

Notions of Realizable Non-Sequential Processes

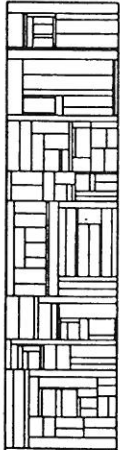
C. Fernandez
M. Nielsen
P.S. Thiagarajan

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Thiagarajan



DATALOGISK AFDELING
Bygning 540 - Ny Munkegade - 8000 Aarhus C
tlf. (06) 12 83 55; telex 66767 aarusci dk
Matematisk Institut Aarhus Universitet

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NOTIONS OF
REALIZABLE NON-SEQUENTIAL PROCESSES

C. Fernandez*

M. Nielsen**

P.S. Thiagarajan**

Abstract

This paper presents some results on non-sequential processes using the language of net theory. The results are concerned with the relationship between various formalizations of the intuition that the causality relation enforced by a process should be in some sense "finitely realizable". The formalizations proposed are of very different flavours, based on notions of observability, approximability, state space covering and discreteness.

* GMD, St. Augustin, W. Germany

** Computer Science Department, Aarhus University, Denmark

0. INTRODUCTION

The aim of this paper is to present some results concerning non-sequential processes using the language of net theory. In this framework a process will consist of partially-ordered holdings of conditions and occurrences of events. The ordering relation associated with a process is to be interpreted as the causality (dependency) relation. One question that then arises is: What restrictions should be placed on such an ordering relation?

Here we wish to concentrate on those restrictions which try to capture the intuition that the causality relation enforced by a process should be in some sense "finitely realizable". For instance, it seems counter-intuitive to admit processes in which the occurrence of an event needs to be preceded by the occurrence of an infinite chain of other events; it would take "for ever" for such an event to occur. (An implicit assumption here is that the focus of interest is "discrete processes" and not "analog processes".)

A number of proposals with very different flavours and origins have been made so far to formalise the intuitive demand that the causality relation associated with a non-sequential process should be in some sense finite. The main thrust of the paper is to argue that these different proposals essentially lead to the same class of objects.

In the next section we introduce the basic terminology and notations. In section 2 we show that the notion of observability due to Winskel [W] is equivalent to the notion of a full state space. Informally, a process is observable if it is possible to give a sequential description of the process; in other words, if it is possible to assign an integer "time" point to every element of the process so that the assignment respects the causal ordering. A process is said to have a full state space if there is a marking starting from which the standard token game (associated with marked nets) can lead to the occurrence of every event in the processes.

In section 3 we consider a proposal due to Goltz and Reisig [GR]. They suggest the restriction that one should consider only those processes that can be "built up" using just finite processes (i.e. processes which have only a finite number of elements). We show that a process is observable iff it can be built-up using chain-bounded processes, i.e. the building blocks are processes for which there is a uniform upper bound on the lengths of chains. We then provide two characterisations of processes that can be approximated by finite processes.

Next we turn to an attractive density property called b-discreteness (bounded-discreteness) proposed first - as far as we know - by Gordon Plotkin. This property states that for every two elements of the process there is a finite upper bound on the lengths of the chains between the two elements. In section 4 we obtain two characterisations of b-discrete processes. Our results show that the essential difference between b-discreteness and observability is countability; and this puts the finishing touch to the nice result due to Winskel [W] which states that for countable processes b-discreteness and observability are equivalent notions.

Since it seems reasonable to consider only those processes that have a countable number of elements, we are then led to conclude that the various proposed restrictions considered in the paper basically give rise to the same class of processes. In the final section we give a more complete summary of results and related work.

1. NOTATIONS AND TERMINOLOGY

In this section we collect together some of the notions of net theory that will be used throughout the paper. In doing so we shall also briefly motivate the net theoretic model of non-sequential processes that will provide the basis for our study. We start with the concept of a net.

Definition 1.1 A (directed) net is a triple $N = (S, T; F)$ satisfying

$$(i) \quad S \cap T = \emptyset \text{ and } S \cup T \neq \emptyset.$$

$$(ii) \quad F \subseteq (S \times T) \cup (T \times S) \text{ so that}$$

$$\text{dom}(F) \cup \text{ran}(F) = S \cup T \text{ where}$$

$$\text{dom}(F) = \{x \in S \cup T \mid \exists y \in S \cup T: (x, y) \in F\}$$

$$\text{ran}(F) = \{y \in S \cup T \mid \exists x \in S \cup T: (x, y) \in F\} \quad \square$$

S is the set of S-elements, T is the set of T-elements and F is the flow relation. $S \cup T$ is the set of elements of N .

In diagrams, the S-elements will be drawn as circles and the T-elements as boxes. If $(x, y) \in F$, we indicate this by drawing a directed arc from x to y . Nets can be used to represent the structure of distributed systems and processes. In such applications the S-elements will denote the local atomic states, the T-elements the local atomic changes-of-states (transitions) and the flow relation will capture the neighbourhood relationship between the local states and transitions. This will become clear in the sequel once we introduce the notions of markings and steps. For now we wish to introduce a very useful notation for representing the neighbourhoods of the elements of a net. Let $N = (S, T; F)$ be a net and $x \in S \cup T$. Then

$$\cdot x = \{y \in S \cup T \mid (y, x) \in F\} - \text{The } \underline{\text{pre-set}} \text{ of } x$$

$$x \cdot = \{y \in S \cup T \mid (x, y) \in F\} - \text{The } \underline{\text{post-set}} \text{ of } x$$

In net theory, non-sequential processes are modelled by a special kind of nets called occurrence nets. Before introducing occurrence nets, it will be convenient to agree on the following conventions.

Through the rest of the paper we let \mathbb{N} denote the set of natural numbers, \mathbb{N}_0 the set of non-negative integers and \mathbb{Z} the set of integers. For the binary relation $R \subseteq Y \times Y$ where Y is a set, we let R^+ denote the transitive closure of R and R^* the reflexive transitive closure of R . Finally $|Y|$ will denote the cardinality of the set Y .

Definition 1.2 An occurrence net is a net $N = (B, E; F)$ such that

$$(i) \quad \forall b \in B: |\cdot b|, |b \cdot| \leq 1.$$

$$(ii) \quad \forall x, y \in B \cup E: (x, y) \in F^+ \Rightarrow (y, x) \notin F^+.$$

$$(iii) \quad \forall e \in E: \cdot e \neq \emptyset \text{ and } e \cdot \neq \emptyset. \quad \square$$

In the literature an occurrence net is often required to satisfy just the clauses (i) and (ii). Here we have thrown in (iii) for technical convenience.

Let $N = (B, E; F)$ be an occurrence net. Then B is the set of conditions, E is the set of events and F is the flow relation of N . Moreover $X_N = B \cup E$ is the set of elements of N .

