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A MODAL LOGIC FOR A SUBCLASS OF EVENT STRUCTURES†

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0. Introduction

A great deal of work has so far gone into developing temporal logic as a tool for specifying and proving the properties of distributed programs. [8, 10, 11] is just a small sample of the literature in this area. Through this line of work it has become clear that the modalities of temporal logic are ideally suited for capturing the dynamic properties of distributed programs.

What is missing in this approach – as we see it – is an explicit treatment of concurrency. Stated differently, in most of the work in the area of temporal logic, concurrency is represented in terms of arbitrary (non-deterministic) interleaving. This by itself is quite acceptable for many purposes. The major consequence however is that one is forced to attach assertions (temporal logic formulas) to the global states of a distributed program. One then reasons about the program in terms of its global states. In general, it is difficult, if not impossible, to "detect" such global states; parts of the global state may be changing concurrently due to independent actions carried out on two "geographically" separated locations. Hence it would be attractive to have a formalism which deals only with the local states of the sequential components that – together with the underlying communication medium – constitute the distributed program.

In this paper we put forward one such formalism. In order to bring the main ideas into sharp focus, we have chosen an abstract model of distributed systems rather than a concrete model of distributed programs. Our model is a subclass of a model of distributed systems called *event structures.*

An event structure is basically a partially ordered set of event occurrences together with a symmetric conflict relation. The ordering relation models causality so that two event occurrences that are neither comparable nor in conflict may occur with no order over their occurrences. In this sense event structures provide an explicit and clearly separated representation of sequence, choice and concurrency. Event structures arise naturally from net theory [7] and Winskel has constructed a substantial theory centered around these objects [13]. In particular, Winskel has obtained the non-interleaved semantics of CCS-like languages using event structures [12]. Hence there is good reason to hope that these objects can serve – at least in theoretical studies – as an adequate model of distributed programs.

As a first step toward constructing temporal logics for distributed programs which emphasise the local states of the sequential constituents, we consider a restricted subclass of event structures called *n-agent* ($n \geq 1$) *event structures.* Here an event structure is viewed as consisting of n sequential event structures each of which can exhibit non-determinism (conflict) but no concurrency in its individual behaviour. The individual agents become aware of the properties of other agents through explicit communication modelled by the global (partial) ordering relation. We attach assertions to the local states of the various agents. This restricted subclass does have a fair amount of modelling power even though we cannot cater for the dynamic creation and destruction of processes. For instance, the formalism used by Chandy and Misra [2], in its "unfolded" version would fit into our framework.

We use indexed modalities to describe the states of knowledge of the agents. Thus our syntax and semantics bear some resemblance to recent work on logics of knowledge

[4, 5, 9]. However there are some fundamental differences – as pointed out in the concluding section – between the two approaches. Hence in the sequel, when we use phrases such as "agent-i knows that agent-j has ... " and so forth, it is meant to be a purely informal use of such phrases.

In the next section we introduce n-agent event structures. They will serve as the frames for the logic with indexed modalities that we propose. The syntax and the semantics of this logic are presented in section 2. Though we view this logic as a version of modal logic it has the flavour of a tense logic (in the sense of Burgess [1]) as well. This is brought out by the axiom system presented in section 3 where we also argue for the soundness of this system w.r.t. the chosen semantics. In section 4 we show the completeness of our axiom system using a Henkin-style proof. In doing so, we rely heavily on Burgess [1]. In the concluding section we discuss in more detail our work in the context of related literature.

1. n-Agent Event Structures

An event structure represents the behaviour of a distributed system through a set of event occurrences, a causality relation that partially orders the event occurrences and a conflict relation which reflects the (behavioural) choices available to the system.

Definition 1.1

An *event structure* is a triple $ES = (E, \leq, \#)$ where

(i) E is a set of *events*.

(ii) $\leq \subseteq E \times E$ is a partial order, called the *causality relation*.

(iii) $\# \subseteq E \times E$ is an irreflexive, symmetric relation, called the *conflict relation*.

(iv) For any e_1, e_2, e_3 in E,

$$e_1 \# e_2 \text{ and } e_2 \leq e_3 \text{ implies } e_1 \# e_3.$$

$\square$

The last clause in the definition captures the intuition that the past of every event should be "consistent". This will become clear once we introduce the notion of configurations.

Here is an example of an event structure. The events have been labelled to reflect the behaviour of a producer communicating via an unbounded buffer to a consumer. The producer can stop after producing zero or more items. Both the producer and consumer are assumed to work sequentially.

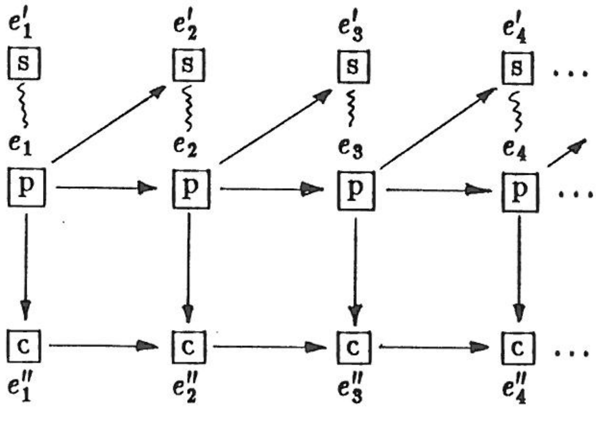

Figure 1

For convenience, in Figure 1 we have just indicated the "minimal" elements of the causality relation (as directed arcs) and the conflict relation (as squiggly lines). Thus $e_1 \leq e_3$ because $e_1 \leq e_2$ and $e_2 \leq e_3$. Moreover, $e'_1 \# e''_2$ because $e'_1 \# e_1 \leq e''_2$. Finally, e''_2 and e_3 can occur concurrently because they are causally incomparable and are not in conflict with one another. We will follow this graphical convention throughout the paper.

The states of an event structure are called configurations.

Definition 1.2

Let $ES = (E, \leq, \#)$ be an event structure and $c \subseteq E$.

(i) c is *#-free* iff $(c \times c) \cap \# = \phi$.

(ii) c is *left-closed* iff for any $e \in c$ and $e' \in E$,

$$e' \leq e \text{ implies } e' \in c.$$

(iii) c is a *configuration* iff c is #-free and left-closed.

(iv) $C_{ES} = \{c \subseteq E \mid c \text{ is a configuration}\}$.

□

For the event structure shown in Figure 1, $\{e_1, e_2\}$, $\{e_1, e_2, e''_1\}$ and $\{e_1, e_2, e_3, \ldots\}$ are configurations, but $\{e_1, e'_1\}$ and $\{e_1, e''_1, e''_2\}$ are not.

A configuration represents a state of affairs that has been reached through the occurrences of a subset of events. The notion of a configuration captures the intuition that an event can occur only after all the events that lie in its past have occurred. Moreover, two events in conflict cannot both occur in any stretch of behaviour. Consequently, two events that are neither causally ordered nor in conflict can occur concurrently. As mentioned earlier, the last clause in the definition of an event structure guarantees that the past of each event is consistent in the sense of being conflict-free. Formally, we say that the left closure of an event in an event structure is a configuration.

Definition 1.3

Let $ES = (E, \leq, \#)$ be an event structure and $e \in E$. The *left closure* of e, denoted $\downarrow e$, is defined by

$$\downarrow e = \{e' \in E \mid e' \leq e\}$$

□

Proposition 1.4

For an event structure $ES = (E, \leq, \#)$ and $e \in E$, $\downarrow e$ is a configuration.

Proof: Follows easily from the definitions.

$\square$

It appears to be very difficult to obtain a logical characterization of event structures in general. Hence in this paper we shall confine our attention to event structures that can be viewed as a collection of a *finite* number of pairwise disjoint "sequential" event structures which may communicate with each other. In a sequential event structure no two events can occur concurrently.

Definition 1.5

An event structure $ES = (E, \leq, \#)$ is said to be *sequential* iff for any two events $e_1, e_2 \in E$,

$$e_1 \leq e_2 \text{ or } e_2 \leq e_1 \text{ or } e_1 \# e_2.$$

$\square$

Here is an example of a sequential event structure.

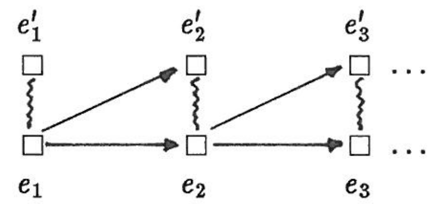

Figure 2

In what follows, we shall let N_0 denote the set of non-negative integers and N the set of positive integers. We can now introduce the subclass of event structures studied in this paper.

Definition 1.6

An *n-agent event structure* is the structure $ES = (E_1, E_2, \ldots, E_n, \leq, \#)$, where $n \in N$ and

(i) $i \neq j$ implies $E_i \cap E_j = \phi$ (for $1 \leq i \leq j \leq n$).

(ii) $(E, \leq, \#)$ is an event structure, where $E = \bigcup_{i=1}^{n} E_i$.

(iii) For $1 \leq i \leq n$, the *agent* $(E_i, \leq_i, \#_i)$ is a sequential event structure, where $\#_i$ (resp. $\leq_i$) is $\#$ (resp. $\leq$) restricted to $E_i \times E_i$.

(iv) $\# = \{(e_1, e_2) \mid \exists (e'_1, e'_2) \in \hat{\#}.\ e'_1 \leq e_1 \text{ and } e'_2 \leq e_2\}$, where

(v) $\hat{\#} = \bigcup_{i=1}^{n} \#_i$.

$\square$

Part (iv) of the definition captures the idea that choices are made "locally" by the individual sequential agents and this information is propagated by the causality relation. Consequently, different agents can influence each other only through explicit communication, which is modelled by the "global" causality relation $\leq$.

The producer-consumer example of Figure 1 can be viewed as a 2-agent event structure $(E_1, E_2, \leq, \#)$ with

$$\begin{aligned} E_1 &= \{e_1, e'_1, e_2, e'_2, \ldots\} \\ E_2 &= \{e''_1, e''_2, \ldots\}. \end{aligned}$$

The agent $(E_1, \leq_1, \#_1)$ was shown in Figure 2.

We conclude this section by sketching in an informal fashion the means for viewing event structures as frames for our modal logic. The standard frames for modal logic have the form (W, R) where W is a set of *worlds* and $R \subseteq W \times W$ is the *accessibility* relation. Given an event structure $ES = (E, \leq, \#)$, the obvious candidate to serve as the set of worlds is C_{ES}, the set of configurations of ES. The accessibility relation should be defined in terms of $\subseteq$, since $(C_{ES}, \subseteq)$ is a poset with $\subseteq$ reflecting the causality ordering over the configurations. The accessibility relation should also reflect the fact that agents acquire knowledge about other agents only through explicit communications received from other agents.

