x_1, x_2 \in X$ such that $x_1 \not\sqsubseteq x_2$ and $x_2 \not\sqsubseteq x_1$. Then $PO' = (X, \sqsubseteq')$ is also a poset where

$$\sqsubseteq' = (\sqsubseteq \cup \{(x_1, x_2)\})^*$$

Proof: Trivial. □

The following intermediate result will be useful for arriving at a correct proof of Lemma 4.19 in DAIMI PB-220.

In what follows whenever we appeal to a result or definition which does not appear here then it is taken to appear in DAIMI PB-220.

Lemma 4.23

Let $ES = (E_1, E_2, \ldots, E_n, \leq)$ be a structure such that $(E, \leq)$ is a *finite* poset (i.e. E is a finite set) where $E = \bigcup_{i=1}^{n} E_i$ and $i \neq j$ implies that $E_i \cap E_j = \emptyset$. Let T be a map which assigns an MCS to each $e \in E$ such that $\forall e \in E.$ [type($T(e)$ $= i$ iff $e \in E_i$] and $\forall e_1, e_2 \in E.$ $[e_1 \leq e_2 \Rightarrow T(e_1) \preceq T(e_2)]$.

Then there exists an n-agent event structure $ES = (E_1, E_2, \ldots, E_n, \leq')$ such that the following conditions are satisfied.

(i) $\leq \subseteq \leq'$.

(ii) $\forall e_1, e_2 \in E.[e_1 \leq' e_2 \Rightarrow T(e_1) \preceq T(e_2)]$.

Proof: Let $k = |(E \times E) - \leq|$. The proof is by induction on k.

<u>$k = 0$</u> Then $(E_1, E_2, \ldots, E_n, \leq)$ is itself an n-agent event structure. In other words we can set $\leq = \leq'$ and be done.

<u>$k > 0$</u> Let $e_0 \in E$ and $i \in \{1, 2, \ldots, n\}$ and $e_1, e_2 \in E_i$ such that $e_1 \not\leq e_2$ and $e_2 \not\leq e_1$ and $e_1 \leq e_0$ and $e_2 \leq e_0$. If no such e_0, i, e_1 and e_2 exist then we can once again set $\leq = \leq'$ and be done.

Now $T(e_1) \preceq T(e_0)$ and $T(e_2) \preceq T(e_0)$ and type($T(e_1)$) $= i =$ type($T(e_2)$). Hence by Lemma 4.16 and Lemma 4.13 (in DAIMI PB-220!) $T(e_1) \preceq T(e_2)$ or $T(e_2) \preceq T(e_1)$. Assume without loss of generality that $T(e_1) \preceq T(e_2)$. Now define $\leq''$ as

$$\leq'' = (\leq \cup \{(e_1, e_2)\})^*$$

Since $e_1 \not\leq e_2$ and $e_2 \not\leq e_1$ we have that $(E\ \leq'')$ is also a poset by Prop. 4.22. Consider now the structure $(E_1, E_2, \ldots, E_n, \leq'')$. As before $\text{type}(T(e)) = i$ iff $e \in E_i$. Now suppose that $e, e' \in E$ such that $e \leq'' e'$.

Then there exists a sequence of elements $x_0, x_1, \ldots, x_n$ such that $e = x_0$ and $x_m = e'$ and $(x_i, x_{i+1}) \in \leq$ or $(x_i, x_{i+1}) = (e_1, e_2)$ for each $i \in \{0, 1, \ldots, m-1\}$. This follows from the definition of $\leq''$. But $(x_i, x_{i+1}) \in \leq$ implies that $T(x_i) \preceq T(x_{i+1})$ by hypothesis. If $(x_i, x_{i+1}) = (e_1, e_2)$ then also $T(x_i) \preceq T(x_{i+1})$ because we have assumed $T(e_1) \preceq T(e_2)$. Moreover $\preceq$ is transitive by Lemma 4.13. Hence $T(e) \preceq T(e')$. Hence the structure $(E_1, E_2, \ldots, E_n, \leq'')$ with the map T satisfies the conditions laid out by the statement of the lemma. Clearly $|(E \times E) - \leq''| \leq k - 1$. The result now follows from the induction hypothesis. □

We can now proceed to present the correct version of Lemma 4.19 in DAIMI PB-220.

