ISSN 0105-8517

An Exact and Efficient Implementation of Threshold Gates with Arbitrary Real Weights

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DAIMI PB – 241
October 1988

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Abstract

We show how to exactly implement an n input threshold gate with arbitrary real weights by a circuit of constant depth and size polynomial in n.

Our circuits use **and**, **or** and **majority** gates as well as unary **negation** gates. The use of **majority** is imperative due to certain lower bound results.

Introduction

Threshold gates have been used as formal models for neurons, for reasoning about neural network behaviour, and for constructing neural networks for tasks such as the learning of simple pattern recognition (see e.g. [Rumelhart 86, Kohonen 88]). The learning process has been modelled by a (continuous) modification of the weights in the individual threshold gates [Jones 87]. One might naively imagine that threshold gates with arbitrary real weights are more powerful than gates with weights restricted to be integers of manageable size.

Surprisingly, this is not the case. We give an exact and efficient hardware implementation of any threshold gate by reducing arbitrary real weights to small integer weights.

In this implementation, the basic hardware units are taken to be majority gates as well as the usual boolean gates. We shall argue both that these basic units are physically reasonable and that the power of majority is needed in any similarly efficient implementation.

Majority is a special kind of threshold gate. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an arbitrary tuple of real numbers. We define $T_k^{\underline{\alpha}}(x_1, \ldots, x_n) \in \{0,1\}$ for arbitrary real k as follows:

$$T_k^{\underline{\alpha}}(x_1, \ldots, x_n) = 1 \text{ iff } \sum_i \alpha_i x_i \geq k$$

where $x_i \in \{0,1\}$ for each i. $T_k^{\underline{\alpha}}$ is then said to be a real weighted threshold function. Rational weighted threshold functions and integer weighted threshold functions are defined analogously, i.e. by restricting $\underline{\alpha}$ and k to be rational numbers and integers respectively.

A majority gate is a threshold gate with $\underline{\alpha} = (1, 1, 1, \ldots, 1)$ and $k = \frac{n}{2}$. We show that the fan-in to these gates remains small in our construction. Thus, super-polynomial amounts of hardware are not required.

The use of majority is essential. It has been shown that boolean circuits using unbounded fan-in **and** and **or** gates and unary **negation** of size $n^{O(1)}$ that compute n-input majority have depth at least $\Omega(\frac{\log n}{\log \log n})$. This even extends to when arbitrary fan-in **exclusive-or** (parity) and generalizations thereof are allowed as gates [Razborov 87, Smolensky 87].

Majority nevertheless seems a physically reasonable gate. (e.g. if we have n switches that when closed each add a standard current to a common wire, then by testing the common current, majority may be computed.) Naively, arbitrary fan-in parity gates seem less physically reasonable than majority gates, so that it is an interesting fact that an n input parity gate can be simulated by an $n^{O(1)}$ size depth 4 circuit using fan-in $2n$ majority gates and unary negation gates [Hajnal 87]. The aforementioned result extends to the simulation of any gate computing a symmetric function.

We show how to implement an arbitrary real weighted threshold gate as a circuit of constant depth and $n^{O(1)}$ size using **majority**, **and**, **or** and **negation** gates.

Definition d1/1

We define an equivalence relation on $\mathbf{R}^n$ as follows: Let $z_1, \ldots, z_n$ be n fixed indeterminants. Define I_z to be the set of formal inequalities of the form $\sum_{i \in I_1} z_i \geq \sum_{i \in I_2} z_i$ or $\sum_{i \in I_1} z_i > \sum_{i \in I_2} z_i$ for all $I_1, I_2 \subseteq \{1, \ldots, n\}$ with $I_1 \cap I_2 = \emptyset$.

We write $\iota(z) \in I_z$ to refer to such an inequality, which we regard as a predicate on $\mathbf{R}^n$ in the obvious way: namely, for $\mathbf{x} \in \mathbf{R}^n$, $\iota(\mathbf{x})$ is true iff $\mathbf{x}$ satisfies $\iota(z)$.