We wish to argue that only a subset of C_{ES} should be chosen to serve the role of worlds. Let $ES = (E_1, E_2, \ldots, E_n, \leq, \#)$ be an n-agent event structure and $c \in C_{ES}$. In general c will represent a global (distributed) state of affairs. It is difficult to justify asserting the truth or falsity of a formula at c without assuming an omnipotent observer capable of recording global states. Hence we only consider those members of C_{ES} which can be regarded as local states belonging to the individual agents. It then becomes natural to assign formulas to such local states. It is then also possible to have situations where different agents (at their individual local states) concurrently differ on the the truth of a formula. These considerations will underlie our formal semantics, to be presented in the next section.

Now the local states of the n-agent event structure $ES = (E_1, E_2, \ldots, E_n, \leq, \#)$ can be defined in a natural fashion as:

$$LC_{ES} = \{\downarrow e \mid e \in E\} \text{ where } E = \cup_{i=1}^{n} E_i$$

It is easy to observe that for two events e and e', $\downarrow e \subseteq \downarrow e'$ iff $e \leq e'$. In other words, $(LC_{ES}, \subseteq)$ and $(E, \leq)$ are isomorphic posets. This justifies our proposal to view the event structures themselves as frames. Thus in what follows when we speak of a formula α being true at an event e, what we mean is that α holds at the local state $\downarrow e$.

2. The Language and Its Models

For the rest of the paper, we fix an $n \in N$ and let i, j, k range over $\{1, 2, \ldots, n\}$.

Syntax

We fix a countable set of atomic propositions $P = \{p_1, p_2, \ldots\}$ and let p, q range over P. We also fix a set consisting of n atomic *type* propositions, $\Gamma = \{\tau_1, \tau_2, ..., \tau_n\}$. We assume that $P \cap \Gamma = \phi$ and set $\hat{P} = P \cup \Gamma$. The type propositions will be used to identify particular agents.

We shall use the logical connectives $\sim$ and $\vee$, as well as n indexed strong future modalities $\Box_1, \Box_2, \ldots, \Box_n$ and n indexed – as it will turn out – not so strong past modalities $\boxminus_1, \boxminus_2, \ldots, \boxminus_n$.

The *formulas* of our language can now be built up inductively.

Definition 2.1

(i) Every member of $\hat{P}$ is a formula.

(ii) If α and β are formulas, then so are $\sim \alpha, \alpha \vee \beta$, $\Box_i \alpha$ and $\boxminus_i \beta$.

$\Box$

We let $\alpha, \beta, \gamma, \delta, \theta$ with or without subscripts, range over F, the set of all formulas. It will be convenient to have available the following derived logical connectives and modalities:

Definition 2.2

(i)	$\alpha \wedge \beta$	$\stackrel{\text{def}}{=}$	$\sim (\sim \alpha \vee \sim \beta)$
(ii)	$\alpha \oplus \beta$	$\stackrel{\text{def}}{=}$	$(\sim \alpha \wedge \beta) \vee (\sim \beta \wedge \alpha)$
(iii)	$\alpha \supset \beta$	$\stackrel{\text{def}}{=}$	$\sim \alpha \vee \beta$
(iv)	$\alpha \equiv \beta$	$\stackrel{\text{def}}{=}$	$(\alpha \supset \beta) \wedge (\beta \supset \alpha)$
(v)	$\Diamond_i \alpha$	$\stackrel{\text{def}}{=}$	$\sim \Box_i \sim \alpha$
(vi)	$\diamondminus_i \alpha$	$\stackrel{\text{def}}{=}$	$\sim \boxminus_i \sim \alpha$

$\Box$

Semantics

Definition 2.3

A *frame* is an n-agent event structure $ES = (E_1, E_2, \ldots, E_n, \leq, \#)$.

$\Box$

We will find it convenient to specify an n-agent event structure as simply $ES = (E, \leq, \#)$ and assume implicitly a partitioning $E_1, \ldots, E_n$ of E.

Definition 2.4

A *model* is an ordered pair $M = (ES, V)$, where

(i) $ES = (E, \leq, \#)$ is a frame,

(ii) $V : E \to 2^{\hat{P}}$ is the *valuation function* satisfying:

$$\text{For any } e \in E, \tau_i \in V(e) \text{ iff } e \in E_i. \text{ (i.e. } V^{-1}(\tau_i) = E_i).$$

□

Let $M = (ES, V)$ be a model and $ES = (E, \leq, \#)$. Let $e \in E$ and α be a formula. Then the notion of α being true at e in the model M is denoted as $e, M \models \alpha$ and is defined inductively:

Definition 2.5

(i) $e, M \models p$ iff $p \in V(e)$

(ii) $e, M \models \sim \alpha$ iff $e, M \not\models \alpha$

(iii) $e, M \models \alpha \vee \beta$ iff $e, M \models \alpha$ or $e, M \models \beta$

(iv) $e, M \models \Box_i \alpha$ iff for some $e' \in E_i$,
$e' \leq e$ and for all $e'' \in E_i$,
$e' \leq e''$ implies $e'', M \models \alpha$

(v) $e, M \models \boxminus_i \alpha$ iff for all $e' \in E_i$,
if $e' \leq e$ then for all $e'' \in E_i$,
if $e'' \leq e'$ then $e'', M \models \alpha$.

□

The first three clauses are standard and require no explanation. The next clause can be expressed informally as:

"At e, it is known that in agent i, α will hold henceforth."

In agent i itself, this is the same as the tense logic $\Box$ operator. In an agent $j \neq i$, the assertion $\Box_i \alpha$ asserts knowledge about another agent. Hence it must be the case that agent i communicated (directly or via other agents) at some stage in the past of agent j the information that α would hold henceforth in agent i.

On the other hand, $e, M \models \boxminus_i \alpha$ expresses something weaker:

"As far as is known, α has always held for the agent i."

Thus if the agent i has never communicated with the agent j, then at any local state belonging to agent j, the formula $\boxminus_i \alpha$ will hold for arbitrary α. In this sense $\boxminus_i$ is weak and $\Box_i$ is strong. We note that the definition of $e, M \models \boxminus_i \alpha$ is somewhat pedantic since it could have been shortened to

$$e, M \models \boxminus_i \alpha \text{ iff for all } e' \in E_i$$
$$\text{if } e' \leq e \text{ then } e', M \models \alpha.$$

But we prefer to keep – in the original definition – the spirit of the definition of $e, M \models \Box_i \alpha$. We also wish to draw attention to the semantics of the derived dual modalities.

$$e, M \models \diamondminus_i \alpha \text{ iff for some } e' \in E_i$$
$$e' \leq e \text{ and for some } e'' \in E_i$$
$$e'' \leq e' \text{ and } e'', M \models \alpha.$$

Once again this can be shortened to:

$$e, M \models \diamondminus_i \alpha \text{ iff for some } e' \in E_i,$$
$$e' \leq e \text{ and } e', M \models \alpha.$$

In any case, the intended meaning is:

"It is known that in agent i, α has held before."

Thus $\diamondminus_i$ is a relatively strong modality asserting something definite about the past of agent i whereas $\Diamond_i$ is relatively weak.

$$e, M \models \Diamond_i \alpha \text{ iff for all } e' \in E_i$$
$$\text{if } e' \leq e \text{ then there is some } e'' \in E_i,$$
$$\text{with } e' \leq e'' \text{ and } e'', M \models \alpha.$$

In agent i itself, $\Diamond_i \alpha$ asserts that α will hold in some future state. However in agent j, where $j \neq i$, $\Diamond_i \alpha$ merely captures:

"As far as is known, α may hold eventually in agent i."

Thus if agent i has never communicated anything to j, $\Diamond_i \alpha$ will hold in j for any α.

We conclude this section by noting down the standard notions of satisfiability and validity.

Definition 2.6

(i) α is *satisfiable* iff there exists a model $M = (ES, V)$ with $ES = (E, \leq, \#)$ and $e \in E$ such that $e, M \models \alpha$.

(ii) For a model $M = (ES, V)$, where $ES = (E, \leq, \#)$,

$$M \models \alpha \text{ iff for all } e \in E.\ e, M \models \alpha$$

(iii) α is *valid*, denoted $\models \alpha$, iff for all models M, $M \models \alpha$.

□

3. The Axiom System

Most of our axioms are indexed versions of standard modal logic axioms taken from [6] and a few tense logic axioms taken from [1]. The axioms that are new are meant to characterize the way in which the agents acquire knowledge concerning other agents through communication. We first present the axiom scheme in full and then provide some explanatorial remarks.

Axioms

(A0) All the substitutional instances of the tautologies of propositional logic.

(A1.a) $\Box_i (\alpha \supset \beta) \supset (\Box_i \alpha \supset \Box_i \beta)$

(Deductive Closure)

(A1.b) $\boxminus_i(\alpha \supset \beta) \supset (\boxminus_i \alpha \supset \boxminus_i \beta)$

(A2.a) $\tau_i \supset (\Box_i \alpha \supset \alpha)$

(Local Reflexivity)

(A2.b) $\tau_i \supset (\alpha \supset \diamondminus_i \alpha)$

(A3.a) $\Box_i \alpha \supset \Box_i \Box_i \alpha$

(Transitivity)

(A3.b) $\diamondminus_i \diamondminus_j \alpha \supset \diamondminus_j \alpha$

(A4) $\diamondminus_i \alpha \wedge \diamondminus_i \beta \supset \diamondminus_i(\alpha \wedge \diamondminus_i \beta) \vee \diamondminus_i(\beta \wedge \diamondminus_i \alpha)$ (Backward Linearity of the Agents)

(A5.a) $\diamondminus_i \alpha \supset \Box_i \diamondminus_i \alpha$

(Relating past and future)

(A5.b) $\Diamond_i \alpha \supset \boxminus_i \Diamond_i \alpha$

(A6.a) $\Box_i \alpha \supset \diamondminus_i(\tau_i \wedge \Box_i \alpha)$
(A6.b) $\diamondminus_i \alpha \supset \diamondminus_i(\tau_i \wedge \diamondminus_i \alpha)$ (Communication axioms)
(A6.c) $\Diamond_i \alpha \supset \boxminus_i(\tau_i \supset \Diamond_i \alpha)$
(A6.d) $\boxminus_i \alpha \supset \boxminus_i(\tau_i \supset \boxminus_i \alpha)$

(A7.a) $\tau_1 \oplus \tau_2 \oplus \ldots \oplus \tau_n$
(A7.b) $\tau_i \supset \Box_i \tau_i$ (type axioms)
(A7.c) $\tau_i \supset \boxminus_i \tau_i$

Inference Rules

(MP)

$$\frac{\alpha, \alpha \supset \beta}{\beta}$$

(TG.a)

$$\frac{\alpha \supset \beta}{\Box_i \alpha \supset \Box_i \beta}$$

(TG.b)

$$\frac{\alpha}{\boxminus_i \alpha}$$

First we note that the standard reflexivity axioms have been modified to ensure that $\sim \alpha \wedge \Box_i \alpha$ is not necessarily inconsistent. After all the agent j might deny α and yet might have received a message stating that the agent i believes α to be true. (A3.a) asserts transitivity *within* agents whereas (A3.b) asserts transitivity *across* agents. (A4) states that the individual agents are tree-like. (A5.a) and (A5.b) are standard axioms and have been adapted from [1].