Lemma 4.24

Let (ES, T) be a *finite* chronicle structure. In other words $ES = (E, \leq)$ is an n-agent event structure with E as a finite set. Let (e, β) be a live communication requirement in (ES, T). Then there exists a chronicle structure (ES', T') with $ES' = (E', \leq')$ such that

(i) $E' = E \cup \{\hat{e}\}$ for some $\hat{e} \notin E$.

(ii) $\leq \subseteq \leq'$.

(iii) T is T' restricted to E.

(iv) (e, β) is not a live communication requirement in (ES', T').

Proof: Assume – as it will turn out – without loss of generality that β is of the form $\diamondminus_i \alpha$. Then $e \notin E_i$. By Lemma 4.11 there exists an MCS A such that $\text{type}(A) = i$ and $\diamondminus_i \alpha \in A$ and $A \preceq_c T(e)$. Pick some $\hat{e} \notin E$

and set for $1 \leq j \leq n$,

$$E'_j = \begin{cases} E_j \cup \{\hat{e}\} & \text{, if } i = j \\ E_j & \text{, otherwise} \end{cases}$$

Set $E' = \bigcup_{j=1}^{n} E'_j$. Now define

$$\text{Pre}(\hat{e}) = \{e' \mid e' \in E_i \text{ and } e' < e\}$$

Now define $\leq''$ as follows.

$$\leq'' = (\leq \cup (\text{Pre}(\hat{e}) \times \{\hat{e}\}) \cup \{(\hat{e}, e)\})^*$$

We shall prove that $(E', \leq'')$ is a poset as follows. For convenience, first set $R_1 = (\text{Pre}(\hat{e}) \times \{\hat{e}\})$ and $R_2 = \{(\hat{e}, e)\}$.

Claim 1

Let $x \in E$ such that $x \leq'' \hat{e}$. Then there exists $y \in \text{Pre}(\hat{e})$ such that $x \leq y$.

Proof of claim 1: Let $x_0, x_1, \ldots, x_n$ be a finite sequence such that $x = x_0$ and $x_m = \hat{e}$ and for each $j \in \{0, 1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$. Such a sequence must exist by the definition of $\leq''$. Now $\hat{e} \notin E$ and $x \in E$. Hence $m \geq 1$. We proceed by induction on m.

$\underline{m = 1}$ Then $(x_0, x_1) \in R_1$ and hence $x_0 \in \text{Pre}(\hat{e})$.

$\underline{m > 1}$ If $(x_0, x_1) \in R_1$ then we are once again done. $(x_0, x_1) \notin R_2$ because this would imply that $x = x_0 = \hat{e}$.
Now suppose that $(x_0, x_1) \in \leq$. By the induction hypothesis there exists $y \in \text{Pre}(\hat{e})$ such that $x_1 \leq y$. But then $\leq$ is transitive so that $x_0 \leq y$ as well.

□

Claim 2

$(e, \hat{e}) \not\in \leq''$.

Proof of claim 2: We know $e \in E$. Hence $e \leq'' \hat{e}$ would imply by claim 1 that there exists $y \in \text{Pre}(\hat{e})$ such that $e \leq y$. But by the definition of $\text{Pre}(\hat{e})$ we know that $y < e$. Since $\leq$ is antisymmetric we can conclude that $(e, \hat{e}) \not\in \leq''$. □

Claim 3

Let $x \in E$ such that $\hat{e} \leq'' x$. Then $e \leq x$.

Proof of claim 3: Let $x_0, x_1, \ldots, x_m$ be a sequence such that $\hat{e} = x_0$ and $x_m = x$ and for each $j = \{0, 1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$. Since $\hat{e} \neq x$ (because $\hat{e} \notin E$ and $x \in E$) we must have $m \geq 1$.