Let $\mathbf{x} \in \mathbf{R}^n$; we define $\mathcal{I}(\mathbf{x}) = \{\iota(z) \in I_z \mid \iota(\mathbf{x})\}$. This is just the set of inequalities satisfied by $\mathbf{x}$ written in terms of the variable z.

For $\underline{\alpha}, \underline{\beta} \in \mathbf{R}^n$, we define $\underline{\alpha} \equiv \underline{\beta}$ iff $\mathcal{I}(\underline{\alpha}) = \mathcal{I}(\underline{\beta})$.

Remark: Our results below will hold for threshold functions with a threshold of zero. The corresponding results for threshold functions with arbitrary thresholds may be inferred directly from the following proposition.

Proposition p1/2

Let $T_k^{\underline{\alpha}}(x_1, \ldots, x_n) = 1$ iff $\sum_{i=1}^n \alpha_i x_i \geq k$, i.e. $T_k^{\underline{\alpha}}$ is an arbitrary threshold function with threshold k.

Define $\underline{\beta} = (\alpha_1, \ldots, \alpha_n, -k)$ and $T_0^{\underline{\beta}}(x_1, \ldots, x_{n+1}) = 1$ iff $\sum_{i=1}^{n+1} \beta_i x_i \geq 0$. Then $T_0^{\underline{\beta}}(x_1, \ldots, x_n, 1) = T_k^{\underline{\alpha}}(x_1, \ldots, x_n)$ for any $x_i \in \{0, 1\}$.

Proof

$T_0^{\underline{\beta}}(x_1, \ldots, x_n, 1) = 1$ iff $\sum_{i=1}^n \underline{\alpha}_i x_i - k \geq 0$ iff $T_k^{\underline{\alpha}}(x_1, \ldots, x_n) = 1$

$\square$

Proposition p2/3

Let $\underline{\alpha}, \underline{\beta} \in \mathbf{R}^n$. Then $\underline{\alpha} \equiv \underline{\beta} \Rightarrow T_0^{\underline{\alpha}} = T_0^{\underline{\beta}}$ (functional equality).

Proof

Assume $\underline{\alpha} \equiv \underline{\beta}$. Since $T_0^{\underline{\alpha}}(x_1, x_2, \ldots, x_n) = 1$ implies that $\sum_{i \in I_1} \alpha_i \geq \sum_{i \in I_2} \alpha_i$ for $I_1 = \{i \mid x_i = 1\}$ and $I_2 = \emptyset$, it must also be the case that $\sum_{i \in I_1} \beta_i \geq \sum_{i \in I_2} \beta_i$ and consequently $T_0^{\underline{\beta}}(x_1, \ldots, x_n) = 1$. By a similar argument for zero, the rest of the proposition follows. $\square$

Proposition p3/4

$\forall \underline{\alpha} \in \mathbf{R}^n \ \exists \underline{\beta} \in \mathbf{Z}^n$ such that $\underline{\alpha} \equiv \underline{\beta}$ and for $1 \leq i \leq n \quad |\beta_i| \leq n^n$.

Proof

Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an arbitrary tuple of reals. An inequality in $\mathcal{I}(\underline{\alpha})$ of the form $\sum_{i \in I_1} \alpha_i \geq \sum_{i \in I_2} \alpha_i$ or $\sum_{i \in I_1} \alpha_i > \sum_{i \in I_2} \alpha_i$ may be written as $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq 0$ or $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq 1$ respectively.

This suggests that we write all of $\mathcal{I}(\underline{\alpha})$ as a single vector equation $\mathbf{A}\underline{\alpha}^t \geq \mathbf{b}$, where $\mathbf{b}$ is a vector with $0, 1$ entries and $\mathbf{A}$ is a matrix with $-1, 0, 1$ entries having 1 row for each inequality in $\mathcal{I}(\underline{\alpha})$ and n columns.