The communication axioms and the type axioms are in some sense the characteristic axioms of our system. (A6.a) and (A6.b) assert that *definite* (i.e. strong future and weak past) information concerning the agent i must have originated from a state belonging to agent i. (A6.c) and (A6.d) assert that – possibly – indefinite information (i.e. weak future and strong past) concerning the agent i must agree with all the information that has so far been received from agent i. (A7.a) captures the idea that each local state belongs to exactly one agent. (A7.b) and (A7.c) guarantee that from any state belonging to agent i, all the other states belonging to agent i can be accessed.

The inference rule (TG.b) is standard whereas the standard future version (i.e. from α infer $\Box_i \alpha$) will not be sound in the present framework. (TG.a) is all what we can soundly infer. It turns out that a slightly modified version of "from α infer $\Box_i \alpha$" which we need can be obtained as a derived inference rule as shown below.

A formula α which can be derived using the axioms and the inference rules will be called a *thesis*. We will use $\vdash \alpha$ to denote the fact that α is a thesis in our system.

Theorem 3.1 (Soundness)

If $\vdash \alpha$ then $\models \alpha$.

Proof: We will just argue for the soundness of three of the axioms and one of the inference rules. It is routine to verify the rest.

Let $M = (ES, V)$ be a model with $ES = (E, \leq, \#)$. Let $e \in E_i$.

Then consider (A3.b). Suppose $e, M \models \diamondminus_j \diamondminus_k \alpha$. Then there exists $e_j \in E_j$ such that $e_j \leq e$ and $e_j, M \models \diamondminus_k \alpha$. This implies that there exists $e_k \in E_k$ such that $e_k \leq e_j$ and $e_k, M \models \alpha$. But $e_k \leq e$ so that $e, M \models \diamondminus_k \alpha$.

Next consider (A4). Let $e \in E_i$ and assume that $e, M \models \diamondminus_j \alpha \wedge \diamondminus_j \beta$. Then there exist $e_1, e_2 \in E_j$ such that $e_1 \leq e, e_1, M \models \alpha, e_2 \leq e$ and $e_2, M \models \beta$. If $e_1 \# e_2$ then from $e_1 \leq e$, we would have $e_2 \# e$. But then $e_2 \leq e$ so that $e \# e$ which is a contradiction because $\#$ is supposed to be irreflexive. Hence $e_1 \leq e_2$ or $e_2 \leq e_1$. Assume without loss of generality that $e_1 \leq e_2$. Then $e_1, M \models \diamondminus_j \beta$, so that $e_1, M \models \alpha \wedge \diamondminus_j \beta$. Clearly, $e, M \models \diamondminus_j(\alpha \wedge \diamondminus_j \beta)$.

Next consider (A6.c). Once again let $e \in E_i$ and assume that $e, M \models \Diamond_j \alpha$. Now suppose that $(\downarrow e) \cap E_j = \phi$. Then clearly $e, M \models \boxminus_j(\tau_j \supset \Diamond_j \alpha)$. So assume that $e_j \in E_j$ with $e_j \leq e$. Clearly $e_j, M \models \tau_j$. So suppose that $e_j, M \not\models \Diamond_j \alpha$. Then

$e_j, M \models \Box_j \sim \alpha$. From $e_j \leq e$ we then have $e, M \models \Box_j \sim \alpha$ so that $e, M \models \sim \Diamond_j \alpha$ and this is a contradiction because we already have $e, M \models \Diamond_j \alpha$.

Next consider the inference rule (TG.a). To show that (TG.a) preserves validity, assume $\models \alpha \supset \beta$. Let $e \in E$ such that $e, M \models \Box_i \alpha$. Then for some $e' \in E_i$ with $e' \leq e$ we must have $e', M \models \Box_i \alpha$. Let $e'' \in E_i$ such that $e' \leq e''$. Then $e'', M \models \alpha$. From $\models \alpha \supset \beta$ we have $e'', M \models \beta$ from which it follows that $e', M \models \Box_i \beta$. Clearly, $e, M \models \Box_i \beta$.

$\Box$

In anticipation of the completeness proof, we now state some derived inference rules and theses.

Derived Rules

(TG.c)

$$\frac{\alpha}{\tau_i \supset \Box_i \alpha}$$

(DR.1)

$$\frac{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m \supset \beta}{\beta}$$

(DR.2)

$$\frac{\alpha \equiv \beta, \theta}{\theta(\alpha \mid \beta)}$$

Here $\theta(\alpha \mid \beta)$ is the formula obtained by uniformly substituting α for β in θ.

(DR.3.a)

$$\frac{\alpha \supset \beta}{\tau_i \supset (\Box_i \alpha \supset \Box_i \beta)}$$

(DR.3.b)

$$\frac{\alpha \supset \beta}{\Diamond\mkern-12mu-_i \alpha \supset \Diamond\mkern-12mu-_i \beta}$$

The derivation associated with (TG.c) is shown in the appendix. The derivation for (DR.1) is easy and we omit it.

The derivation for (DR.2) is by induction on the complexity of θ and uses (TG.a), (TG.b) and (A1.b). (DR.3.a) follows at once from (TG.a) and the derivation for (DR.3.b) is shown in the appendix. Note that thanks to (DR.2), double negations can be introduced and removed freely in derivations.

Theses

(T1.a) $\Diamond_i (\alpha \wedge \beta) \supset (\Diamond_i \alpha \wedge \Diamond_i \beta)$
(T1.b) $\Diamond\mkern-12mu-_i (\alpha \wedge \beta) \supset (\Diamond\mkern-12mu-_i \alpha \wedge \Diamond\mkern-12mu-_i \beta)$

(T2) $\boxminus_i (\alpha \wedge \beta) \equiv \boxminus_i \alpha \wedge \boxminus_i \beta$

(T3.a) $(\Box_i\alpha \wedge \Diamond_i\beta) \supset \Diamond_i(\beta \wedge \alpha)$
(T3.b) $\boxminus_i\alpha \wedge \diamondminus_i\beta \supset \diamondminus_i(\alpha \wedge \beta)$

(T4.a) $\tau_i \supset (\alpha \wedge \Diamond_i\beta \supset \Diamond_i(\beta \wedge \diamondminus_i\alpha))$
(T4.b) $\tau_i \supset (\alpha \wedge \diamondminus_i\beta \supset \diamondminus_i(\beta \wedge \Diamond_i\alpha))$

(T5.a) $\Box_i\alpha \equiv \diamondminus_i \Box_i\alpha$
(T5.b) $\diamondminus_i\alpha \equiv \Box_i\diamondminus_i\alpha$

(T6.a) $\Box_i\alpha \wedge \Diamond_i\beta \supset \diamondminus_i(\Box_i\alpha \wedge \tau_i \wedge \Diamond_i\beta)$
(T6.b) $\diamondminus_i\alpha \wedge \boxminus_i\beta \supset \diamondminus_i(\diamondminus_i\alpha \wedge \tau_i \wedge \boxminus_i\beta)$

(T7.a) $\tau_i \wedge \Diamond_i\alpha \supset \Diamond_i(\tau_i \wedge \alpha)$
(T7.b) $\tau_i \wedge \diamondminus_i\alpha \supset \diamondminus_i(\tau_i \wedge \alpha)$

(T8) $\diamondminus_i\alpha_1 \wedge \diamondminus_i\alpha_2 \ldots \wedge \diamondminus_i\alpha_m \supset \bigvee_{f \in P_m}(\diamondminus_i\widehat{\alpha_f})$ where $m \geq 1$ and P_m is given by:

$P_m = \{f \mid f : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, m\}$is a bijection$\}$.
And for $f \in P_m$ we define $\diamondminus_i\widehat{\alpha_f}$ as,

$$\diamondminus_i\widehat{\alpha_f} \stackrel{\text{def}}{=} \diamondminus_i(\alpha_{f(1)} \wedge (\diamondminus_i\alpha_{f(2)} \wedge (\diamondminus_i\alpha_{f(3)} \wedge \ldots \diamondminus_i\alpha_{f(m)})) \ldots).$$

Thus (T8) is a generalization of (A4).

The derivations of the theses are presented in the appendix.

4. Completeness of the Axiom System

We now wish to show that the axiom system presented in the previous section is complete. Our proof both in spirit and in its details is strongly guided by Burgess [1].

As usual by a *consistent* formula we shall mean a formula whose negation is not a thesis. Of course thesishood is to be understood relative to our axiom system. Our proof of completeness will establish that every consistent formula is satisfiable.

The finite set of formulas $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is *consistent* iff $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_m$ is consistent. A set for formulas is consistent iff every finite subset is. By an MCS (Maximal Consistent Set) we mean a consistent set of formulas which is not properly included in any other consistent set. We assume the next two results. Proofs can be easily extracted from [6].

Proposition 4.1

Any consistent set of formulas can be extended to an MCS.

□

Proposition 4.2

Let A be an MCS.

(i) $\sim \alpha \in A$ iff $\alpha \notin A$.

(ii) $\alpha \vee \beta \in A$ iff $\alpha \in A$ or $\beta \in A$.

(iii) $\alpha \wedge \beta \in A$ iff $\alpha \in A$ and $\beta \in A$.

(iv) If α is a thesis then $\alpha \in A$.

(v) If $\alpha_1, \alpha_2, \ldots, \alpha_m \in A$ and $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_m \supset \beta$ is a thesis then $\beta \in A$.

□

The notion of the *type* of an MCS will play a crucial role in what follows.

Definition 4.3

Let C denote the class of all MCSs. Then *type* : $C \rightarrow \{1, \ldots, n\}$ is given by:

$$\forall A \in C.\ \mathit{type}(A) = i \text{ iff } \tau_i \in A.$$

□

By (A7.a) and Prop. 4.2, we are assured that type as specified above is a well-defined function.