Now consider the pair (x_0, x_1). Then $(x_0, x_1) \in R_2$. We now have that $x_1 = e$. From claim 2 it follows that $x_j \neq \hat{e}$ for each $j \in \{2, \ldots, m\}$. Hence $(x_j, x_{j+1}) \in \leq$ for each $j = \{1, \ldots, m-1\}$. Hence $e \leq x$. □

Claim 4

Let $(x, y) \in (\leq'' - \leq)$. Then $x \leq'' \hat{e}$ and $\hat{e} \leq'' y$.

Proof of claim 4: Let $x_0, x_1, \ldots, x_m$ be a sequence such that $x = x_0$ and $x_m = y$ and for each $j \in \{0, 1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$. Since $(x, y) \in (\leq'' - \leq)$ it must be the case that $x_j = \hat{e}$ for some $j \in \{0, 1, \ldots, m\}$. The claim now follows at once. □

Claim 5

$(E', \leq'')$ is a poset.

Proof of claim 5: $\leq''$ is reflexive and transitive by definition. So consider $x, y \in E'$ so that $x \leq'' y$ and $y \leq'' x$. If $x \leq y$ and $y \leq x$ we have at once $x = y$ because $\leq$ is anti-symmetric.

Suppose that $(x, y) \in (\leq'' - \leq)$. Then by claim 4 we know that $x \leq'' \hat{e}$ and $\hat{e} \leq'' y$. We can now analyze four different possibilities.

If $x = \hat{e}$ and $\hat{e} = y$ then we have at once $x = y$.

If $x \neq \hat{e}$ and $\hat{e} = y$ then by claim 1 we know that for some $x' \in \mathrm{Pre}(\hat{e})$, $x \leq x'$. But then $y \leq'' x$ would imply by claim 3 that $e \leq x$. We now have the contradiction $e \leq x'$.

If $x = \hat{e}$ and $\hat{e} \neq y$ then from $x \leq'' y$ we can deduce – using claim 3 – that $e \leq y$. But then $y \leq'' x$ implies that $e \leq'' x = \hat{e}$ and this contradicts claim 2.

If $x \neq \hat{e}$ and $y \neq \hat{e}$ then by claim 4 we know that $x \leq'' \hat{e}$ and $\hat{e} \leq'' y$. From $\hat{e} \leq'' y$ it follows by claim 3 that $e \leq y$. From $x \leq'' \hat{e}$ it follows by claim 1 that $x \leq x'$ for some $x \in \mathrm{Pre}(\hat{e})$. Now $x \leq'' y$ and $y \leq'' x$ together lead to the contradiction $e \leq x'$.

So finally assume that $(y, x) \in (\leq'' - \leq)$. The proof of the fact that $x \leq'' y$ and $y \leq'' x$ implies $x = y$ is exactly the same as the case where $(x, y) \in (\leq'' - \leq)$ (with the roles of x and y interchanged) and hence we omit it. $\square$

At this stage we have the poset $(E', \leq'')$ with E' partitioned into $(E'_1, E'_2, \ldots, E'_n)$. Now extend T to E' as follows.

$$\forall x \in E'.\ T'(x) = \begin{cases} A, \text{ if } x = \hat{e} \\ T(x), \text{ otherwise} \end{cases}$$

Unfortunately $(E', \leq'')$ might not be an n-agent event structure because part (iii) of Def. 4.21 might not be fulfilled. We can however *augment* $\leq''$

with additional pairs to arrive at $\leq'$ so that $(E', \leq')$ is indeed an event structure. Lemma 4.23 will be used for this purpose.

Claim 6

$\forall x \in E'.\ [\text{type}(T'(x)) = j \text{ iff } x \in E'_j]$.

Proof of claim 6: Trivial. $\square$

Claim 7

$\forall x, y \in E'.\ [x \leq'' y \Rightarrow T'(x) \preceq T'(y)]$.

Proof of claim 7: Let $x, y \in E'$ such that $x \leq'' y$. Then there exists $x_0, x_1, \ldots, x_m$ such that $x = x_0$ and $x_m = y$ and $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$ for each $j \in \{0, 1, \ldots, m-1\}$. We now proceed by induction on m.

$\underline{m=0}$ Then $x = y$ and we know from Lemma 4.13 that $\preceq$ is reflexive.