A simple - but for our purposes crucial - observation is this:

With the occurrence net $N = (B, E; F)$ we can associate the poset $PO_N = (X_N; \leq_N)$ where $\leq_N \stackrel{\text{def}}{=} F^*$. It is this feature of occurrence nets that make them a candidate for modelling non-sequential processes. Now for some terminology concerning posets.

Definition 1.3 Let $PO: (X; \leq)$ be a poset and s, l two non-empty sub-sets of X .

(i) $\underline{li} \stackrel{\text{def}}{=} \{(x, y) \in X \times X \mid x \leq y \text{ or } y \leq x\}$.

(ii) l is a chain (li-set) iff $\forall x, y \in l: x \underline{li} y$
 $|l|$ is the length of l .

Let l be a chain and $x, y \in X$.

Then l is said to be a chain from x to y iff
 $(\forall x' \in l) [x \leq x' \leq y]$.

(iii) $\underline{co} = \{(x, y) \in X \times X \mid x \not\leq y \text{ and } y \not\leq x\}$

(iv) s is a co-set (anti-chain) iff $\forall x, y \in s: x \underline{co} y$.

(v) s is a slice iff it is a maximal co-set. In other words, s is a co-set and $(\forall x \in X - s) (\exists y \in s) [x < y \text{ or } y < x]$.

(vi) For $A \subseteq X$,

$\downarrow A \stackrel{\text{def}}{=} \{x \in X \mid \exists a \in A: x \leq a\}$

$\uparrow A \stackrel{\text{def}}{=} \{x \in X \mid \exists a \in A: a \leq x\}$.

If $A = \{a\}$ is a singleton, we shall write $\downarrow a$ ($\uparrow a$) instead of $\downarrow\{a\}$ ($\uparrow\{a\}$). □

In this paper we shall assume the axiom of choice. In particular we shall assume that every co-set of a poset can be extended to a slice.

Let $N = (B, E; F)$ be an occurrence net and $PO_N = (X_N; \leq_N)$ the associated poset. Then SL_N will denote the set of slices of PO_N . Where N is clear from the context, we will often write SL instead of SL_N . Here is an example of an occurrence net where we have shown some of the slices with the help of dashed lines passing through the elements contained in a slice.

Fig. 1.1

In order to interpret an occurrence net as a model of the underlying (causality) structure of a non-sequential process, we need to introduce a restricted kind of markings and an associated token game.

Definition 1.4 Let $N = (B, E; F)$ be an occurrence net. Then a permissible marking of N is a function $M: B \rightarrow \{0, 1\}$ such that $s_M = \{b \in B \mid M(b) = 1\}$ is a slice of N . \square

In what follows we will deal with only permissible markings. Hence for brevity we will drop the qualifying adjective 'permissible' and just talk about markings. Let $N = (B, E; F)$ be an occurrence net, M a marking of N . Since $M: B \rightarrow \{0, 1\}$ it will be convenient to identify M with the set of conditions that hold at M , i.e. we shall identify $\{b \in B \mid M(b) = 1\}$ with M .

Keeping this convention in mind, we can now introduce one of the central ideas of this paper.

Definition 1.5 Let $N = (B, E; F)$ be an occurrence net and M a marking of N .

(i) Let $\emptyset \neq u \subseteq E$. Then u is enabled at M (u is a step at M) iff $(\forall e \in u) [\cdot e \subseteq M]$.

$M[u \rangle$ will denote the fact that u is enabled at M .

If $u = \{e\}$ is a singleton we write $M[e \rangle$ instead of $M[\{e\} \rangle$.

(ii) Suppose $\emptyset \neq u \subseteq E$ is enabled at M .

Then events in u can occur concurrently at M to lead to the marking M' given by:

$$M' = (M - \cdot u) \cup u' \text{ where } \cdot u = \bigcup_{e \in u} \cdot e \text{ and } u' = \bigcup_{e \in u} e'$$

The transformation of M into M' by the occurrence of the step u (i.e. by the concurrent occurrences of the events in u) at M will be denoted as $M[u \rangle M'$.

If $u = \{e\}$ is a singleton, we write $M[e \rangle M'$. By convention, $M[\emptyset \rangle M$. □

Thus for an event e to occur at a marking M , its pre-conditions ($\cdot e$) must hold. Since M is a slice, if e can occur at M , none of the post-conditions of e will hold at M . When e occurs at M , its pre-conditions cease to hold and its post-conditions begin to hold.

In order to establish our main results in a fairly general setting, we will include both the past and the future in defining the state space generated by a marking.

Definition 1.6 Let M be a marking of the occurrence net $N = (B, E; F)$. Then $[M]$, the state space generated by M is the least set of markings of N given by:

- (i) $M \in [M]$
- (ii) If $M' \in [M]$, $u \subseteq E$ and M'' is a marking of N such that $M'[u > M''$ or $M''[u > M'$ then $M'' \in [M]$. \square

Markings of an occurrence net can be (causally) ordered as follows.

Let M' and M'' be two markings of the occurrence net $N = (B, E; F)$. Then $M' \sqsubseteq M''$ iff $\uparrow M'' \subseteq \uparrow M'$ (or equivalently $\downarrow M' \subseteq \downarrow M''$). For the marking M of N , consider $M' \in [M]$. If $M' \sqsubseteq M$ then M' lies in the "past" of M , i.e. the state M' must have preceded M . If $M \sqsubseteq M'$ then M' lies in the "future" of M ; M must precede M' in the process modelled by the quadruple (B, E, F, M) . It is also possible, that two markings in $[M]$ are incomparable. Consequently the state space $[M]$ is partially ordered under \sqsubseteq . This is the justification for viewing (B, E, F, M) as a model of a non-sequential process.

The question addressed in this paper is: What is the class of occurrence nets that should be chosen to serve as the underlying nets of a non-sequential process?

That we must exclude some occurrence nets can be brought out through an example.

Fig. 1.2

The trouble with this net is that for the "natural" initial marking M^0 shown, the event e will not be enabled at any marking in the state space $[M^0]$. Indeed there is no marking M for this occurrence net such that every event is enabled to occur at some marking in $[M]$. The point is that in some sense, the underlying

causal ordering \leq_N of this net is not "realizable". Our aim is to consider various proposals for evaluating the "goodness" of the ordering relation \leq_N . The first one is in terms of state spaces. We propose that one should consider only those occurrence nets that have a full state space.

Definition 1.7 Let $N = (B, E; F)$ be an occurrence net.

- (i) Let M be a marking of N . Then $[M]$ is said to be a full state space of N iff $(\forall e \in E) (\exists M' \in [M]) [e \text{ is enabled at } M'; M'[e >]]$.
- (ii) N has a full state space iff for some marking M of N , $[M]$ is a full state space. □

The second proposal is to consider only those occurrence nets that are observable.

Definition 1.8 Let $N = (B, E; F)$ be an occurrence net and $PO_N = (X_N; \leq_N)$ the associated poset.