Let us establish that $\{\beta \mid \mathcal{I}(\underline{\alpha}) = \mathcal{I}(\underline{\beta})\} = \{\beta \mid \mathbf{A}\underline{\beta} \geq \mathbf{b}\}$. First we observe that any solution $\mathbf{x} = \underline{\beta}$ to $\mathbf{Ax} \geq \mathbf{b}$ satisfies that $\mathcal{I}(\underline{\alpha}) \subseteq \mathcal{I}(\underline{\beta})$. Assume that $\mathbf{A}\underline{\beta} \geq \mathbf{b}$ and $\iota(z) \in \mathcal{I}(\underline{\beta}) \setminus \mathcal{I}(\underline{\alpha})$. If $\iota(z)$ is a sharp inequality of the form $\sum_{i \in I_1} z_i > \sum_{i \in I_2} z_i$ then $\mathcal{I}(\alpha)$ must contain the negation of $\iota(z)$, which is a weak inequality $\sum_{i \in I_1} z_i \leq \sum_{i \in I_2} z_i$, and so must $\mathcal{I}(\underline{\beta})$ by the observation above. Hence $\mathcal{I}(\underline{\beta})$ contains both $\iota(z)$ and the negation of $\iota(z)$, which is a contradiction. A similar argument can be made in the case of $\iota(z)$ being a weak inequality of the form $\sum_{i \in I_1} z_i \geq \sum_{i \in I_2} z_i$.

At this point, we have proven that $\{\underline{\beta} \mid \underline{\alpha} \equiv \underline{\beta}\} = \{\underline{\beta} \mid \mathbf{A}\underline{\beta} \geq \mathbf{b}\}$. We note that $\mathbf{A}$ has full column rank n and the set $\{\mathbf{x} \mid \mathbf{Ax} \geq \mathbf{b}\}$ is nonempty. In this case it is a fundamental fact of linear programming theory that there exists a basic feasible solution $\mathbf{x} = \underline{\gamma}$ for $\mathbf{Ax} \geq \mathbf{b}$ [Grötschel 88]. Such a basic feasible solution satisfies $\mathbf{B}\underline{\gamma} = \mathbf{d}$, where $\mathbf{B}$ is some $n \times n$ nonsingular submatrix of $\mathbf{A}$ and $\mathbf{d}$ is the corresponding subvector of $\mathbf{b}$.

By Cramer's rule $\underline{\gamma}^t = \frac{1}{det\mathbf{B}}(det\mathbf{B}_1, det\mathbf{B}_2, \ldots, det\mathbf{B}_n)$, where $\mathbf{B}_i$ is $\mathbf{B}$ with the i'th column replaced by $\mathbf{d}$. We note that the vector $\underline{\gamma}$ has rational entries and from $\underline{\gamma}$ we may obtain an integer valued solution $\mathbf{x} = \underline{\beta}$ to $\mathbf{Ax} \geq \mathbf{b}$ by multiplying through by the common denominator: $\underline{\beta} = (det\mathbf{B} \cdot det\mathbf{B}_1, \ldots, det\mathbf{B} \cdot det\mathbf{B}_n)$.

In order to bound the absolute values of the integers $\beta_1, \beta_2, \ldots, \beta_n$, we observe that $\mathbf{B}, \mathbf{B}_1, \ldots, \mathbf{B}_n$ are matrices with $-1, 0, 1$ entries. The determinant of such a matrix can be interpreted as the volume of a hyperparallepiped each of whose edges is at most $\sqrt{n}$ in length, implying that the absolute value of such a determinant is bounded by $\sqrt{n}^n$.

It follows that each entry β_i of the integer valued solution $\underline{\beta}$ defined above satisfies $|\beta_i| \leq (\sqrt{n}^n)^2 = n^n$.