We shall now define a pre-order relation over MCSs of the same type.

Definition 4.4

Let A and B be two MCSs with $type(A) = i = type(B)$. Then

$$A \preceq_t B \stackrel{\text{def}}{\Longleftrightarrow} \{\diamondminus_i \alpha \mid \alpha \in A\} \subseteq B.$$

$\square$

Proposition 4.5

Let A and B be two MCSs with $type(A) = i = type(B)$. Then the following statements are equivalent.

(i) $A \preceq_t B$

(ii) $\{\Diamond_i \alpha \mid \alpha \in B\} \subseteq A$

(iii) $\{\alpha \mid \Box_i \alpha \in A\} \subseteq B$

(iv) $\{\alpha \mid \boxminus_i \alpha \in B\} \subseteq A$

$\square$

Proof:

(i) $\Rightarrow$ (iii) Suppose $\Box_i \alpha \in A$. Then by (i), $\diamondminus_i \Box_i \alpha \in B$. From (A5.b) it follows that $\diamondminus_i \Box_i \alpha \supset \Box_i \alpha$ is a thesis. Hence $\Box_i \alpha \in B$. Since $\tau_i \in B$, we now have from (A2.a), $\alpha \in B$.

(ii) $\Rightarrow$ (iv) Suppose $\boxminus_i \alpha \in B$. Then by (ii), $\Diamond_i \boxminus_i \alpha \in A$. From (A5.a) it follows that $\Diamond_i \boxminus_i \alpha \supset \boxminus_i \alpha$ is a thesis. Hence $\boxminus_i \alpha \in A$. As before, from (A2.b), $\alpha \in A$ because $\tau_i \in A$.

(iii) $\Rightarrow$ (ii) Suppose $\alpha \in B$. Then $\sim \alpha \notin B$. By (iii), $\Box_i \sim \alpha \notin A$. Thus $\Diamond_i \alpha \in A$.

(iv) $\Rightarrow$ (i) Similar to the proof of (iii) $\Rightarrow$ (ii).

$\square$

Before we explore one additional property of $\preceq_t$, we observe the following.

Lemma 4.6

(i) If $\diamondminus_i \alpha$ is consistent then so is α.

(ii) If $\tau_i \wedge \Diamond_i \alpha$ is consistent, then so is $\alpha \wedge \tau_i$.

(iii) If $\tau_i \wedge \diamondminus_i \alpha$ is consistent, then so is $\alpha \wedge \tau_i$.

Proof:

(i) Assume that $\diamondminus_i\alpha$ is consistent. Suppose that α is not consistent. Then,

$\vdash \sim \alpha$

$\vdash \boxminus_i \sim \alpha$ (TG.b)

$\vdash \sim \diamondminus_i\alpha$

This contradicts the consistency of $\diamondminus_i\alpha$.

(ii) Assume that $\tau_i \wedge \Diamond_i\alpha$ is consistent.
Then so is $\tau_i \wedge \Box_i\tau_i \wedge \Diamond_i\alpha$ (A7.b).
Hence $\tau_i \wedge \Diamond_i(\alpha \wedge \tau_i)$ is consistent (T3.a). (*)
Now suppose that $\tau_i \wedge \alpha$ is not consistent.

$\vdash \sim (\tau_i \wedge \alpha)$

$\vdash \tau_i \supset \Box_i \sim (\alpha \wedge \tau_i)$ (TG.c)

$\vdash \sim (\tau_i \wedge \Diamond_i(\alpha \wedge \tau_i))$ (PC, Def. of $\Diamond$)

which contradicts the earlier result (*).

(iii) Assume that $\tau_i \wedge \diamondminus_i\alpha$ is consistent, and $\tau_i \wedge \alpha$ is not.
Then

$\vdash \sim (\tau_i \wedge \alpha)$

$\vdash \sim \diamondminus_i(\tau_i \wedge \alpha)$ (Part (i) of this lemma)

$\vdash \sim (\diamondminus_i\alpha \wedge \tau_i)$ (T7.b)

which contradicts the assumption.

$\Box$

Lemma 4.7

Let A be an MCS with $type(A) = i$. Then for any formula α,

(i) If $\diamondminus_i\alpha \in A$ then there exists an MCS B with $type(B) = i$ such that $\alpha \in B$ and $B \preceq_t A$.

(ii) If $\Diamond_i\alpha \in A$ then there exists an MCS B with $type(B) = i$ such that $\alpha \in B$ and $A \preceq_t B$.

Proof:

(i) By Prop. 4.5 and Prop. 4.1, it suffices to show that $B' = \{\Diamond_i\beta \mid \beta \in A\} \cup \{\alpha, \tau_i\}$ is consistent. Since $\beta_1, \beta_2, \ldots, \beta_m \in A$ implies that $\Diamond_i (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_m) \in B'$, by (T1.a) it suffices to show that $\Diamond_i \beta \wedge \alpha \wedge \tau_i$ is consistent for every $\beta \in A$. So consider $\beta \in A$. By Lemma 4.6 it suffices to show that $\diamondminus_i(\Diamond_i\beta \wedge \alpha) \wedge \tau_i$ is consistent.

Now $\beta \wedge \diamondminus_i\alpha \in A$ and $\tau_i \in A$. Hence by (T4.b), $\diamondminus_i(\alpha \wedge \Diamond_i\beta) \in A$. Since $\tau_i \in A$ we have that $\tau_i \wedge \diamondminus_i(\alpha \wedge \Diamond_i\beta)$ is consistent.

(ii) It suffices to show that $\{\diamondminus_i\beta \mid \beta \in A\} \cup \{\alpha, \tau_i\}$ is consistent. As in part (i) of this lemma, it is easy to observe that in fact it suffices to show that $\diamondminus_i\beta \wedge \alpha \wedge \tau_i$ is consistent for every $\beta \in A$. So assume that $\beta \in A$. Once again from Lemma 4.6, it is enough to show that $\Diamond_i\,(\diamondminus_i\beta \wedge \alpha) \wedge \tau_i$ is consistent. Now $\tau_i \in A$ and $\beta\wedge\Diamond_i\alpha \in A$. Hence by (T4.a), $\Diamond_i\,(\alpha \wedge \diamondminus_i\beta) \in A$. From $\tau_i \in A$ we obtain that $\tau_i\wedge\Diamond_i\,(\alpha \wedge \diamondminus_i\beta)$ is indeed consistent.

□

Lemma 4.8

$\preceq_t$ is transitive.

Proof: Follows easily from the definition of $\preceq_t$ and (A3.b).

□

Now we shall introduce a pre-order relation for MCSs of different types.

Definition 4.9

Let A, B be two MCSs with $type(A) = i$ and $type(B) = j \neq i$. Then

$$A \preceq_c B \overset{\text{def}}{\Longleftrightarrow} \{\diamondminus_i\alpha \mid \diamondminus_i\alpha \in A\} = \{\diamondminus_i\beta \mid \diamondminus_i\beta \in B\}.$$

□

Lemma 4.10

Let A, B be two MCSs with $type(A) = i \neq j = type(B)$. Then the following statements are equivalent.

(i) $A \preceq_c B$

(ii) $\Diamond_i\alpha \in A$ iff $\Diamond_i\alpha \in B$

(iii) $\Box_i\alpha \in A$ iff $\Box_i\alpha \in B$

(iv) $\boxminus_i\alpha \in A$ iff $\boxminus_i\alpha \in B$.

Proof:

(i)⇒(iv) $\boxminus_i\alpha \in A$ iff $\sim \boxminus_i\alpha \notin A$. Now $\diamondminus_i \sim \alpha \notin A$ iff $\diamondminus_i \sim \alpha \notin B$ by (i). But $\diamondminus_i \sim \alpha \notin B$ iff $\sim \diamondminus_i \sim \alpha \in B$.

(iv)⇒(i) Similar to the proof of (i)⇒(iv).

(ii)⇒(iii) $\Box_i\alpha \in A$ iff $\sim\Box_i\alpha \notin A$ iff $\Diamond_i \sim \alpha \notin A$. But $\Diamond_i \sim \alpha \notin A$ iff $\Diamond_i \sim \alpha \notin B$ by (ii). And $\Diamond_i \sim \alpha \notin B$ iff $\sim\Diamond_i \sim \alpha \in B$.

(iii)⇒(ii) Similar to the proof of (ii) ⇒ (iii).

(i)⇒(iii) $\Box_i\alpha \in A$ iff $\diamondminus_i\Box_i\alpha \in A$ by T5.a. But $\diamondminus_i\Box_i\alpha \in A$ iff $\diamondminus_i\Box_i\alpha \in B$ by (i). Once again by T5.a, $\diamondminus_i\Box_i\alpha \in B$ iff $\Box_i\alpha \in B$.

(iii)⇒(i) $\diamondminus_i\alpha \in A$ iff $\Box_i \diamondminus_i\alpha \in A$ by T5.b. $\Box_i \diamondminus_i\alpha \in A$ iff $\Box_i \diamondminus_i\alpha \in B$ by (iii). Once again from T5.b, we get $\Box_i \diamondminus_i\alpha \in B$ iff $\diamondminus_i\alpha \in B$.

□

Lemma 4.11

Let B be an MCS with $type(B) \neq i$. Then for any formula α,

(i) If $\diamondminus_i\alpha \in B$ then there exists an MCS A with $type(A) = i$ such that $A \preceq_c B$ and $\diamondminus_i\alpha \in A$.

(ii) If $\Box_i\alpha \in B$ then there exists an MCS A with $type(A) = i$ such that $A \preceq_c B$ and $\Box_i\alpha \in A$.