- (i) An observer of N is a function

$$O : X_N \rightarrow Z \text{ such that}$$

$$(\forall x, y \in X_N) [x <_N y \Rightarrow O(x) < O(y)]$$

- (ii) N is observable iff it has an observer. □

It is not difficult to verify that the net shown in fig. 1.2 is not observable. Indeed our first task will be to show that the notions of full state space and observability coincide.

2. THE EQUIVALENCE OF OBSERVABILITY AND FULL STATE SPACE

In order to establish that the notions of observability and full state space coincide, it will be convenient to rework the notion of full state space. The intuitive idea is to place tokens on the arcs of an occurrence net and play a token game in which both the conditions and events have occurrences.

Formally, let $N = (B, E, F)$ be an occurrence net and $s \subseteq X_N$ a slice of N . Then

$$\text{act}^+(s) = \{x \in s \mid x^* \neq \emptyset \wedge \cdot(x^*) \subseteq s\}$$

$$\text{act}^-(s) = \{x \in s \mid \cdot x \neq \emptyset \wedge (\cdot x)^* \subseteq s\}.$$

Consider the following occurrence net. For the slice s shown, $\text{act}^+(s) = \{e\}$ and $\text{act}^-(s) = \{b_1, b_2, e\}$

Fig. 2.1

Note that we can replace e (in s) by its post-condition to obtain a new slice; we can also replace both b_1 and b_2 (in s) by e' to obtain a new slice. In other words, $\text{act}^+(s)$ and $\text{act}^-(s)$, the set of forward active and backward active elements of a slice, can be used to define a transition relation over slices.

Definition 2.1 Let $s, s' \in SL_N$ for the occurrence net $N = (B, E, F)$. Then $s \rightarrow s'$ iff there exists $Y \subseteq \text{act}^+(s)$ such that $s' = (s - \cdot(Y\cdot)) \cup Y\cdot$. \square

The following facts are easy to verify.

Theorem 2.2 Let $N = (B, E, F)$ be an occurrence net.

- (i) Suppose s is a slice and $Y_1 \subseteq \text{act}^+(s)$ and $Y_2 \subseteq \text{act}^-(s)$. Then both

$$(s - \cdot(Y_1\cdot)) \cup Y_1\cdot \text{ and } \\ (s - (\cdot Y_2)\cdot) \cup \cdot Y_2 \text{ are slices.}$$

- (ii) Let s and s' be a pair of slices. Then $s \rightarrow s'$ iff there exists $Y \subseteq \text{act}^-(s')$ such that $s = (s' - (\cdot Y)\cdot) \cup \cdot Y$. \square

The modified notion of a full state space can now be stated.

Definition 2.3 Let $N = (B, E, F)$ be an occurrence net and s a slice.

- (i) Then $\llbracket s \rrbracket$, the slice space generated by s is the least sub-set of SL_N given by

a) $s \in \llbracket s \rrbracket$

b) If $s' \in \llbracket s \rrbracket$ and $s'' \in SL_N$ such that $s' \rightarrow s''$ or $s'' \rightarrow s'$ then $s'' \in \llbracket s \rrbracket$.

- (ii) $\llbracket s \rrbracket$ is a full slice space iff $\cup \{s' \in SL_N \mid s' \in \llbracket s \rrbracket\} = X_N$.

(iii) N has a full slice space iff there exists a slice $s \in SL_N$ such that $\llbracket s \rrbracket$ is full. □

Remark It will be useful to observe that $s' \in \llbracket s \rrbracket$ is equivalent to saying that there exists a sequence of slices $s_0 s_1 \dots s_n$ such that $s = s_0$ and $s_n = s'$ and for $0 \leq i < n$, $s_i \rightarrow s_{i+1}$ or $s_{i+1} \rightarrow s_i$.

Theorem 2.4 The occurrence net $N = (B, E, F)$ has a full state space iff it has a full slice space.

Proof

\Rightarrow Assume that M is a marking of N such that $[M]$ is a full state space. By definition of a (permissible) marking, M is a slice. We shall argue that $\llbracket M \rrbracket$ is a full slice space.

Let $x \in X_N$. We must show that for some $s \in \llbracket M \rrbracket$, $x \in s$. First assume that x is an event. (Using the fact that $\cdot x \cup x \neq \emptyset$ because N is a net, we will later dispose off the case where $x \in B$). Since $[M]$ is a full state space, we can find a sequence of markings $M_0, M_1, \dots, M_n \in [M]$ and a sequence of steps $u_0, u_1, \dots, u_{n-1} \subseteq E$ such that $M = M_0$, for $0 \leq i < n$, $M_i[u_i > M_{i+1}$ or $M_{i+1}[u_i > M_i$ and $M_n[x >$ (i.e. x is enabled at M_n). We shall first prove by induction on n that $M_n \in \llbracket M \rrbracket$.

$n = 0$ Trivial.

$n > 0$ By the induction hypothesis $M_{n-1} \in \llbracket M \rrbracket$. Assume that $M_{n-1}[u_{n-1} > M_n$. (The proof in case $M_n[u_{n-1} > M_{n-1}$ is completely symmetric and we shall omit it.)

Then $\cdot u_{n-1} \subseteq \text{act}^+(M_{n-1})$ and therefore

$s_n = (M_{n-1} - \cdot u_{n-1}) \cup u_{n-1}$ is a slice. Moreover $M_{n-1} \rightarrow s_n$.

But then $u_{n-1} \subseteq \text{act}^+(s_n)$ and $M_n = (s_n - u_{n-1}) \cup u_{n-1}^*$ so that $s_n \rightarrow M_n$. From $M_{n-1} \in \llbracket M \rrbracket$ it now follows at once that $M_n \in \llbracket M \rrbracket$.

To complete the argument that $\llbracket M \rrbracket$ is a full slice space, recall that $M_n[x\rangle$. Then $\cdot x \subseteq \text{act}^+(M_n)$ so that $s = (M_n - \cdot x) \cup \{x\}$ is a slice with $M_n \rightarrow s$. Hence we have found $s \in \llbracket M \rrbracket$ with $x \in s$.

In case $x \in B$, we can pick an event $x' \in \cdot x \cup x'$ and apply the above argument to obtain $s' \in \llbracket M \rrbracket$ such that $x' \in s'$. If $x' \in \cdot x$, then $s = (s' - \{x'\}) \cup (x')^*$ will be a slice satisfying $x \in s$ and $s' \rightarrow s$. If on the other hand $x' \in x'$, then $s = (s' - \{x'\}) \cup \cdot(x')$ will be a slice satisfying $x \in s$ and $s \rightarrow s'$. In either case $s \in \llbracket M \rrbracket$.