$\square$

Theorem th1/5

Let $T_k^{\underline{\alpha}}(x_1, \ldots, x_n)$ be an arbitrary threshold function with $\underline{\alpha} \in \mathbf{R}^n$ and $k \in \mathbf{R}$. Then there exists $\underline{\beta} \in \mathbf{Z}^n$ and $\ell \in \mathbf{Z}$ such that for $1 \leq i \leq n$ the following propositions are true: $|\beta_i| \leq (n+1)^{n+1}, |\ell| \leq (n+1)^{n+1}$ and $T_k^{\underline{\alpha}} = T_\ell^{\underline{\beta}}$ (functional equality).

Proof

In accordance with p1/2, we introduce an additional variable x_{n+1} and consider the zero threshold function $T_0^{\underline{\gamma}}(x_1, \ldots, x_{n+1})$ satisfying $T_0^{\underline{\gamma}}(x_1, \ldots, x_n, 1) = T_k^{\underline{\alpha}}(x_1, \ldots, x_n)$.

By p3/4, there is some $\underline{\gamma}' \in \mathbf{Z}^{n+1}$ with $\underline{\gamma}' \equiv \underline{\gamma}$ and for $1 \leq i \leq n+1$ having the property that $|\gamma_i'| \leq (n+1)^{n+1}$. By p2/3, we have $T_0^{\underline{\gamma}} = T_0^{\underline{\gamma}'}$.

Finally, by setting x^{n+1} to 1, we have $T_k^{\underline{\alpha}} = T_\ell^{\underline{\beta}}$ where $\underline{\gamma}' = (\beta_1, \ldots, \beta_n, -\ell)$.

$\square$

Remark: This theorem asserts that all threshold functions can be obtained by using integer weights and thresholds of magnitude at most $(n+1)^{n+1}$. It is not possible to improve this result so as to obtain a polynomial bound on the values of the weights and the threshold. Taking

$\underline{\alpha} = (f_1, f_2, \ldots, f_n)$ and $k = f_{n+1}$ where f_i is the i'th Fibonacci-number, yields a threshold function that is not equivalent to any other threshold function with a smaller magnitude of the weights and the threshold. However, since the exponentially bounded values are implementable using around $n \log n$ bits, the possibility of implementing all threshold functions directly from the definition is open. The next theorem asserts how this may be done with great parallel efficiency.

The following lemma was first noted in [?].

Lemma 11/6

Given any threshold function $T_k^{\underline{\alpha}}(x_1, \ldots, x_n)$, define a map $\theta : \{1, \ldots, n\} \to \{0, 1\}$ given by $\theta(i) = 1$ iff $\alpha_i < 0$. Define $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ by $\beta_i = |\alpha_i|$ for each i. Let $\oplus$ denote exclusive-or. Then, $\exists \ell \in \mathbf{Z}^+ \cup \{0\}$ such that $T_k^{\underline{\alpha}}(x_1 \oplus \theta(1), x_2 \oplus \theta(2), \ldots, x_n \oplus \theta(n)) = T_\ell^{\underline{\beta}}(x_1, \ldots, x_n)$.

Proof

$T_k^{\underline{\alpha}}(x_1, \ldots, x_n) = 1$ iff $\sum_{i=1}^n \alpha_i x_i \geq k$. Since $\beta_i = \alpha_i \cdot (-1)^{\theta(i)}$, this inequality is equivalent to the condition $\sum_{i=1}^n (-1)^{\theta(i)} \cdot \beta_i \cdot x_i \geq k$. Since $x_i \in \{0, 1\}$, we have $x_i \oplus \theta(i) = \theta(i) + (-1)^{\theta(i)} \cdot x_i$ where the right hand expression uses integer arithmetic, as can easily be verified by checking the two cases, $\theta(i) = 0$ and $\theta(i) = 1$. Thus, we have $T_k^{\underline{\alpha}}(x_1 \oplus \theta(1), x_2 \oplus \theta(2), \ldots, x_n \oplus \theta(n)) = 1$ iff $\sum_{i=1}^n \beta_i (-1)^{\theta(i)} (\theta(i) + (-1)^{\theta(i)} x_i) \geq k$. Expanding the bracket by the distributive law and subtracting terms independent of x from both sides yields the equivalent condition $\sum_{i=1}^n \beta_i x_i \geq k + \sum_{i=1}^n \beta_i \theta(i)$. Since the left hand side of this condition is always non-negative, then if the right hand side is negative it may be replaced by zero without changing the condition. Thus, define $\ell = max(0, k + \sum_{i=1}^n \beta_i \theta(i))$ and we have $T_k^{\underline{\alpha}}(x_1 \oplus \theta(1), x_2 \oplus \theta(2), \ldots, x_n \oplus \theta(n)) = T_\ell^{\underline{\beta}}(x_1, \ldots, x_n)$.