$\underline{m>0}$ Suppose $(x_0, x_1) \in \leq$. Then $T'(x_0) = T(x_0)$ and $T'(x_1) = T(x_1)$. Hence $T'(x_0) \preceq T'(x_1)$ because $T(x_0) \preceq T(x_1)$ is implied by $x_0 \leq x_1$. Now $T'(x_1) \preceq T'(x_m)$ by the induction hypothesis. By Lemma 4.13, $\preceq$ is transitive. Hence $T'(x_0) \preceq T'(x_m)$.
Next suppose that $(x_0, x_1) \in R_1$. Then $x_0 \in \text{Pre}(\hat{e})$ and $x_1 = \hat{e}$. Hence $x_0 < e$ and this at once implies that $T(x_0) \preceq T(e)$. By the definition of A $(= T'(\hat{e}))$ we know that $A \preceq_c T(e)$. We claim that this implies that $T(x_0) = T'(x_0) \preceq T'(\hat{e})$. To see this assume that $\nu \in T(x_0) = T'(x_0)$. Since $T(x_0) \preceq T(e)$ and $\text{type}(T(x_0)) = i$ we then have that $\ominus_i \nu \in T(e)$. But then $A \preceq_c T(e)$ and $\text{type}(A) = i$. Hence $\ominus_i \nu \in A$. Thus $T'(x_0) \preceq A = T'(\hat{e}) = T'(x_1)$. Once again by the induction hypothesis, $T'(x_1) \preceq T'(x_m)$. Hence $T'(x_0) \preceq T'(x_m)$.
Finally assume that $(x_0, x_1) \in R_2$. Then $x_0 = \hat{e}$ and $x_1 = e$. By definition of A we know that $T'(x_0) \preceq_c T'(x_1)$. But this implies that $T'(x_0) \preceq T'(x_1)$ as well due to Lemma 4.13. Now once again the required result follows from the induction hypothesis.

□

We now have that the poset $(E', \leq'')$ together with the map T' fulfills the hypothesis of Lemma 4.23. Hence there exists $\leq'$ such that the following conditions are met.

- $(E', \leq')$ is an n-agent event structure.
- $\leq'' \subseteq \leq'$.
- $\forall x, y \in E'. \ [x \leq' y \Rightarrow T'(x) \preceq T'(y)]$.

Clearly (ES', T'), where $ES' = (E', \leq')$, is a chronicle structure in which (e, β) is no longer a live communication requirement. Moreover, $\leq \subseteq \leq'' \subseteq \leq'$ and T is T' restricted to E. □

Lemma 4.25

Let (ES, T) be a chronicle structure with $ES = (E, \leq)$. Let (e, β) be a live historic requirement in (ES, T). Then there exists a chronicle structure (ES', T') with $ES' = (E', \leq')$ such that:

(i) $E' = E \cup \{\hat{e}\}$ for some $\hat{e} \notin E$.

(ii) $\leq \subseteq \leq'$.

(iii) T is T' restricted to E.

(iv) (e, β) is not a live historic requirement in (ES', T').

Proof: Assume that $e \in E_i$ and that β is of the form $\ominus_i \alpha$. Then by Lemma 4.7 (and Lemma 4.13) we can find an MCS A such that $\text{type}(A) = i$ and $\alpha \in A$ and $A \preceq T(e)$. Pick some $\hat{e} \notin E$ and set for $1 \leq j \leq n$

$$E'_j = \begin{cases} E_j \cup \{\hat{e}\} & \text{, if } i = j \\ E_j & \text{, otherwise} \end{cases}$$

Set $E' = \bigcup_{j=1}^{n}$. Now define

$$\begin{aligned} \text{Pre}(\hat{e}) &= \{x \mid x \in E_i \text{ and } x < e \text{ and } T(x) \preceq A\} \\ \text{Post}(\hat{e}) &= \{x \mid x \in E_i \text{ and } x < e \text{ and } T(x) \not\preceq A \text{ and } A \preceq T(x)\} \\ &\quad \cup \{e\} \end{aligned}$$

Claim 1

(i) $E_i \cap \downarrow e = \text{Pre}(\hat{e}) \cup \text{Post}(\hat{e})$.