\Leftarrow Let s be a slice such that $\llbracket s \rrbracket$ is full. Let $u = s \cap E$ and $M = (s - u) \cup \cdot u$. Clearly M is a (permissible) marking and viewed as a slice, we have $M \rightarrow s$. We claim that $\llbracket M \rrbracket$ is a full state space. To see this, let $e \in E$. Then from the fact that $\llbracket s \rrbracket$ is a full slice space it follows that there exists a sequence of slices s_0, s_1, \dots, s_n such that $s = s_0$, for $0 \leq i < n$ $s_i \rightarrow s_{i+1}$ or $s_{i+1} \rightarrow s_i$ and $e \in s_n$.

Again by induction on n it is easy to prove that for some $M' \in \llbracket M \rrbracket$, e is enabled at M' . □

Remark In what follows we shall write $[s]$ instead of $\llbracket s \rrbracket$ for the sake of convenience.

To prove the main result of this section we start with the notion of the distance of a net element from a co-set.

Definition 2.5 Let s be a co-set of the occurrence net $N = (B, E; F)$ and $x \in X$. Then

$$d_F(x, s) = \begin{cases} \text{Sup}\{n \mid y \in s \wedge yF^n x\}, & \text{if } x \in \uparrow s \\ \text{Sup}\{n \mid y \in s \wedge xF^n y\}, & \text{if } x \in \downarrow s \\ 0, & \text{otherwise} \end{cases} \quad \square$$

Remark Since s is a co-set and F^0 is the identity relation over X_N by convention, we have that $d_F: X_N \rightarrow \mathbb{N}_0 \cup \{\infty\}$. (The usual ordering over \mathbb{N}_0 denoted as \leq is extended to $\mathbb{N}_0 \cup \{\infty\}$ in the obvious way.) For the sake of convenience, we will also often write \leq instead of \leq_N in dealing with the ordering relationship associated with the occurrence net N . From the context it should be clear as to which ordering relation is meant. We will also often write X instead of X_N . Finally, we let $|k|$ denote the absolute value of the integer k .

Lemma 2.6 Let s be a slice of the occurrence net $N = (B, E; F)$. Then $[s]$ is a full slice space of N iff $\forall x \in X: d_F(x, s) \in \mathbb{N}_0$.

Proof

\Rightarrow Let $x \in X$. Since $[s]$ is full, there exists a sequence $s = s_0, s_1, \dots, s_k$ such that for $0 \leq i < k$, $s_i \rightarrow s_{i+1}$ or $s_{i+1} \rightarrow s_i$ and $x \in s_k$. We wish to show that $d_F(x, s) \leq k$. To this end assume that for some $y \in s_0$, $yF^n x$ so that it is sufficient to prove that $n \leq k$. (The proof for the case $xF^n y$ is completely symmetrical and we shall omit it.)

Since $yF^n x$ we have $y = x_0 F x_1 F x_2 \dots F x_n = x$ where for $0 \leq i \leq n$, $x_i \in X$. As a first step we shall show that each x_i belongs to at least one of the slices s_0, s_1, \dots, s_k .

Suppose that there exists $x' \in \{x_0, x_1, \dots, x_n\}$ such that $x' \notin s_j$ for $0 \leq j \leq k$. We claim that for $0 \leq j \leq k$, there exists $z_j \in s_j$ such that $z_j < x'$. The claim can be established

by induction on j . First note that $x' \neq x_0$ because $y = x_0 \in s_0$.

$j = 0$ Then $z_0 = y \in s_0$ and we know that $y = x_0 < x'$.

$j > 0$ By the induction hypothesis, there exists $z_{j-1} \in s_{j-1}$ such that $z_{j-1} < x'$. If $z_{j-1} \in s_j$ then we are done. So suppose $z_{j-1} \notin s_j$.

Then either $\emptyset \neq (z_{j-1})^\circ \subseteq s_j$ or $\emptyset \neq \cdot(z_{j-1}) \subseteq s_j$. If $\emptyset \neq \cdot(z_{j-1}) \subseteq s_j$ then there exists $z_j \in s_j$ such that $z_j \mathbb{F} z_{j-1}$ which implies that $z_j < x'$ because $z_{j-1} < x'$ by the induction hypothesis.

So assume that $\emptyset \neq (z_{j-1})^\circ \subseteq s_j$. Since we are assuming that $x' \notin s_j$, we are assured that $x' \notin (z_{j-1})^\circ$. But then $\leq_{\mathbb{N}} = \mathbb{F}^*$ and $z_{j-1} <_{\mathbb{N}} x'$ together guarantee that for some $z_j \in (z_{j-1})^\circ$, $z_j < x'$. But then $z_j \in s_j$ and we are done.

Since the claim is now proved we have that for some $z_k \in s_k$, $z_k < x'$. Recall that $x' \in \{x_1, \dots, x_n = x\}$. Since $x' \leq x_n = x$ we have that $z_k < x$ which is a contradiction because both z_k and x are in the slice s_k .

Hence each x_i ($0 \leq i \leq n$) belongs to one of the slices s_0, s_1, \dots, s_k . If $n > k$ then at least two different elements in $\{x_0, x_1, \dots, x_n\}$ would belong to the same slice which is of course a contradiction because $x_0 \mathbb{F} x_1 \mathbb{F} x_2 \dots \mathbb{F} x_n$. Therefore $n \leq k$.

\Leftarrow We want to prove the following:

$\forall x \in X$, there exists a sequence $s = s_0, s_1, \dots, s_k$ such that $x \in s_k$ and $s_i \rightarrow s_{i+1}$ or $s_{i+1} \rightarrow s_i$ for all $0 \leq i < k$.

The proof is by induction on $d_{\mathbb{F}}(x, s) = n$.

$n = 0$ $d_{\mathbb{F}}(x, s) = 0$ implies $x \in s$ and the result follows immediately.

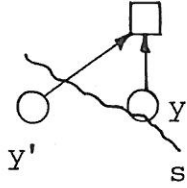
$n > 0$ We only consider the case $x \in \uparrow s$ (the proof for the case $x \in \downarrow s$ is similar and we omit it).

Define $a \subseteq s$ by

$$a = \{y \in s \mid yF^n x\}$$

a is by definition a non-empty subset of s , and we claim that $a \subseteq \text{act}^+(s)$.

Assume $y \in a$ and $\cdot(y') \not\subseteq s$. From the definition of occurrence nets $y \in B$ and there exists another B-element $y' \in \cdot(y')$ not in s .



$y' \notin s$ implies $\exists z \in s$ such that $y' < z$ or $z < y'$.

If $y' < z$ then $y < z$, a contradiction since $y, z \in s$.

If $z < y'$ then $zF^m x$ with $m > n$, which contradicts

the fact that $d_F(x, s) = n$.

So, $a \subseteq \text{act}^+(s)$. Define $s_1 = (s - \cdot(a')) \cup a'$.