$\square$

Remark: The lemma simply shows that a threshold function with positive and negative weights may be expressed in terms of a threshold function with the absolute values of the weights and possibly a different threshold value. This can be done by negating those variables of negative weight.

Lemma l2/7

There is a constant depth $n^{O(1)}$ size majority/negation circuit to compare 2 n-bit binary numbers.

Proof

See [Chandra 84].

$\square$

Lemma l3/8

There is a constant depth $n^{O(1)}$ size majority/negation circuit to compute the sum of n n-bit numbers.

Proof

See [Chandra 84].

$\square$

Theorem th2/9

There exists a constant depth $n^{O(1)}$ size majority/negation circuit to compute $T_k^{\underline{\alpha}}(x_1, \ldots, x_n)$ for any $\underline{\alpha} \in \mathbf{R}^n$ and $k \in \mathbf{R}$.

Proof

By theorem 1/5 and lemma 1/6, we may take $\underline{\alpha} \in (\mathbf{Z}^+)^n$ and $k \in \mathbf{Z}^+$ without loss of generality. Lemma 1/6 then may necessitate at most n negations in parallel in addition in depth 1 (inputs with weights of zero are ignored). What is more, theorem 1/5 guarantees that these weights are at most $(n+1)^{n+1}$ and thus can be implemented in $n^{O(1)}$ bits. For each i, there is a depth 1 size $n^{O(1)}$ circuit to compute $\alpha_i x_i$ since $x_i \in \{0,1\}$. The $\sum \alpha_i x_i$ is computed from the outputs of these circuits in constant depth and $n^{O(1)}$ size. This is done by using the circuits from lemma 3/8 with input size $n^{O(1)}$ instead of n. The results of this computation may be compared to k in constant depth and $n^{O(1)}$ size by using the circuit from lemma 2/7 with input size $n^{O(1)}$ instead of n. Thus the predicate $\sum \alpha_i x_i \geq k$ may be tested in constant depth and $n^{O(1)}$ size.

□

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Errata
december 1988

Please make the following corrections upon reading:

Page 4, line 4-6: The paragraph

> Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an arbitrary tuple of reals. An inequality in $\mathcal{I}(\underline{\alpha})$ of the form $\sum_{i \in I_1} \alpha_i \geq \sum_{i \in I_2} \alpha_i$ or $\sum_{i \in I_1} \alpha_i > \sum_{i \in I_2} \alpha_i$ may be written as $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq 0$ or $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq 1$ respectively.

is replaced by

> Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an arbitrary tuple of reals. For such $\underline{\alpha}$ there exists $\epsilon > 0$ such that every inequality in $\mathcal{I}(\underline{\alpha})$ of the form $\sum_{i \in I_1} \alpha_i \geq \sum_{i \in I_2} \alpha_i$ or $\sum_{i \in I_1} \alpha_i > \sum_{i \in I_2} \alpha_i$ may be written as $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq 0$ or $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq \epsilon$ respectively.
>
> If we replace $\underline{\alpha}$ by $1/\epsilon \cdot \underline{\alpha}$ then an inequality of the form $\sum_{i \in I_1} \alpha_i > \sum_{i \in I_2} \alpha_i$ may be written as $\sum_{i \in I_1} \alpha_i - \sum_{i \in I_2} \alpha_i \geq 1$.
>
> We asssume henceforth that $\underline{\alpha}$ has been replaced by $1/\epsilon \cdot \underline{\alpha}$. This implies no loss of generality since such a transformation preserves the equivalence class generated by $\underline{\alpha}$.