(ii) $\forall x \in \text{Pre}(\hat{e}).\ \forall y \in \text{Post}(\hat{e}).\ [x < y]$.

Proof of claim 1: Clearly $\text{Pre}(\hat{e}), \text{Post}(\hat{e}) \subseteq E_i \cap \downarrow e$. So consider $x \in E_i \cap \downarrow e$. Since $x \leq e$, $T(x) \preceq T(e)$. Since $A \preceq T(e)$ by the choice of A, we have that $T(x) \preceq A$ (in which case $x \in \text{Pre}(\hat{e})$) or $A \preceq T(x)$ by Lemma 4.16. To establish the second part of the claim assume that $x \in \text{Pre}(\hat{e})$ and $y \in \text{Post}(\hat{e})$. By the definitions of $\text{Pre}(\hat{e})$ and $\text{Post}(\hat{e})$, it is clear that $x \neq y$.

Suppose that $y = e$. Then clearly $x < y$. So assume that $y \in \text{Post}(\hat{e}) - \{e\}$. Now $x \leq e$ and $y \leq e$ and $x, y \in E_i$. Hence $x < y$ or $y < x$. If $y < x$ then $T(y) \preceq T(x)$. But $x \in \text{Pre}(\hat{e})$ implies that $T(x) \preceq A$. Since $\preceq$ is transitive this would imply that $T(y) \preceq A$. This contradicts the fact that $y \in \text{Post}(\hat{e})$. Hence $x < y$. □

Now define $\leq'$ as follows.

$$\leq' = (\leq \cup (\text{Pre}(\hat{e}) \times \{\hat{e}\}) \cup (\{\hat{e}\} \times \text{Post}(\hat{e})))^*$$

As before, for convenience we set $R_1 = \text{Pre}(\hat{e}) \times \{\hat{e}\}$ and $R_2 = \{\hat{e}\} \times \text{Post}(\hat{e})$. Our first aim is to show that $ES' = (E', \leq')$ is an n-agent event structure. $(E'_1, E'_2, \ldots, E'_n)$ is clearly a partitioning of E'. Hence we need to show first that $(E', \leq')$ is a poset.

Claim 2

Let $x \in E$ such that $x \leq \hat{e}$. Then there exists $y \in \text{Pre}(\hat{e})$ such that $x \leq y$.

Proof of claim 2: Let $x_0, x_1, \ldots, x_m$ be a sequence such that $x = x_0$ and $x_m = \hat{e}$ and for each $j \in \{0, 1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$. Since $x \in E$ and $\hat{e} \notin E$ we have that $x \neq \hat{e}$. Hence $m \geq 1$. We now proceed by induction on m.

$\underline{m=1}$ $(x_0, x_1) \notin \leq$ because $x_1 = \hat{e} \notin E$. $(x_0, x_1) \notin R_2$ because $x_0 = x \neq \hat{e}$. Hence $(x_0, x_1) \in R_1$ which at once implies that $x_0 \in \text{Pre}(\hat{e})$.

$\underline{m>1}$ Consider (x_0, x_1). As before $(x_0, x_1) \notin R_2$ because $x_0 = x \neq \hat{e}$. If $(x_0, x_1) \in R_1$ then $x_0 \in \text{Pre}(\hat{e})$ and we are done. So suppose that $(x_0, x_1) \in \leq$. Then $x_1 \in E$ and $x_1 \leq' \hat{e}$. Hence by the induction hypothesis there exists $y \in \text{Pre}(\hat{e})$ such that $x_1 \leq y$. This implies that $x = x_0 \leq y$ as well.

□

Claim 3

Let $x, y \in E$ such that $x \leq' y$. Then $x \leq y$.

Proof of claim 3: Let $x_0, x_1, \ldots, x_m$ be a sequence such that $x = x_0$ and $x_m = y$ and for each $j \in \{0, 1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$. We now proceed by induction on m.

$\underline{m=0}$ Clearly $x = y$ and $x \leq y$.