Proof:

(i) Let $B^- = \{\diamondminus_i\beta \mid \diamondminus_i\beta \in B\} \cup \{\sim \diamondminus_i\gamma \mid \diamondminus_i\gamma \notin B\}$. Then it suffices to show that $B^- \cup \{\tau_i\}$ is consistent. Consider

$$\{\diamondminus_i\beta_1, \diamondminus_i\beta_2, \ldots, \diamondminus_i\beta_m, \sim \diamondminus_i\gamma_1, \sim \diamondminus_i\gamma_2, \ldots, \sim \diamondminus_i\gamma_l\} \subseteq B^-$$

Since $B^- \subseteq B$ we have that $\{\diamondminus_i\alpha_1, \ldots, \diamondminus_i\alpha_m, \boxminus_i \sim \gamma_1, \boxminus_i \sim \gamma_2, \ldots, \boxminus_i \sim \gamma_l\}$ is consistent. By T2, $\{\diamondminus_i\alpha_1, \ldots, \diamondminus_i\alpha_m, \boxminus_i\gamma\}$ is consistent where $\gamma = \sim \gamma_1 \wedge \sim \gamma_2 \wedge \ldots \wedge \sim \gamma_l$. From T8, we have, without loss of generality,

$\diamondminus_i\beta_1 \wedge \diamondminus_i\beta_2 \ldots \wedge \diamondminus_i\beta_m \supset \diamondminus_i(\beta_1 \wedge \diamondminus_i(\beta_2 \wedge \ldots \wedge \diamondminus_i\beta_m)) \ldots)$. Define $\hat{\beta}_m = \beta_m$ and $\hat{\beta}_{m'} = \beta_{m'} \wedge \diamondminus_i\hat{\beta}_{m'+1}$ for $1 \leq m' < m$. Then we have that $\{\diamondminus_i\hat{\beta}_1, \boxminus_i\gamma\}$ is consistent.

By T6.b and Lemma 4.6.i, $\{\diamondminus_i\hat{\beta}_1, \boxminus_i\gamma, \tau_i\}$ is consistent. Let A_0 be an MCS containing $\{\diamondminus_i\hat{\beta}_1, \boxminus_i\gamma, \tau_i\}$. Then by lemma 4.7, there exists an MCS A_1 with $type(A_1) = i$ such that $\hat{\beta}_1 = \beta_1 \wedge \diamondminus_i\hat{\beta}_2 \in A_1$ and $A_1 \preceq_t A_0$. We can apply the same argument at A_1 to obtain an MCS A_2 such that $type(A_2) = i$ and $\hat{\beta}_2 \in A_2$ and

$A_2 \preceq_t A_1$. Applying this argument m times we get m MCSs $A_m, A_{m-1}, \ldots, A_1, A_0$ such that $A_m \preceq_t A_{m-1} \cdots \preceq_t A_0$ with $type(A_l) = i$, $\hat{\beta}_l \in A_l$ for $1 \leq l \leq m$. Since $\preceq_t$ is transitive it is easy to verify that $\{\Diamond_i\beta_1, \Diamond_i\beta_2, \ldots, \Diamond_i\beta_l\} \subseteq A_0$. Thus $\{\Diamond_i\beta_1, \ldots, \Diamond_i\beta_l, \boxminus_i\gamma, \tau_i\}$ is consistent.

(ii) Follows easily from the thesis (T5.a) and the proof of part (i).

□

Finally we can introduce the pre-order relation we are really after.

Definition 4.12

Let A and B be two MCSs with $type(A) = i$. Then

$$A \preceq B \stackrel{\text{def}}{\Longleftrightarrow} \{\Diamond_i\alpha \mid \alpha \in A\} \subseteq B$$

□

Lemma 4.13

$\preceq = (\preceq_t \cup \preceq_c)*$.

<u>Proof</u>: Let A, B be two MCSs with $type(A) = i$ and $A \preceq B$. Then $\{\Diamond_i\alpha \mid \alpha \in A\} \subseteq B$.

<u>case 1</u> $type(B) = i$.
Then $A \preceq_t B$ by definition.

<u>case 2</u> $type(B) \neq i$.
Let $C^- = \{\Diamond_i\alpha, \sim \Diamond_i\beta \mid \Diamond_i\alpha \in B \text{ and } \Diamond_i\beta \notin B\}$. By the proof of the previous result, we have that $C^- \cup \{\tau_i\}$ is consistent. Let C be an MCS containing $C^- \cup \{\tau_i\}$. Clearly $A \preceq_t C$ and $C \preceq_c B$. Hence $\preceq \subseteq (\preceq_t \cup \preceq_c)*$.

For showing containment in the other direction, we first claim that $\preceq$ is transitive. To see this, let $A \preceq B$ and $B \preceq C$ with $type(A) = i$ and $type(B) = j$. Consider $\alpha \in A$. Then $\Diamond_i\alpha \in B$ and $\Diamond_j\Diamond_i\alpha \in C$. From A3.b, it follows that $\Diamond_i\alpha \in C$. Now let $A_1, A_2, \ldots, A_m$ be MCSs such that $(A_i, A_{i+1}) \in \preceq_t \cup \preceq_c$ for $1 \leq i < m$. The proof is by induction on m.

<u>$m = 1$</u> Then $\alpha \in A_1$ and $type(A_1) = i$ implies $\Diamond_i\alpha \in A_1$. Hence $A_1 \preceq A_1$.

<u>$m > 1$</u> By the induction hypothesis $A_2 \preceq A_n$. Assume that $A_1 \preceq_t A_2$. Then $A_1 \preceq A_2$ by definition. Since $\preceq$ is transitive, we have $A_1 \preceq A_m$ so assume that $A_1 \preceq_c A_2$. Let $type(A_1) = i$ and $type(A_2) = j$ with $i \neq j$. Consider $\alpha \in A_1$. Then $\Diamond_i\alpha \in A_1$. Hence $\Diamond_i\alpha \in A_2$. Hence $\Diamond_j\Diamond_i\alpha \in A_m$ since $A_2 \preceq A_m$. But this implies $\Diamond_i\alpha \in A_m$ by A2.b.

□

We need the notion of a perfect chronicle for completing the completeness proof.

Definition 4.14

Let $ES = (E, \leq, \#)$ be a frame. Then

(i) A *chronicle* on ES is a function T which assigns an MCS to each $e \in E$.

Let T be a chronicle on the frame $ES = (E, \leq, \#)$. Then

(ii) T is *coherent* iff it satisfies the following requirements

(a) $\forall e, e' \in E.\ e \leq e'$ implies $T(e) \preceq T(e')$ and

(b) $\forall e \in E.\ \tau_i \in T(e)$ iff $e \in E_i$.

(iii) T is *prophetic* iff $e \in E_i$ and $\Diamond_i \alpha \in T(e)$ implies that there exists $e' \in E_i$ such that $e \leq e'$ and $\alpha \in T(e')$.

(iv) T is *historic* iff $e \in E_i$ and $\diamondminus_i \alpha \in T(e)$ implies that there exists $e' \in E_i$ such that $e' \leq e$ and $\alpha \in T(e')$.

(v) T is *prophetically informed* iff $e \in E_j$ and $j \neq i$ and $\Box_i \alpha \in T(e)$ implies that there exists $e' \in E_i$ such that $e' \leq e$ and $\Box_i \alpha \in T(e')$.

(vi) T is *historically informed* iff $e \in E_j$ and $j \neq i$ and $\diamondminus_i \alpha \in T(e)$ implies that there exists $e' \in E_i$ such that $e' \leq e$ and $\diamondminus_i \alpha \in T(e')$.

(vii) T is *informed* iff it is prophetically and historically informed.

(viii) T is *perfect* iff it is coherent, prophetic, historic and informed.

□

Given a valuation V on the frame $ES = (E, \leq, \#)$ the *chronicle induced by* V is denoted as T_V and is given by:

$$\forall e \in E.\ T_V(e) = \{\alpha \mid e, M \models \alpha\} \text{ where } M = (ES, V).$$

It is easy to verify that the chronicle induced by a valuation is always perfect. On the other hand every perfect chronicle induces a valuation. To see this let T be a perfect chronicle on the frame $ES = (E, \leq, \#)$. Then V_T, the *valuation induced by* T is given by:

$$\forall e \in E.\ V_T(e) = \{p \in \hat{P} \mid p \in T(e)\}.$$

Lemma 4.15

Let T be a perfect chronicle on the frame $ES = (E, \leq, \#)$. Then $T_{(V_T)} = T$.

Proof: Let $M = (ES, V_T)$ and $e \in E$. We must show that $T(e) = \{\alpha \mid e, M \models \alpha\}$. To this end let α be a formula. Then we proceed by structural induction on α.

If $\alpha \in \hat{P}$ then by definition of V_T, $e, M \models \alpha$ iff $\alpha \in T(e)$.

If α is of the form $\sim \beta$, then $\sim \beta \in T(e)$ iff $\beta \notin T(e)$. But by the induction hypothesis $\beta \notin T(e)$ iff $e, M \not\models \beta$ which is the same as $e, M \models \sim \beta$.

If α is of the form $\beta_1 \vee \beta_2$ then $\beta_1 \vee \beta_2 \in T(e)$ iff $\beta_1 \in T(e)$ or $\beta_2 \in T(e)$. Once again the required result follows from the induction hypothesis.

So assume that α is of the form $\Box_i\beta$.

case 1 $\quad e \in E_i$.

Suppose that $\Box_i \beta \in T(e)$. Consider $e' \in E_i$ such that $e \leq e'$. Then since T is coherent, we have that $T(e) \preceq T(e')$. Clearly $T(e) \preceq_t T(e')$. From Prop. 4.5 we have that $\beta \in T(e')$. By the induction hypothesis, $e', M \models \beta$. Hence $e, M \models \Box_i\beta$.

Now suppose that $\Box_i \beta \notin T(e)$. Then $\Diamond_i \sim \beta \in T(e)$. Since T is perfect, we can find $e' \in E_i$ such that $e \leq e'$ and $\sim \beta \in T(e')$. By the induction hypothesis $e', M \not\models \beta$. Hence $e, M \not\models \Box_i\beta$.

case 2 $\quad e \notin E_i$.

Suppose that $\Box_i\beta \in T(e)$. Since T is perfect, there exists $e' \in E_i$ such that $e' \leq e$ and $\Box_i\beta \in T(e')$. By the proof of the preceding case we have that $e', M \models \Box_i\beta$. Clearly $e, M \models \Box_i\beta$.

On the other hand, if $e, M \models \Box_i\beta$, then $\exists e' \in E_i$ such that $e' \leq e$ and $e', M \models \Box_i\beta$. Once again by the proof of case 1 it follows that $\Box_i \beta \in T(e')$. Now $T(e') \preceq T(e)$. Hence $\ominus_i\Box_i\beta \in T(e)$. By T5.a, $\Box_i \beta \in T(e)$.

The proof for the case where α is of the form $\boxminus_i\beta$ is very similar to the previous case and we shall omit it.

□

We now wish to show that a frame with a coherent but not perfect chronicle can be extended to a frame with an improved chronicle. We start with a useful lemma.

Lemma 4.16

Let A, B, C be MCSs such that $type(A) = i = type(B)$, $A \preceq C$ and $B \preceq C$. Then $A \preceq_t B$ or $B \preceq_t A$.