Page 4, line 10: The sentence

Let us establish that $\{\beta \mid \mathcal{I}(\underline{\alpha}) = \mathcal{I}(\underline{\beta})\} = \{\beta \mid \mathbf{A}\underline{\beta} \geq \mathbf{b}\}$.

is replaced by

Let us establish that $\{\beta \mid \mathcal{I}(\underline{\alpha}) = \mathcal{I}(\underline{\beta})\} \supseteq \{\beta \mid \mathbf{A}\underline{\beta} \geq \mathbf{b}\}$.

Page 4, line 18: The sentence

At this point, we have proven that $\{\underline{\beta} \mid \underline{\alpha} \equiv \underline{\beta}\} = \{\underline{\beta} \mid \mathbf{A}\underline{\beta} \geq \mathbf{b}\}$.

is replaced by

At this point, we have proven that $\{\underline{\beta} \mid \underline{\alpha} \equiv \underline{\beta}\} \supseteq \{\underline{\beta} \mid \mathbf{A}\underline{\beta} \geq \mathbf{b}\}$.

Page 4, line 28: The expression

$(det\mathbf{B} \cdot det\mathbf{B}_1, \ldots, det\mathbf{B} \cdot det\mathbf{B}_1)$.

is replaced by the expression

$\frac{|det\mathbf{B}|}{det\mathbf{B}} \cdot (det\mathbf{B}_1, \ldots, det\mathbf{B}_1)$.

Page 5, line 6:

satisfies $|\beta_i| \leq (\sqrt{n}^n)^2 = n^n$.

is replaced by

satisfies $|\beta_i| \leq \sqrt{n}^n = n^{n/2}$, which is far better than stated in the proposition that was to be proved.

Page 6, between line 3 and line 4: Insert

Let us make the statement of this lower bound more precise and provide a short argument to support its truth: If $T_l^{\underline{\beta}} \equiv T_k^{\underline{\alpha}}$ for $\underline{\beta} \in \mathbf{Z}^n$ and $l \in \mathbf{Z}$ then $l \geq f_{n+1}$ and $\beta_i \geq f_i$ for $i = 1, \ldots, n$.

In order to prove $\beta_h \geq f_h$ it suffices to prove that

(i) $\beta_{2h+1} \geq 1 + \Sigma_{i=1}^{h} \beta_{2i}$

(ii) $\beta_{2h} \geq \Sigma_{i=1}^{h} \beta_{2i-1}$

since

(iii) $f_{2h+1} = 1 + \Sigma_{i=1}^{h} f_{2i}$

(iv) $f_{2h} = \Sigma_{i=1}^{h} f_{2i-1}$

In the case of n being even it follows by (iii) and (iv) that $T_k^{\underline{\alpha}}(0,1,0,1,0,1,\ldots,0,1) = 0$ but $T_k^{\underline{\alpha}}(0,0,0,\ldots,0,1,1,0,1,0,1,0,1,\ldots,0,1) = 1$. Since $T_k^{\underline{\alpha}} = T_l^{\underline{\beta}}$ this implies that $\Sigma_{i=1}^{n/2} \beta_{2i} < l$ and $\beta_{2h+1} + \Sigma_{i=h+1}^{n/2} \beta_{2i} \geq l$. These two inequalities imply in combination the truth of (i).

The proof of (ii), $l \geq f_{n+1}$ and the case of n being odd may be treated similarly.

Page 6, line 8: The reference

[?]

is replaced by

[Hajnal 87]

Page 8, line 14: Add a new item to the list of references

[Chandra 84]CHANDRA, A., K., STOCKMEYER, L. and VISHKIN, U., Constant Depth Reducibility. *SIAM Journal on Computing* **13** *(1984), pp. 423-439.*