$\underline{m>0}$ Consider (x_0, x_1). If $(x_0, x_1) \in \leq$ then $x_1 \leq x_m$ by the induction hypothesis and hence $x_0 \leq x_m$. We know that $(x_0, x_1) \notin R_2$ because $x_0 \in E$ and $\hat{e} \notin E$.
So suppose that $(x_0, x_1) \in R_1$. Then $x_0 \in \text{Pre}(\hat{e})$. Since $x_1 = \hat{e} \notin E$ we must have $m > 1$. So consider the pair (x_1, x_2). Then $(x_1, x_2) \in R_2$. Hence $x_2 \in \text{Post}(\hat{e})$. Now $x_2 \leq x_m$ by the induction hypothesis. And $x_0 < x_2$ by claim 1.

□

Claim 4

Let $x \in E$ such that $\hat{e} \leq' x$. Then there exists $y \in \text{Post}(\hat{e})$ such that $y \leq x$.

Proof of claim 4: Let $x_0, x_1, \ldots, x_m$ be a sequence such that $\hat{e} = x_0$ and $x_m = x$ and for each $j \in \{0, 1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in \leq \cup R_1 \cup R_2$. As before $m \geq 1$ because $\hat{e} \neq x$. Clearly $(x_0, x_1) \in R_2$. Hence $x_1 \in \mathrm{Post}(\hat{e})$. This implies that $x_1 \leq' x_m$ and by the previous claim we have that $x_1 \leq x_m$. □

Claim 5

$(E', \leq')$ is a poset.

Proof of claim 5: $\leq'$ is reflexive and transitive by definition. So assume that $x, y \in E'$ such that $x \leq' y$ and $y \leq' x$. If $x \in E$ and $y \in E$ then we have from claim 3 that $x \leq y$ and $y \leq x$ and hence $x = y$. If $x \notin E$ and $y \notin E$ then clearly $x = y = \hat{e}$. So assume that $x \in E$ and $y \notin E$. Then $y = \hat{e}$. From $x \leq' y$ it follows from claim 2 that there exists some $x' \in \mathrm{Pre}(\hat{e})$ such that $x \leq' x'$. Since $y \leq' x$ and $y = \hat{e}$ and $x \in E$ it follows from claim 4 that for some $y' \in \mathrm{Post}(\hat{e})$ it is the case that $y' \leq x$. But this would imply that $y' \leq x'$ which contradicts claim 1. The argument for the case $x = \hat{e}$ and $y \in E$ is entirely symmetric and hence we omit it. □

Claim 6

$(E', \leq')$ is an n-agent event structure.

Proof of claim 6: As observed before $(E'_1, E'_2, \ldots, E'_m)$ is a partitioning of E'. By claim 5, $(E', \leq')$ is a poset. Hence it suffices to check that part (iii) of Def. 4.21 holds.

So assume that $e_1, e_2 \in E'_j$ and $e_0 \in E'$ such that $e_1 \leq' e_0$ and $e_2 \leq' e_0$. We must show that $e_1 \leq' e_2$ or $e_2 \leq' e_1$.

Case 1 $e_1 \in E$ and $e_2 \in E$.
Suppose that $e_0 \in E$ as well. Then $e_1 \leq e_0$ and $e_2 \leq e_0$ by claim 3. Hence $e_1 \leq e_2$ or $e_2 \leq e_1$ because $(E, \leq)$ is an n-agent event structure. The required result now follows from the fact $\leq \subseteq \leq'$.
So suppose that $e_0 \notin E$. Then $e_0 = \hat{e}$. But then $\hat{e} \leq' e$. Hence by the transitivity $\leq'$ we have $e_1 \leq' e$ and $e_2 \leq' e$ with $e \in E$. We now have the situation considered above.