Proof: Suppose that $A \not\preceq_t B$ and $B \not\preceq_t A$. Then there exist formulas α and β such that $\alpha \wedge \sim \ominus_i\beta \in A$ and $\beta \wedge \sim \ominus_i\alpha \in B$. Since $A \preceq C$ and $B \preceq C$, we then have $\ominus_i(\alpha \wedge \boxminus_i \sim \beta), \ominus_i(\beta \wedge \boxminus_i \sim \alpha) \in C$. Hence by A4, $\ominus_i(\alpha \wedge \boxminus_i \sim \beta \wedge \ominus_i(\beta \wedge \boxminus_i \sim \alpha)) \in C$ or $\ominus_i(\beta \wedge \boxminus_i \sim \alpha \wedge \ominus_i(\alpha \wedge \boxminus_i \sim \beta)) \in C$. Assume without loss of generality that $\ominus_i(\alpha \wedge \boxminus_i \sim \beta \wedge \ominus_i(\beta \wedge \boxminus_i \sim \alpha)) \in C$. Then $\alpha \wedge \boxminus_i \sim \beta \wedge \ominus_i(\beta \wedge \boxminus_i \sim \alpha)$ is consistent by lemma 4.6. Now $\boxminus_i\delta \wedge \ominus_i\gamma \supset \ominus_i(\boxminus_i\delta \wedge \tau_i \wedge \ominus_i\gamma)$ by T6.b. Hence $\ominus_i(\boxminus_i \sim \beta \wedge \tau_i \wedge \ominus_i(\beta \wedge \boxminus_i \sim \alpha))$ is consistent which leads – once again by lemma 4.6 – to the fact that $\boxminus_i \sim \beta \wedge \tau_i \wedge \ominus_i(\beta \wedge \boxminus_i \sim \alpha)$ is consistent. But now from T3.b we have that $\ominus_i(\sim \beta \wedge \beta \wedge \boxminus_i \sim \alpha)$ is consistent which – yet again by lemma 4.6 – leads to the contradiction that $\sim \beta \wedge \beta$ is consistent.

□

To show that a coherent but not perfect chronicle can be improved it will be convenient to work with chronicle structures.

Definition 4.17

A *chronicle structure* is a pair (ES, T) where $ES = (E, \leq, \#)$ is a frame and T is a coherent chronicle on ES which respects conflict in the following sense.

$$\forall i.\ \forall e, e' \in E_i.\ e\#e' \Rightarrow T(e) \not\leq_t T(e') \wedge T(e') \not\leq_t T(e).$$

$\square$

The imperfections of a chronicle can be judged with the help of live requirements.

Definition 4.18

Let (ES, T) be a chronicle structure with $ES = (E, \leq, \#)$. Let $e \in E$.

(i) $(e, \ominus_i\alpha)$ is a *live historic requirement* in (ES, T) iff $e \in E_i$ and $\ominus_i\alpha \in T(e)$ and there does not exist $e' \in E_i$ for which $e' \leq e$ and $\alpha \in T(e')$ holds.

(ii) $(e, \Diamond_i\alpha)$ is a *live prophetic requirement* in (ES, T) iff $e \in E_i$ and $\Diamond_i\alpha \in T(e)$ and there does not exist $e' \in E_i$ for which $e \leq e'$ and $\alpha \in T(e')$ holds.

(iii) $(e, \ominus_i\alpha)$ (resp. $(e, \Box_i\alpha)$) is a *live communication requirement* in (ES, T) iff $e \notin E_i$ and $\ominus_i\alpha \in T(e)$ (resp. $\Box_i\alpha \in T(e)$) and there does not exist $e' \in E_i$ for which $e' \leq e$ and $\ominus_i\alpha \in T(e')$ (resp. $\Box_i\alpha \in T(e')$) holds.

(iv) (e, β) is a *live requirement* in (ES, T) iff it is a live prophetic or historic or communication requirement in (ES, T).

$\square$

Lemma 4.19

Let (ES, T) be a chronicle structure with $ES = (E, \leq, \#)$. Let (e, β) be a live requirement in (ES, T). Then there exists a chronicle structure (ES', T') with $ES' = (E', \leq', \#')$ such that:

(i) $E' = E \cup \{\hat{e}\}$ for some $\hat{e} \notin E$.

(ii) $\leq$ (resp. $\#$) is $\leq'$ (resp. $\#'$) restricted to $E \times E$.

(iii) T is T' restricted to E.

(iv) (e, β) is not a live requirement in (ES', T').

Proof:
case 1 (e, β) is a live communication requirement.

Assume – as it will turn out – without loss of generality that β is of the form $\ominus_i \alpha$. Then $e \notin E_i$. By lemma 4.11, there exists an MCS A such that $type(A) = i$ and $\ominus_i \alpha \in A$ and $A \preceq_c T(e)$. Pick some $\hat{e} \notin E$ and set for $1 \leq j \leq n$,

$$E'_j = \begin{cases} E_j \cup \{\hat{e}\}, & \text{if } j = i \\ E_j, \text{otherwise} \end{cases}$$

Set $E' = \bigcup_{j=1}^{n} E'_j$. Now define

$$Pre(\hat{e}) = \{e' \in E_i \mid T(e') \preceq_t A \wedge A \not\preceq_t T(e')\}$$

and

$$Post(\hat{e}) = \{e' \in E_i \mid A \preceq_t T(e')\} \cup \{\hat{e}\}$$

Let $\leq'$ be the least subset of $E' \times E'$ given by:

(i) $\leq \subseteq \leq'$

(ii) $(\hat{e}, e) \in \leq'$

(iii) $Pre(\hat{e}) \times \{\hat{e}\} \subseteq \leq'$ and $\{\hat{e}\} \times Post(\hat{e}) \subseteq \leq'$

(iv) $(\leq')* = \leq'$.

Next define $\hat{\#}'$ to be

$$\hat{\#}' = \bigcup_{j=1}^{n} \#'_j$$

where for $1 \leq j \leq n$,

$$\#'_j = \begin{cases} \#_j \cup \{(\hat{e}, x), (x, \hat{e}) \mid x \in E_i \wedge A \not\preceq_t T(x) \wedge T(x) \not\preceq_t A\}, & \text{if } j = i \\ \#_j, \text{otherwise} \end{cases}$$

Then define $\# \subseteq E' \times E'$ as

$$\# = \{(x, y) \mid \exists (x', y') \in \hat{\#}'.\ x' \leq' x \text{ and } y' \leq' y\}$$

Finally extend T to E' as follows.

$$\forall x \in E'.\ T'(x) = \begin{cases} A, & \text{if } x = \hat{e} \\ T(x), & \text{otherwise} \end{cases}$$

We claim that (ES', T') is a chronicle structure with $ES' = (E', \leq', \#')$ in which (e, β) is no longer a live requirement.
To see this, we first observe that $\leq'$ is reflexive and transitive by definition. Next we note that $\forall x \in Pre(\hat{e})$ and $\forall y \in Post(\hat{e})$, $x \leq y$ (where $y \neq \hat{e}$). This is so because for $x \in Pre(\hat{e})$ and $y \in Post(\hat{e})$, if $y \leq x$ then $T(y) \preceq_t T(x)$. But this would imply that $A \preceq_t T(x)$ which contradicts the definition of $Pre(\hat{e})$. (Recall that $\preceq_t$ is transitive by lemma 4.8.) On the other hand $y \# x$ is ruled out by $T(x) \preceq_t A \preceq_t T(y)$ and the fact that (ES, T) is a chronicle structure. Hence $x \leq y$.
Now suppose $x, y \in E'$ such that $x \leq' y$ and $y \leq' x$. First consider the case where $x = \hat{e}$. From $y \leq' \hat{e}$ it follows that for some $z_1 \in Pre(\hat{e})$, $y \leq z_1$. From $\hat{e} \leq' y$ it follows that for some $z_2 \in Post(\hat{e}) \cup \{e\}, z_2 \leq y$. But this implies that $z_2 \leq z_1$ so that $T(z_2) \preceq T(z_1)$. Moreover $A \preceq T(z_2)$, (recall that $(\preceq_c \cup \preceq_t)* = \preceq$) so that $A \preceq T(z_1)$ which contradicts the definition of $Pre(\hat{e})$. Hence $x = y$.
By a similar argument we can show that $\leq'$ is – in general – antisymmetric.
$\#'$ is symmetric by definition. It is easy to check that $\#'$ is irreflexive with the help of lemma 4.16. Now it is routine to verify that (ES', T') is a chronicle structure in which (e, β) is no longer a live requirement.

<u>case 2</u> (e, β) is a live prophetic requirement.

Assume that $e \in E_i$ and β is of the form $\Diamond_i \alpha$. By lemma 4.7 there exists an MCS A such that $T(e) \preceq_t A$ and $type(A) = i$ and $\alpha \in A$. Pick some $\hat{e} \notin E$ and define for $1 \leq j \leq n$,

$$E'_j = \begin{cases} E_j \cup \{\hat{e}\}, & \text{if } j = i \\ E_j, \text{otherwise} \end{cases}$$

Set $E' = \bigcup_{j=1}^{n} E'_j$. Let

$$\begin{aligned} Pre(\hat{e}) &= \{x \in E_i \mid T(x) \preceq_t A\} \cup \{\hat{e}\} \\ Post(\hat{e}) &= \{x \in E_i \mid A \preceq_t T(x) \wedge T(x) \not\preceq_t A\} \end{aligned}$$

Define $\leq'$ to be the least subset of $E' \times E'$ which satisfies the following conditions.

(i) $\leq \subseteq \leq'$

(ii) $Pre(\hat{e}) \times \{\hat{e}\} \subseteq \leq$ and $\{\hat{e}\} \times Post(\hat{e}) \subseteq \leq'$

(iii) $(\leq')* = \leq'$

Next set $\hat{\#}' = \bigcup_{j=1}^{n} \#'_j$ where for $1 \leq j \leq n$,

$$\#'_j = \begin{cases} \#_j \cup \{(x, \hat{e}), (\hat{e}, x) \mid x \in E_i \wedge T(x) \not\preceq_t A \wedge A \not\preceq_t T(x)\}, & \text{if } j = i \\ \#_j, \text{otherwise} \end{cases}$$

Then define $\#' \subseteq E' \times E'$ as

$$\#' = \{(x, y) \mid \exists (x', y') \in \hat{\#}'.\, x' \leq' x \text{ and } y' \leq' y\}$$

Finally extend T to E' through,

$$\forall x \in E'.\, T'(x) = \begin{cases} A, & \text{if } x = \hat{e} \\ T(x), & \text{otherwise} \end{cases}$$

It is easy now to check that (ES', T') – where $ES' = (E', \leq', \#')$ – is a chronicle structure in which (e, β) is no longer a live requirement.