We then have $s \rightarrow s_1$ and $d(x, s_1) < n$. The result now follows from the induction hypothesis. \square

We come now, with the next three lemmas, to the main theorem of this section, that is: that the class of occurrence nets with full state (slice) spaces coincides with the class of observable occurrence nets. We start with

Definition 2.7 Let $N = (B, E; F)$ be an occurrence net, $s \in SL$ and \hat{O} an observer for N . \hat{O} is synchronized on s iff $s = \{x \in X \mid \hat{O}(x) = 0\}$.

Lemma 2.8 Let $N = (B, E; F)$ be an occurrence net with $([s])$ as a full slice space. Then N has an observer which is synchronized on s .

Proof Define $O: N \rightarrow Z$ as follows.

$$\forall x \in X : O(x) = \begin{cases} d_F(x, s) & \text{if } x \in \uparrow s \\ -d_F(x, s) & \text{if } x \in \downarrow s \end{cases}$$

O is a well defined mapping by Lemma 2.6. From the definition of d_F it follows that O is an observer which is synchronized on s . \square

Lemma 2.9 Let $N = (B, E; F)$ be an observable occurrence net. Then N has an observer which is synchronized on some slice of N .

Proof Let O be an observer for N . We first modify O as follows. Define:

$$\forall x \in X \quad \eta(x) = \begin{cases} 2 \times O(x) & \text{if } O(x) \geq 0 \\ 2 \times O(x) + 1 & \text{if } O(x) < 0. \end{cases}$$

η is clearly an observer for N , with the property that all positive values are even and all negative values odd. From this it follows that all sets $\{x \in X \mid |\eta(x)| = i\}$ with $i \geq 0$ are co-sets. This property is used in the following. Define inductively

$$X_0 = \{x \in X \mid \eta(x) = 0\} \quad \text{and for } i > 0,$$

$$X_i = \{x \in X \mid |\eta(x)| = i \wedge x \text{ co } X_j \quad 0 \leq j < i\}$$

(where $x \text{ co } X_j$ iff $\forall y \in X_j: x \text{ co } y$.)

We claim that $s = \bigcup_{i \geq 0} X_i$ is a slice.

From the definitions s is clearly a co-set. We have to prove that it is a maximal co-set.

Assume $x \notin s$. For some i , $|\eta(x)| = i$ and $x \notin X_i$ which means that there exists $y \in X_j \subseteq s$ $j < i$ such that $x \text{ li } y$. Hence s is a maximal co-set; a slice.

Define now:

$$\tau(x) = \begin{cases} 0 & \text{if } x \in s \\ \eta(x) & \text{otherwise} \end{cases}$$

We claim that τ is an observer which by definition is synchronized on s .

Let $x < y$, we know that $\eta(x) < \eta(y)$ and we want to prove that $\tau(x) < \tau(y)$.

If $\eta(x) = \tau(x)$ and $\eta(y) = \tau(y)$ then the result is obviously true.

If $\eta(x) \neq \tau(x)$ and $\eta(y) \neq \tau(y)$ it follows that

$\tau(x) = \tau(y)$ which implies $x, y \in s$ a contradiction since $x < y$.

So, the only possible cases in which "things could go wrong" are:

1) $\tau(x) \neq \eta(x) < \eta(y) = \tau(y)$

From $\tau(x) \neq \eta(x)$, it follows that $x \in s$. Since $x < y$, we then have $y \notin s$ and hence $\tau(y) \neq 0$. This implies that $\eta(y) \neq 0$.

Suppose that $\eta(y) > 0$. We then at once have $\tau(x) < \tau(y)$ because $\tau(x) = 0$ by virtue of $x \in s$ and $\tau(y) = \eta(y)$.

Suppose that $\eta(y) < 0$. Since $y \notin s$, we must have for some $z \in X_j$, $y < z$ or $z < y$ where $0 \leq j < |\eta(y)|$. We claim that $z < y$ is impossible. This is because 0 being an observer, $z < y$ would imply that $\eta(z) < \eta(y)$ which in turn would imply that $|\eta(y)| < |\eta(z)|$. But we know that $|\eta(z)| = j$ and $j < |\eta(y)|$. Hence it must be the case that $y < z$. From $x < y$ we would then have $x < z$ which is a contradiction because both x and z are supposed to be members of the slice s . Hence $\eta(y) < 0$ is impossible.

2) $\tau(x) = \eta(x) < \eta(y) \neq \tau(y)$

As before $y \in s$ and $\tau(y) = 0$ follow from $\eta(y) \neq \tau(y)$. $x \notin s$ follows from $x < y$. Hence $\tau(x) = \eta(x) \neq 0$. If $\eta(x) < 0$ we have at once $\eta(x) = \tau(x) < \tau(y) = 0$.

We can rule out $\eta(x) > 0$ as follows. Since $\eta(x) = \tau(x)$ we have that $z < x$ or $x < z$ for some $z \in X_j$ with $0 \leq j < |\eta(x)|$. $x < z$ would imply that $\eta(x) < \eta(z)$ which in turn would imply that $|\eta(x)| < |\eta(z)|$ which is ruled out by $|\eta(z)| = j$ and $j < |\eta(x)|$. But then $z < x$ is also not possible because this would lead to $z < y$ and we know that $z, y \in s$. \square

Lemma 2.10 Let $N = (B, E; F)$ be an occurrence net with an observer which is synchronized on a slice s . Then $[s]$ is a full slice space.

Proof Let O be an observer which is synchronized on s . It follows from the definitions that

$$\forall x \in X: d_F(x, s) \leq |O(x)|$$

The desired result now follows immediately from Lemma 2.6. \square

To sum up, we have:

Theorem 2.11 Let $N = (B, E; F)$ be an occurrence net. N is observable iff it has a full slice (state) space. \square

3. CHAIN-BOUNDED APPROXIMATIONS

In this section we consider another way of formalising the intuition that the causality relation associated with an occurrence net ought to be finitely realizable. The proposal is to admit only those occurrence nets that can be "built up" using occurrence nets of "finite length". This is a generalisation of the proposal made by U. Goltz and W. Reisig [GR] to the effect that one should consider only those occurrence nets that can be built up using finite occurrence nets. We will discuss the implications of this stronger restriction towards the end of this section.

Occurrence nets of "finite length" can be formalised as follows.

Definition 3.1 Let $N = (B, E; F)$ be an occurrence net. N is chain-bounded iff there exists an integer $k_N \in \mathbb{N}_0$ such that for every chain $l \subseteq X$, $|l| \leq k_N$.

Definition 3.2 Let $N_1 = (B_1, E_1; F_1)$ and $N_2 = (B_2, E_2; F_2)$ be a pair of occurrence nets with $X_1 = B_1 \cup E_1$, $X_2 = B_2 \cup E_2$, $\leq_1 = F_1^*$ and $\leq_2 = F_2^*$.

- i) N_1 is a subnet of N_2 , denoted $N_1 \subseteq N_2$, iff $B_1 \subseteq B_2$, $E_1 \subseteq E_2$ and $F_1 = F_2 \cap [(B_1 \times E_1) \cup (E_1 \times B_1)]$.
- ii) N_1 is a convex subnet of N_2 iff $N_1 \subseteq N_2$ and $(\forall x, y \in X_1)$
 $(\forall z \in X_2): [x \leq_2 z \leq_2 y \Rightarrow z \in X_1]$.