Case 2 $e_1 \in E$ and $e_2 \notin E$.
Then $e_2 = \hat{e}$ and $e_1 \in E_i$. Suppose $e_0 \notin E$. Then $e_0 = \hat{e} = e_2$ so that $e_1 \leq' e_2$.
So assume that $e_0 \in E$. Since $e_2 = \hat{e} \leq' e_0$ we have from claim 4 that there exists $y \in \text{Post}(\hat{e})$ such that $y \leq e_0$. Now $e_1 \leq' e_0$ implies that $e_1 \leq e_0$ by claim 3. We now have $e_1, y \in E_i$ and $e_1 \leq e_0$ and $y \leq e_0$. Hence $e_1 \leq y$ or $y \leq e_1$. Suppose that $e_1 \leq y$. Then $e_1 \in E_i \cap \downarrow e$. Hence by claim 1 we have that $e_1 \in \text{Pre}(\hat{e}) \cup \text{Post}(\hat{e})$. This at once implies that $e_1 \leq' \hat{e}$ or $\hat{e} \leq' e_1$. If on the other hand, we have $y \leq e_1$ then from $\hat{e} \leq' y$ we at once have $\hat{e} \leq' e_1$.

Case 3 $e_1 \notin E$ and $e_2 \in E$.
The argument for this case is the same as the one for the previous case with the roles of e_1 and e_2 interchanged.

Case 4 $e_1 \notin E$ and $e_2 \notin E$.
Then $e_1 = e_2 = \hat{e}$.

□

We now extend T to E' as follows:

$$\forall x \in E'.\, T'(x) = \begin{cases} A, & \text{if } x = \hat{e} \\ T(x), & \text{otherwise} \end{cases}$$

Claim 7

Let $x, y \in E'$ such that $x \leq' y$. Then $T'(x) \preceq T'(y)$.

Proof of claim 7:

Case 1 $x \in E$ and $y \in E$.
Then $x \leq y$ by claim 3. Moreover $T'(x) = T(x)$ and $T'(y) = T(y)$. But then $T(x) \preceq T(y)$ because $x \leq y$.

Case 2 $x \in E$ and $y \notin E$.
Then $y = \hat{e}$. By claim 2 we have that for some $x' \in \text{Pre}(\hat{e})$ it is the case that $x \leq x'$. As before $T'(x) = T(x)$ and $T'(x') = T(x')$ and $T(x) \preceq T(x')$. Hence $T'(x) \preceq T'(x')$. But then $x' \in \text{Pre}(\hat{e})$ implies that $T(x') \preceq A = T'(\hat{e})$. The required result now follows from the transitivity of $\preceq$.

Case 3 $x \notin E$ and $y \in E$.
Then $x = \hat{e}$. By claim 4 we know that $x' \leq y$ for some $x' \in \text{Post}(\hat{e})$. But $x' \in \text{Post}(\hat{e})$ implies that $A \preceq T'(x')$. On the other hand, $x' \leq y$ implies that $T(x') \preceq T(y)$. Hence $T'(x) \preceq T'(y)$ by the transitivity of $\preceq$.

Case 4 $x \notin E$ and $y \notin E$.
Then $x = y = \hat{e}$ and the result follows from the reflexivity of $\preceq$.

□

It is now easy to conclude that (ES', T'), where $ES' = (E', \leq')$, is a chronicle structure with the required properties. □

Lemma 4.26

Let (ES, T) be a chronicle structure with $ES = (E, \leq)$. Let (e, β) be a live prophetic requirement in (ES, T). Then there exists a chronicle structure (ES', T') with $ES' = (E', \leq')$ such that

(i) $E' = E \cup \{\hat{e}\}$ with $\hat{e} \notin E$.

(ii) $\leq \subseteq \leq'$.

(iii) T is T' restricted to E.

(iv) (e, β) is not a live prophetic requirement in (ES', T').

Proof: Assume that $e \in E_i$ and that β is of the form $\Diamond_i \alpha$. By Lemma 4.7 (and Lemma 4.13) there exists an MCS A such that type$(A) = i$ and $\alpha \in A$ and $T(e) \preceq A$. Pick some $\hat{e} \notin E$ and set for $1 \leq j \leq n$,

$$E'_j = \begin{cases} E_j \cup \{\hat{e}\} & \text{, if } i = j \\ E_j & \text{, otherwise} \end{cases}$$

Set $E' = \bigcup_{j=1}^{n}$. Now define $\leq'$ as follows.

$$\leq' = (\leq \cup \{(e, \hat{e})\})^*$$

The following claims are easy to verify and hence we omit the proofs.