<u>case 3</u> (e, β) is a live historic requirement.

Let $e \in E_i$ and β be of the form $\diamondminus_i \alpha$. Then once again by lemma 4.7 we can find an MCS A such that $A \preceq_t T(e)$ and $type(A) = i$ and $\alpha \in A$. Pick some $\hat{e} \notin E_i$ and define

$$\begin{aligned} Pre(\hat{e}) &= \{x \in E_i \mid T(x) \preceq_t A \wedge A \not\preceq_t T(x)\} \text{ and} \\ Post(\hat{e}) &= \{x \in E_i \mid A \preceq_t T(x)\} \cup \{\hat{e}\} \end{aligned}$$

The rest of the proof is very similar to the proof of case 1 and we omit it.

□

Theorem 4.20 (completeness)

If $\models \alpha$ then $\vdash \alpha$.

<u>Proof</u>: We will show that every consistent formula is satisfiable. Let $\hat{E}$ be a countable set of events. Fix an enumeration $e_1, e_2, \ldots$ of $\hat{E}$ and fix an enumeration $\alpha_1, \alpha_2, \ldots$ of F, the set of formulas. Fix an *injective* function $f : \hat{E} \times F \rightarrow N$. Since $\hat{E} \times F$ is a countable set, there will be no trouble in finding such an injective function. In what follows, for $(e, \alpha) \in \hat{E} \times F$, we will refer to $f((e, \alpha))$ as the *code number* of (e, α).

Now assume that α is a consistent formula. Pick an MCS A which contains α. Let $CS^1 = (ES^1, T^1)$ where $ES^1 = (\{e_1\}, \{(e_1, e_1)\}, \phi)$ and $T^1(e_1) = A$. Clearly CS^1 is a chronicle structure. For $m \geq 1$, suppose the chronicle structure $CS^m = (ES^m, T^m)$ is defined with $ES^m = (E^m, \leq^m, \#^m)$ where $E^m = \{e_1, e_2, \ldots, e_m\}$. Suppose CS^m does not have any live requirements. Then set $CS^{m+1} = CS^m$. Otherwise consider a live requirement (e, β) in CS^m which has – among all the live requirements in CS^m – the least code number. Then by the previous lemma CS^m can be extended to the chronicle structure $CS^{m+1} = (ES^{m+1}, T^{m+1})$ with $ES^{m+1} = (E^{m+1}, \leq^{m+1}, \#^{m+1})$ and $E^{m+1} = E^m \cup \{e_{m+1}\}$ so that (e, β) is no longer a live requirement in CS^{m+1}. Finally set $CS = (ES, T)$ where $ES = (E, \leq, \#)$, $E = \bigcup_{m=1}^{\infty} E^m$, $\leq = \bigcup_{m=1}^{\infty} \leq^m$ and $\# = \bigcup_{m=1}^{\infty} \#^m$. Moreover T is given by:

$$\forall e \in E.\, T(e) = T^m(e) \text{ where } e \in E^m.$$

It is routine to verify that T is a perfect chronicle on ES. Hence by lemma 4.15, $M = (ES, V_T)$ is a model in which $e_1, M \models \alpha$.

$\square$

5. Discussion

In this paper we have obtained a modal logic characterization of a subclass of event structures. Our major goal has been to use modal logic to characterize the flow of information between the sequential components of a distributed system. Admittedly, the work reported here constitutes merely a first step toward achieving this goal.

Our language needs to be expanded to include Next-State and Until operators. It is not entirely clear what the semantics of the Until operator should be and what – if any – additional expressive power is got by introducing this operator. Of course we will have to then look for the right axioms corresponding to the semantics of these new operators.

As for our frames, for many applications we will have to impose additional restrictions such as discreteness, well-foundedness etc. Here we feel quite hopeful that the corresponding axioms can be quickly found thanks to Burgess [1]. More seriously, we permit at present only asynchronous communication between the agents via message passing. In many situations, it will be necessary to admit synchronized (i.e. hand-shake) communication as well. This would amount to dropping the assumption that the sets of events belonging to the agents are pair-wise disjoint. Clearly our axiom system will have to be modified (and a new completeness proof will have to be constructed) to reflect this. In particular, (A7.a) will now have to read: $\bigvee_{i=1}^{n} \tau_i$. There is good reason to hope that not much else will have to be changed and that the completeness proof will be essentially the same.

As for our semantics, we have considered a standard interpretation of $\Box$ and $\Diamond$ where the individual agents are viewed as basically partial orders in which the predecessors of any element are totally ordered. We have done so in order to carry out our first study in as simple a setting as possible. However the individual agents represent sequential and – in general – *non-deterministic* behaviours. Hence a more natural semantics would be to view the agents as branching time structures. Then $\Box$ and $\Diamond$ would have to be interpreted over the paths (i.e. rooted maximal chains) in the agents. In such a set-up $\Diamond$ would become much stronger and $\Box$ would be much weaker (compared to the current set-up) but this is how it should be. We think that it will be a non-trivial task to construct a sound and complete axiom system w.r.t. this branching time semantics. Fortunately, the existing literature on branching time logics (see, for example [3]) provides a solid basis for attacking this problem.

It will also be interesting to develop a syntax for building up n-agent event structures and see whether the insights gained through the present work leads to a proof system which is compositional w.r.t. the chosen syntax.

In an informal sense, our formulas may be viewed as assertions concerning the states of knowledge or belief of the individual agents. However, there are two issues that make it difficult to establish a clear link between our work and recent work on logics of knowledge and belief. Firstly, the presence of past operators is crucial to our approach and it is not obvious how to interpret $\boxminus$ as a knowledge operator. The second problem is that, in dealing with knowledge one always assumes that an agent knows only true facts which is formalized by demanding $\Box_i\alpha \supset \alpha$ to be an axiom. In our system however, $(\Box_i\alpha) \wedge \sim \alpha$ will be – in general – consistent. Hence we will have to leave it to future work to discover whether or not the work reported here can yield useful tools for reasoning about knowledge and belief in distributed environments.

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7. Appendix

We first establish that the derived inference rules TG.c and DR.3b preserve thesishood. (In what follows, PC is an abbreviation for Propositional Calculus.)

(TG.c)	(1)	$\vdash \alpha$	(Given)
	(2)	$\vdash \alpha \supset (\tau_i \supset \alpha)$	(PC)
	(3)	$\vdash \tau_i \supset \alpha$	(1, 2, MP)
	(4)	$\vdash \Box_i \tau_i \supset \Box_i \alpha$	(3, TG.a)
	(5)	$\vdash (\tau_i \supset \Box_i \tau_i) \supset ((\Box_i \tau_i \supset \Box_i \alpha) \supset (\tau_i \supset \Box_i \alpha))$	(PC)
	(6)	$\vdash \tau_i \supset \Box_i \tau_i$	(A7.b)
	(7)	$\vdash (\Box_i \tau_i \supset \Box_i \alpha) \supset (\tau_i \supset \Box_i \alpha)$	(5, 6, MP)
	(8)	$\vdash \tau_i \supset \Box_i \alpha$	(4, 7, MP)

(DR3.b)	$\vdash \alpha \supset \beta$	
	$\vdash \sim \beta \supset \sim \alpha$	(PC)
	$\vdash \boxminus_i \sim \beta \supset \boxminus_i \sim \alpha$	(TG.b, A1.b)
	$\vdash \diamondminus_i \alpha \supset \diamondminus_i \beta$	(PC)

We will now derive the theses. In doing so we will use a numbered sequence of steps and label the theses as and when we encounter them.

	(1)	$\vdash \alpha \supset \alpha \vee \beta$	(PC)
	(2)	$\vdash \Box_i \alpha \supset \Box_i (\alpha \vee \beta)$	(TG.a)
	(3)	$\vdash \Box_i \beta \supset \Box_i (\alpha \vee \beta)$	(Subst. in 2)
	(4)	$\vdash \Box_i \alpha \vee \Box_i \beta \supset \Box_i (\alpha \vee \beta)$	(2, 3, PC)
	(5)	$\vdash \Box_i \sim \alpha \vee \Box_i \sim \beta \supset \Box_i (\sim \alpha \vee \sim \beta)$	(Subst. in 4)
<u>(T1.a)</u>	(6)	$\vdash \Diamond_i (\alpha \wedge \beta) \supset (\Diamond_i \alpha \wedge \Diamond_i \beta)$	(PC)
	(1)	$\vdash \alpha \supset \alpha \vee \beta$	(PC)
	(2)	$\vdash \boxminus_i \alpha \supset \boxminus_i (\alpha \vee \beta)$	(TG.b,A1.b)
	(3)	$\vdash \boxminus_i \beta \supset \boxminus_i (\alpha \vee \beta)$	(Subst. in 2)
	(4)	$\vdash \boxminus_i \alpha \vee \boxminus_i \beta \supset \boxminus_i (\alpha \vee \beta)$	(2, 3)
	(5)	$\vdash \boxminus_i \sim \alpha \vee \boxminus_i \sim \beta \supset \boxminus_i (\sim \alpha \vee \sim \beta)$	(Subst. in 4)
<u>(T1.b)</u>	(6)	$\vdash \diamondminus_i (\alpha \wedge \beta) \supset (\diamondminus_i \alpha \wedge \diamondminus_i \beta)$	(PC)
	(1)	$\vdash \alpha \supset (\beta \supset \alpha \wedge \beta)$	(PC)
	(2)	$\vdash \boxminus_i \alpha \supset (\boxminus_i \beta \supset \boxminus_i (\alpha \wedge \beta))$	(TG.b, A1.b, A1.b)
	(3)	$\vdash \alpha \wedge \beta \supset \alpha$	(PC)
	(4)	$\vdash \boxminus_i (\alpha \wedge \beta) \supset \boxminus_i \alpha$	(TG.b, A1.b)
	(5)	$\vdash \boxminus_i (\alpha \wedge \beta) \supset \boxminus_i \beta$	(Subst. in 4)
	(6)	$\vdash \boxminus_i (\alpha \wedge \beta) \supset (\boxminus_i \alpha \wedge \boxminus_i \beta)$	(4, 5)
<u>(T2)</u>	(7)	$\vdash \boxminus_i (\alpha \wedge \beta) \equiv (\boxminus_i \alpha \wedge \boxminus_i \beta)$	(6, 2)