Our next goal is to characterise the class of observable occurrence nets as the class of those occurrence nets which can be approximated by chain-bounded occurrence nets in the following technical sense.

Definition 3.3 Let $N = (B, E; F)$ be an occurrence net. N is said to be approximated by the sequence of occurrence nets N_1, N_2, \dots iff

- i) Each N_i is a convex subnet of N_{i+1}
- ii) $N = \left(\bigcup_{i \geq 1} B_i, \bigcup_{i \geq 1} E_i; \bigcup_{i \geq 1} F_i \right)$ □

Lemma 3.4 Let $N = (B, E; F)$ be an occurrence net. If N is approximated by a sequence of chain-bounded occurrence nets N_1, N_2, \dots , then N is observable.

Proof The idea is to define an observer for N_1 and then inductively one for N_i and finally construct an observer for N .

First, note that if N is chain-bounded then the set of maximal elements, $\max N$ of N , constitutes a slice, with the property that:

$$\forall x \in X: d_F(x, \max N) \leq k$$

where $k \in \mathbb{N}_0$ is the bound of all chain lengths.

Assume that all chain lengths of N_i are bounded by k_i , $1 \leq i$. Assume that $N_i = (B_i, E_i; F_i)$ and $X_i = B_i \cup E_i$ for $i \geq 1$. For N_1 define:

$$\forall x \in X_1 \quad O_1(x) = k_1 - d_{F_1}(x, \max N_1)$$

It is not difficult to verify that O_1 is an observer for N_1 with the property that $|O_1(x)| \leq k_1$ for all $x \in X_1$.

We now define an observer for N_i inductively. We begin with a "partial" observer τ_i for N_i .

$$\forall i > 1: \quad \forall x \in X_i \quad \tau_i(x) = \begin{cases} O_{i-1}(x) & \text{if } x \in X_{i-1} \\ \sum_{j=1}^i k_j & \text{if } x \in \max N_i \setminus X_{i-1} \\ \text{undefined,} & \text{otherwise} \end{cases}$$

Intuitively, O_{i-1} is just extended with some large value for possibly new maximal elements in X_i (part of the induction hypothesis is that $|O_{i-1}(x)| \leq \sum_{j=1}^{i-1} k_j$).

Now we extend τ_i to what will turn out to be an observer for N_i .

$$\forall x \in X_i \quad O_i(x) = \begin{cases} \tau_i(x), & \text{if } \tau_i(x) \text{ is defined} \\ \min\{\tau_i(y) - n \mid xF^n y \text{ and } \tau_i(y) \text{ is defined}\}, & \\ \text{otherwise} & \end{cases}$$

First of all it follows from the fact that $\max N_i$ is a slice and that N_i is chain-bounded that $O_i: X_i \rightarrow Z$ is a well defined mapping. Furthermore from the hypothesis $|O_{i-1}(x)| \leq \sum_{j=1}^{i-1} k_j$ it follows $|O_i(x)| \leq \sum_{j=1}^i k_j$.

Now let $x, y \in X_i$ with $x < y$, we have to prove that $O_i(x) < O_i(y)$. If $\tau_i(x)$ is undefined the desired property follows immediately from definition.

If $\tau_i(x)$ is defined then $x \in X_{i-1}$, if also $y \in X_{i-1}$, the property follows for the fact that O_{i-1} is an observer for N_{i-1} .

If $x \in X_{i-1}$ and $y \notin X_{i-1}$, then it follows from the fact that N_{i-1} is a convex subset of N_i that for no $z \in X_{i-1}$, $yF^n z$, and hence from definition $O_i(y) > \sum_{j=0}^{i-1} k_j$. But $x \in X_{i-1}$, and from the induction hypothesis we have that

$$O_i(x) = O_{i-1}(x) \leq \sum_{j=1}^{i-1} k_j \text{ and so } O_i(x) < O_i(y)$$

Finally define $O: X \rightarrow Z$ as

$$\forall x \in X \quad O(x) = O_i(x) \quad \text{where } x \in X_i.$$

It is straightforward from the above to verify that O is an observer for N . □

Lemma 3.5 Let $N = (B, E; F)$ be an occurrence net. If N is observable then N is approximated by a sequence of chain-bounded occurrence nets.

Proof Since N is observable, there is a slice s such that $[s]$ is full. Then from the proof of lemma 2.8 it follows that O defined below is an observer of N which is synchronised on s .

$$\forall x \in X: \quad O(x) = \begin{cases} d_F(x, s), & \text{if } x \in \uparrow s \\ -d_F(x, s), & \text{if } x \in \downarrow s. \end{cases}$$

Now define inductively for $i \geq 1$,

$$\begin{aligned} Y_i &= \{y \in X \mid 0 < |O(y)| \leq i\} \\ Z_i &= \{z \in s \mid \exists y \in Y_i: z \text{ li } y\} \\ X_i &= Y_i \cup Z_i \\ N_i &= (X_i \cap B, X_i \cap E; F|_{X_i}). \end{aligned}$$

We claim that the N_i 's constitute an approximating sequence of chain-bounded occurrence nets. So, we have to prove

Claim 1 Each N_i is a chain-bounded occurrence net.

Claim 2 Each N_i is a convex-sub-net of N_{i+1} .

Claim 3 $\cup X_i = X$.

Proof of claim 1 That each N_i is an occurrence net is merely an observation. The only non-trivial part is to verify that $\text{dom}(F_i) \cup \text{ran}(F_i) = X_i$.

Let $y \in Y_i$ and $|O(y)| = k$. Then $0 < k \leq i$. If $d_F(y, s) > 0$ then there exists $y' \in Y_i \cup Z_i$ such that $y' \in \cdot y$. If $d_F(y, s) < 0$ then there exists $y' \in Y_i \cup Z_i$ such that $y' \in y \cdot$. In any case N_i is a net and hence an occurrence net. The fact that each N_i is chain-bounded follows

from the observation that all 0-values on X_i are numerically bounded by i .

Proof of Claim 2 $N_i \subseteq N_{i+1}$ from definition, so all we have to prove is convexity. It is sufficient to prove that each N_i is a convex subnet of N . So, assume $x, y \in X_i$ and $x < z < y$ for some $z \in X$. If $O(z) = 0$ then $z \in s$ and $z \in Z_i \subseteq X_i$ from definition. If $O(z) \neq 0$ then $|O(z)| < \max(|O(x)|, |O(y)|)$ and hence $z \in Y_i \subseteq X_i$.