Claim 1

Let $x \in E$. Then $(\hat{e}, x) \notin \leq'$.

Claim 2

Let $x \in E$ such that $x \leq' \hat{e}$. Then $x \leq e$.

Claim 3

Let $x, y \in E$ such that $x \leq' y$. Then $x \leq y$.

Claim 4

$(E', \leq')$ is a poset.

Claim 5

$(E', \leq')$ is an n-agent event structure with $(E'_1, E'_2, \ldots, E'_n)$ as a partition of E.

Now extend T to E' as follows.

$$\forall x \in E'. \, T'(x) = \begin{cases} A, & \text{if } x = \hat{e} \\ T(x), & \text{otherwise} \end{cases}$$

It is now easy to verify that (ES', T') (with $ES' = (E', \leq')$) is a chronicle structure which enjoys the required properties. □

Theorem 4.27 (Completeness)

If $\models \alpha$ then $\vdash \alpha$.

Proof: We will show that every consistent formula is satisfiable. Let $\hat{E}$ be a countably infinite set of events. Fix an enumeration $e_1, e_2, \ldots$ of $\hat{E}$. Fix an enumeration $\alpha_1, \alpha_2, \ldots$ of F, the set of formulas. Fix an *injective* function $f : \hat{E} \times F \rightarrow \mathbf{N}$. Since $\hat{E} \times F$ is a countable set there will be no trouble in finding such an injective function. In what follows, for $(e, \alpha) \in \hat{E} \times F$, we will refer to $f((e, \alpha))$ as the *code number* of (e, α).

Now assume that α is a consistent formula. Pick as MCS A which contains α. Let $CS^1 = (ES^1, T^1)$ where $ES^1 = (\{e_1\}, \{(e_1, e_1)\})$ and $T^1(e_1) = A$. Define for each $j \in \{1, \ldots, n\}$,

$$E^1_j = \begin{cases} \{e_1\}, & \text{if type}(A) = j \\ \emptyset, & \text{otherwise} \end{cases}$$

Set $E^1 = \{e_1\}$ and $\leq^1 = \{(e_1, e_1)\}$. Clearly (ES^1, T^1) is a chronicle structure. For $m \geq 1$ assume inductively that the chronicle structure $CS^m = (ES^m, T^m)$ is defined with $ES^m = (E^m, \leq^m)$ where $E^m = \{e_1, e_2, \ldots, e_m\}$. Suppose CS^m does not have any live requirements. Then set $CS^{m+1} = CS^m$. Otherwise consider a live requirement (e, β) in CS^m which has – among all the live requirements in CS^m – the least code number.

Now CS^m is clearly a *finite* chronicle structure. Hence by the previous three lemmas CS^m can be extended to the chronicle structure $CS^{m+1} = (ES^{m+1}, T^{m+1})$ with $ES^{m+1} = (E^{m+1}, \leq^{m+1})$ and $E^{m+1} = E^m \cup \{e_{m+1}\}$ so that (e, β) is no longer a live requirement in ES^{m+1} and $\leq^m \subseteq \leq^{m+1}$ and T^m is T^{m+1} restricted to E^m.

Finally set $CS = (ES, T)$ where $ES = (E, \leq)$ and $E = \bigcup_{m=1}^{\infty} E^m$ and $\leq = \bigcup_{m=1}^{\infty} \leq^m$ and T is given by:

$$\forall e \in E.\, T(e) = T^m(e) \text{ where } e \in E^m.$$

It is routine to verify that T is a perfect chronicle on ES. Hence by Lemma 4.15 $M = (ES, V_T)$ is a model in which $e_1, M \models \alpha$. □

3. Concluding Remarks

It turns out that the axiomatization of our logical system can be considerably simplified. The completeness proof can also be more succinct. With some additional axioms one can also obtain a sound and complete axiomatization of *finitary* n-agent event structures, i.e. those n-agent event structures in which $|\downarrow e| < \infty$ for every event. These details will appear in a forthcoming paper.

The decidability of our logical system (i.e. the decidability of the question whether or not an arbitrary formula is satisfiable) is still open. Lemma 4.23 has given us fresh hope that indeed our system is decidable.