	(1)	$\vdash (\alpha \supset \beta) \supset (\sim \beta \supset \sim \alpha)$	(PC)
	(2)	$\vdash \Box_i(\alpha \supset \beta) \supset (\Box_i \sim \beta \supset \Box_i \sim \alpha)$	(TG.a, A1.a)
	(3)	$\vdash \Box_i(\alpha \supset \beta) \supset (\Diamond_i\alpha \supset \Diamond_i\beta)$	(PC)
	(4)	$\vdash \Box_i(\beta \supset (\alpha \wedge \beta)) \supset (\Diamond_i\beta \supset \Diamond_i(\alpha \wedge \beta))$	(Subst. in 3)
	(5)	$\vdash \alpha \supset (\beta \supset \alpha \wedge \beta)$	(PC)
	(6)	$\vdash \Box_i\alpha \supset \Box_i(\beta \supset \alpha \wedge \beta)$	(TG.a, A1.a)
	(7)	$\vdash \Box_i\alpha \supset (\Diamond_i\beta \supset \Diamond_i(\alpha \wedge \beta))$	(4, 6)
<u>(T3.a)</u>	(8)	$\vdash \Box_i\alpha \wedge \Diamond_i\beta \supset \Diamond_i(\alpha \wedge \beta)$	(PC)
	(1)	$\vdash (\alpha \supset \beta) \supset (\sim \beta \supset \sim \alpha)$	(PC)
	(2)	$\vdash \boxminus_i(\alpha \supset \beta) \supset (\boxminus_i \sim \beta \supset \boxminus_i \sim \alpha)$	(TG.b, A1.b, A1.b)
	(3)	$\vdash \boxminus_i(\alpha \supset \beta) \supset (\diamondminus_i\alpha \supset \diamondminus_i\beta)$	(PC)
	(4)	$\vdash \boxminus_i(\beta \supset \alpha \wedge \beta) \supset (\diamondminus_i\beta \supset \diamondminus_i(\alpha \wedge \beta))$	(Subst. in 3)
	(5)	$\vdash \alpha \supset (\beta \supset \alpha \wedge \beta)$	(PC)
	(6)	$\vdash \boxminus_i\alpha \supset \boxminus_i(\beta \supset \alpha \wedge \beta)$	(TG.b, A1.b)
	(7)	$\vdash \boxminus_i\alpha \supset (\diamondminus_i\beta \supset \diamondminus_i(\alpha \wedge \beta))$	(4, 6)
<u>(T3.b)</u>	(8)	$\vdash \boxminus_i\alpha \wedge \diamondminus_i\beta \supset \diamondminus_i(\alpha \wedge \beta)$	(PC)
	(1)	$\vdash \tau_i \supset (\alpha \supset \diamondminus_i\alpha)$	(A2.b)
	(2)	$\vdash \tau_i \supset (\alpha \supset \Box_i\diamondminus_i\alpha)$	(A5.a, PC)
	(3)	$\vdash \Box_i\diamondminus_i\alpha \wedge \Diamond_i\beta \supset \Diamond_i(\diamondminus_i\alpha \wedge \beta)$	(Subst. in T3.a)
<u>(T4.a)</u>	(4)	$\vdash \tau_i \supset (\alpha \wedge \Diamond_i\beta \supset \Diamond_i(\beta \wedge \diamondminus_i\alpha))$	(2, 3, PC)
	(1)	$\vdash \tau_i \supset (\alpha \supset \Diamond_i\alpha)$	(A2.a)
	(2)	$\vdash \tau_i \supset (\alpha \supset \boxminus_i\Diamond_i\alpha)$	(A5.b, PC)
	(3)	$\vdash \boxminus_i\Diamond_i\alpha \wedge \diamondminus_i\beta \supset \diamondminus_i(\beta \wedge \Diamond_i\alpha)$	(Subst. in T3.b)
<u>(T4.b)</u>	(4)	$\vdash \tau_i \supset ((\alpha \wedge \diamondminus_i\beta) \supset \diamondminus_i(\beta \wedge \Diamond_i\alpha))$	(2, 3, PC)
	(1)	$\vdash \Box_i\alpha \supset \diamondminus_i(\tau_i \wedge \Box_i\alpha)$	(A6.a)
	(2)	$\vdash \Box_i\alpha \supset \diamondminus_i\tau_i \wedge \diamondminus_i\Box_i\alpha$	(T1.b)
	(3)	$\vdash \diamondminus_i\Box_i\alpha \supset \Box_i\alpha$	(A5.b)
<u>(T5.a)</u>	(4)	$\vdash \Box_i\alpha \equiv \diamondminus_i\Box_i\alpha$	(2, 3)
	(1)	$\vdash \tau_i \supset (\Box_i\alpha \supset \alpha)$	(A2.a)
	(2)	$\vdash \diamondminus_i(\tau_i \wedge \Box_i\alpha) \supset \diamondminus_i\alpha$	(PC, DR.3.b)
	(3)	$\vdash \Box_i\diamondminus_i\alpha \supset \diamondminus_i(\tau_i \wedge \Box_i\diamondminus_i\alpha)$	(A6.a)
	(4)	$\vdash \Box_i\diamondminus_i\alpha \supset \diamondminus_i\diamondminus_i\alpha$	(3, Subst. in 2)
	(5)	$\vdash \Box_i\diamondminus_i\alpha \supset \diamondminus_i\alpha$	(A3.b, PC)
<u>(T5.b)</u>	(6)	$\vdash \Box_i\diamondminus_i\alpha \equiv \diamondminus_i\alpha$	(5, A5.a)
	(1)	$\vdash \Diamond_i\alpha \wedge \Box_i\beta \supset \diamondminus_i(\tau_i \wedge \Box_i\beta) \wedge \boxminus_i(\tau_i \supset \Diamond_i\alpha)$	(A6.a, A6.b, PC)
	(2)	$\vdash \Diamond_i\alpha \wedge \Box_i\beta \supset \diamondminus_i(\tau_i \wedge \Box_i\beta \wedge (\tau_i \supset \Diamond_i\alpha))$	(T3.b)
<u>(T6.a)</u>	(3)	$\vdash \Diamond_i\alpha \wedge \Box_i\beta \supset \diamondminus_i(\tau_i \wedge \Box_i\beta \wedge \Diamond_i\alpha)$	(PC, DR.3.b)
	(1)	$\vdash \diamondminus_i\alpha \wedge \boxminus_i\beta \supset \diamondminus_i(\tau_i \wedge \diamondminus_i\alpha) \wedge \boxminus_i(\tau_i \supset \boxminus_i\beta)$	(A6.b, A6.d, PC)

	(2)	$\vdash \diamondminus_i\alpha \wedge \boxminus_i\beta \supset \diamondminus_i(\tau_i \wedge \diamondminus_i\alpha \wedge (\tau_i \supset \boxminus_i\beta))$	(T3.b)
<u>(T6.b)</u>	(3)	$\vdash \diamondminus_i\alpha \wedge \boxminus_i\beta \supset \diamondminus_i(\tau_i \wedge \diamondminus_i\alpha \wedge \boxminus_i\beta)$	(PC, Dr.3.b)
	(1)	$\vdash \tau_i \wedge \Diamond_i\alpha \supset \tau_i \wedge \Box_i\tau_i \wedge \Diamond_i\alpha$	(A7.b)
	(2)	$\vdash \tau_i \wedge \Diamond_i\alpha \supset \tau_i \wedge \Diamond_i(\alpha \wedge \tau_i)$	(T3.a)
<u>(T7.a)</u>	(3)	$\vdash \tau_i \wedge \Diamond_i\alpha \supset \Diamond_i(\alpha \wedge \tau_i)$	(PC)
	(1)	$\vdash \tau_i \wedge \diamondminus_i\alpha \supset \tau_i \wedge \boxminus_i\tau_i \wedge \diamondminus_i\alpha$	(A7.c)
	(2)	$\vdash \tau_i \wedge \diamondminus_i\alpha \supset \tau_i \wedge \diamondminus_i(\alpha \wedge \tau_i)$	(T3.b)
<u>(T7.b)</u>	(3)	$\vdash \tau_i \wedge \diamondminus_i\alpha \supset \diamondminus_i(\alpha \wedge \tau_i)$	(PC)

It is necessary to develop some notational machinery for deriving T8 in its general form. We will merely show the main idea by establishing T8 for the concrete case where $m = 3$.

(1)	$\diamondminus_i\alpha_1 \wedge \diamondminus_i\alpha_2 \wedge \diamondminus_i\alpha_3$	$\supset (\diamondminus_i\alpha_1 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3)) \vee$ $(\diamondminus_i\alpha_1 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_2))$	(PC, A4)
(2)	$\diamondminus_i\alpha_1 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3)$	$\supset \diamondminus_i(\alpha_1 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3)) \vee$ $\diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3 \wedge \diamondminus_i\alpha_1)$	(A4)
(3)	$\diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3 \wedge \diamondminus_i\alpha_1)$	$\supset \diamondminus_i(\alpha_2 \wedge \diamondminus_i(\alpha_1 \wedge \diamondminus_i\alpha_3)) \vee$ $\diamondminus_i(\alpha_2 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_1))$	(A4, DR3.b)
(4)	$\diamondminus_i\alpha_1 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3)$	$\supset \diamondminus_i(\alpha_1 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3)) \vee$ $\diamondminus_i(\alpha_2 \wedge \diamondminus_i(\alpha_1 \wedge \diamondminus_i\alpha_3)) \vee$ $\diamondminus_i(\alpha_2 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_1))$	(2, 3)
(5)	$\diamondminus_i\alpha_1 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_2)$	$\supset \diamondminus_i(\alpha_1 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_2)) \vee$ $\diamondminus_i(\alpha_3 \wedge \diamondminus_i(\alpha_1 \wedge \diamondminus_i\alpha_2)) \vee$ $\diamondminus_i(\alpha_3 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_1))$	(Subst. in 4)
(6)	$\diamondminus_i\alpha_1 \wedge \diamondminus_i\alpha_2 \wedge \diamondminus_i\alpha_3$	$\supset \diamondminus_i(\alpha_1 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_3)) \vee$ $\diamondminus_i(\alpha_1 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_2)) \vee$ $\diamondminus_i(\alpha_2 \wedge \diamondminus_i(\alpha_1 \wedge \diamondminus_i\alpha_3)) \vee$ $\diamondminus_i(\alpha_2 \wedge \diamondminus_i(\alpha_3 \wedge \diamondminus_i\alpha_1)) \vee$ $\diamondminus_i(\alpha_3 \wedge \diamondminus_i(\alpha_1 \wedge \diamondminus_i\alpha_2)) \vee$ $\diamondminus_i(\alpha_3 \wedge \diamondminus_i(\alpha_2 \wedge \diamondminus_i\alpha_1))$	(1, 4, 5)