Proof of Claim 3 We want to prove that each $x \in X$ is a member of some X_i .
 If $O(x) \neq 0$ then obviously $x \in Y_{|O(x)|} \subseteq X_{|O(x)|}$.
 If $O(x) = 0$ then $x \in s$ and since N is an occurrence net there exists a $y \in X$ such that either $x F y$ or $y F x$ and $O(y) \neq 0$. In both cases $x \in Z_{|O(y)|} \subseteq X_{|O(y)|}$. \square

Summing up the main result of the last section with the last two lemmas, we get:

Theorem 3.6 Let $N = (B, E; F)$ be an occurrence net. The following three characterisations are equivalent:

1. N has a full state (slice) space.
2. N is observable.
3. N is approximated by chain-bounded occurrence nets.

As promised at the beginning of the section we shall now examine the proposal that one should consider only those occurrence nets that can be approximated by finite occurrence nets. It turns out that such occurrence nets are characterised by two properties:

Definition 3.7 Let $N = (B, E; F)$ be an occurrence net.

- (a) N is countable iff X_N is a countable set.
- (b) N is interval-finite iff $\forall x, y \in X_N: |[x, y]| < \infty$ where $[x, y] = \{z \in X_N \mid x \leq z \leq y\}$

A useful concept which will be used in the next theorem and also in the next section is that of convex closure.

Definition 3.8 Let $N = (B, E; F)$ be an occurrence net and $X' \subseteq X = B \cup E$. The convex closure of X' , denoted by $\langle X' \rangle$, is given by $\langle X' \rangle = \{y \in X \mid \exists y_1, y_2 \in X' : y_1 \leq y \leq y_2\}$.

Theorem 3.9 Let $N = (B, E; F)$ be an occurrence net. N can be approximated by finite occurrence nets iff N is countable and interval-finite.

Proof (\Rightarrow) Trivial.

(\Leftarrow) Let x_0, x_1, x_2, \dots be an enumeration of X_N and $g: X_N \rightarrow X_N$ a function which satisfies, $\forall x \in X_N: g(x) \in {}^*x \cup x^*$. Since N is a net, the existence of g is assured. Let $X_0 = \{x_0, g(x_0)\}$ and for $i \geq 0$ define inductively,

$$X_{i+1} = \langle X_i \cup \{x_{i+1}, g(x_{i+1})\} \rangle$$

It is easy to prove that for each $i \geq 0$

$N_i = (X_i \cap B, X_i \cap E, (X_i \times X_i) \cap F)$ is an occurrence net.

We shall first verify that each X_i is finite. Clearly X_0 is finite and so assume that X_i is finite and we shall prove that X_{i+1} is finite.

Setting $X'_i = \{x_{i+1}, g(x_{i+1})\}$ for convenience, we then have

$$X_{i+1} = X_i \cup (\cup \{[x, y] \mid x \in X_i \wedge y \in X'_i\}) \\ \cup (\cup \{[x, y] \mid x \in X'_i \wedge y \in X_i\})$$

X_{i+1} must be finite because X_i is finite and N is interval-finite.

Since $\langle X_i \rangle = X_i$ and $X_i \subset X_{i+1}$ we have at once that for each i , N_i is a convex sub-net of N_{i+1} .

$N = (\bigcup_{i \geq 0} B_i, \bigcup_{i \geq 0} E_i, \bigcup_{i \geq 0} F_i)$ where $N_i = (B_i, E_i; F_i)$, because

X_N is countable. □

We conclude this section with a second characterisation of those occurrence nets that have finite approximations. This characterisation is in terms of injective observers.

$N = (B, E; F)$ is said to have an injective observer iff it has an observer O such that $O: X_N \rightarrow Z$ is an injective function.

Theorem 3.10 The occurrence net $N = (B, E; F)$ can be approximated by finite occurrence nets iff it has an injective observer.

Proof

\Rightarrow Assume that N is approximated by the sequence of occurrence nets N_1, N_2, \dots such that each N_i is finite.

Let $N_i = (B_i, E_i; F_i)$, $X_i = B_i \cup E_i$, $<_i = F_i^+$ and co_i the co-relation associated with N_i for all $i \geq 1$. We start by constructing an injective observer $O_1: X_1 \rightarrow Z$ for N_1 . Let $x_0 \in X_1$ and define

$$\begin{aligned} X_1^0 &= \{x_0\} \\ X_1^- &= \{x \in X_1 \mid x <_1 x_0\} \\ X_1^+ &= X_1 - (X_1^0 \cup X_1^-) \end{aligned}$$

Choose an enumeration y_1, y_2, \dots, y_n of X_1^+ (with $|X_1^+| = n$) such that for $1 \leq l \leq n$ and $1 \leq j \leq n$, $l < j$ implies $y_l co_1 y_j$ or $y_l <_1 y_j$. Choose an enumeration z_1, z_2, \dots, z_m of X_1^- such that for $1 \leq l \leq m$ and $1 \leq j \leq m$, $l < j$ implies $z_j co_1 z_l$ or $z_j <_1 z_l$.

Define $O_1: X_1 \rightarrow Z$ as follows.

$$\begin{aligned} O_1(x_0) &= 0 \\ O_1(y_j) &= j \text{ for } 1 \leq j \leq n \\ O_1(z_j) &= -j \text{ for } 1 \leq j \leq m. \end{aligned}$$

Clearly O_1 is an injective observer for N_1 . The reason for constructing O_1 in this elaborate fashion is to ensure that with $k_1 = |X_1|$, $O_1(x) \in [-k_1, k_1]$ for every $x \in X_1$. It is easy to see that O_1 satisfies this requirement.

Assume inductively that we have an injective observer O_i for N_i such that $\forall x \in X_i: O_i(x) \in [-k_i, k_i]$ where $k_i = |X_i|$ (for $i \geq 1$). Then

$$\begin{aligned} X_{i+1}^0 &= X_i \\ X_{i+1}^- &= \{x \in X_{i+1} - X_i \mid x <_{i+1} x' \text{ for some } x' \in X_i\} \\ X_{i+1}^+ &= X_{i+1} - (X_{i+1}^0 \cup X_{i+1}^-). \end{aligned}$$

Choose an enumeration y_1, y_2, \dots, y_n of X_{i+1}^+ (with $n = |X_{i+1}^+|$) such that $1 \leq l \leq n$ and $1 \leq j \leq n$, $l < j$ implies $y_l \text{ co}_{i+1} y_j$ or $y_l <_{i+1} y_j$.

Choose an enumeration z_1, z_2, \dots, z_m of X_{i+1}^- such that for $1 \leq l \leq m$ and $1 \leq j \leq m$, $l < j$ implies $y_j \text{ co}_{i+1} y_l$ or $y_j <_{i+1} y_l$. Now define $O_{i+1}: X_{i+1} \rightarrow \mathbb{Z}$ as follows.

$$\begin{aligned} O_{i+1}(x) &= O_i(x) && \text{for every } x \in X_i \\ O_{i+1}(y_j) &= k_i + j && \text{for } 1 \leq j \leq n \\ O_{i+1}(z_j) &= -(k_i + j) && \text{for } 1 \leq j \leq m. \end{aligned}$$

It is easy to show that O_{i+1} is an injective observer of N_{i+1} . Moreover $\forall x \in X_{i+1}, O_{i+1}(x) \in [-k_{i+1}, k_{i+1}]$ where $k_{i+1} = |X_{i+1}|$. Finally define $O: X = \bigcup_{i \geq 1} X_i \rightarrow \mathbb{Z}$ as

$$\forall x \in X: O(x) = O_i(x) \quad \text{if } x \in X_i.$$

It is routine to verify that O is an injective observer of N .

⇐ Suppose that O is an injective observer of N . Then N must be countable and interval-finite. By theorem 3.9 we have that N can be approximated by finite occurrence nets. \square

An obvious consequence of the preceding two results is

Corollary 3.11 The occurrence net N has an injective observer iff it is countable and interval-finite. \square

This result has been independently obtained by E. Smith [S] without appealing to the notion of finite approximations. In fact we were motivated to characterise finitely approximable occurrence nets in terms of injective observers by Smith's result.

4. B-DISCRETENESS

In this section we look at an attractive density property called bounded discreteness or b-discreteness for short. An occurrence net is said to have this property, if for any two elements, not only are all chains between them finite (discreteness), but there is also a finite upper bound on the length of such chains (boundedness).

Definition 4.1 An occurrence net N is said to be b-discrete iff for every $x, y \in X_N$ there exists a natural number $d_{x,y}$ such that the length of any chain from x to y is bounded by $d_{x,y}$. □

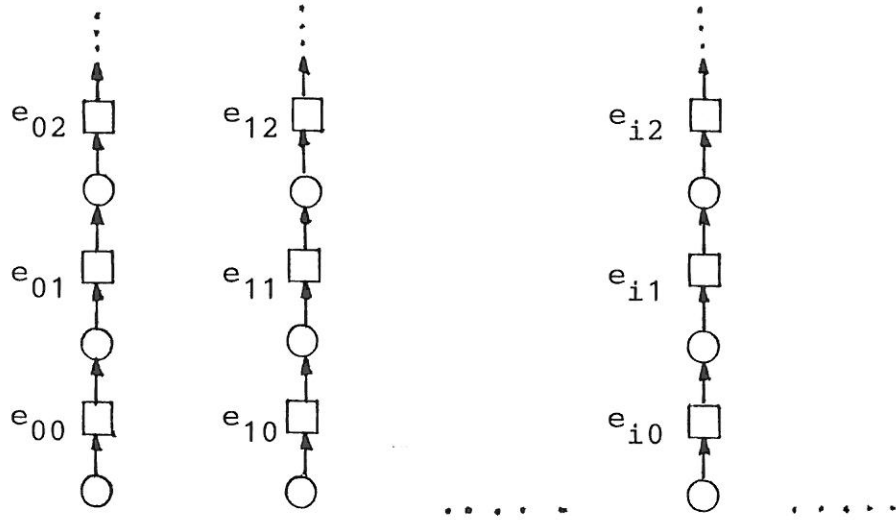
Now, what is the relationship between this density property and the properties we have discussed in the previous sections? All the examples we have considered so far don't bring out any difference between b-discreteness and say observability. However, Winskel [W] has proven the following result (in a slightly different setting).

Theorem 4.2

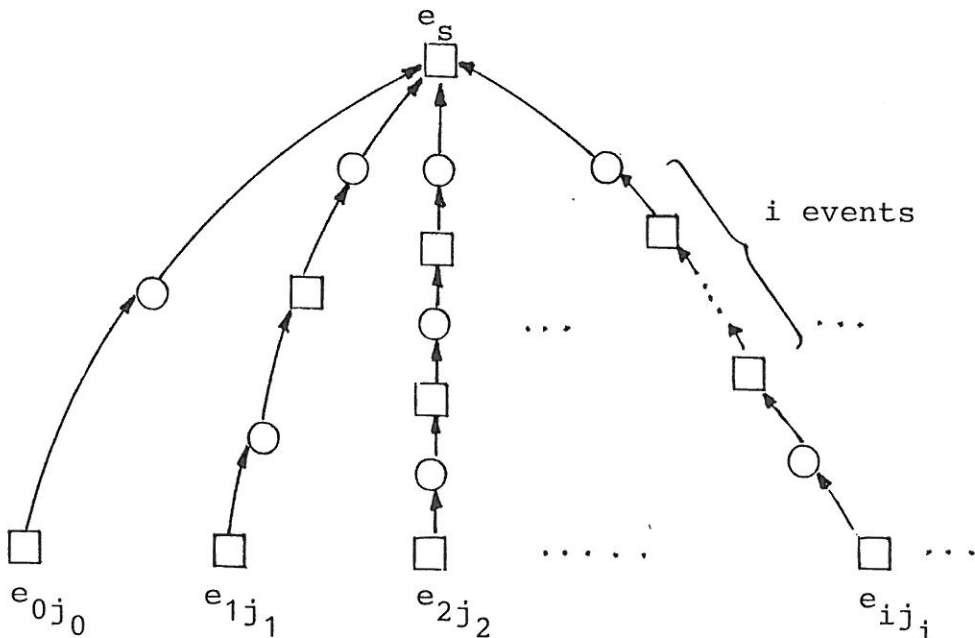
- 1) All observable occurrence nets are b-discrete.
- 2) There exist b-discrete occurrence nets which are not observable.
- 3) All countable, b-discrete occurrence nets are observable. □

In this section we shall look for general characterisations of b-discreteness in terms of the other properties under consideration. To put this in perspective let us just present an example of a non-observable, b-discrete occurrence net - essentially taken from Winskel [W].

Example 4.3 The example is built around a countable set of chains as follows.



For each slice of this set consisting of events only $s = e_{0j_0}, e_{1j_1}, \dots$, our example has a distinct event e_s with distinct chains from e_{ij_i} to e_s of length $2 \times i + 1$.



This occurrence net is clearly b-discrete (and uncountable). Assume that the net has an observer, O . There must exist a slice,

s, as above with only positive $O(e_{ij_i})$ -values (trivial). But this means that $O(e_s)$ cannot take any finite value - a contradiction.

Modifying Winskel's proof of Theorem 4.2 3) slightly, we get the following result. In this result we shall work with an extended notion of observability. More specifically, the poset $PO' = (X'; \leq')$ is observable iff there exists $O: X' \rightarrow \mathbb{Z}$ such that $\forall x, y \in X': x <' y \Rightarrow O(x) < O(y)$.

Theorem 4.4 Let $N = (B, E; F)$ be an occurrence net. Then the following statements are equivalent.

- 1) N is b-discrete
- 2) The convex closure of every countable subset of X_N (viewed as a poset) is observable.
- 3) The convex closure of every finite subset of X_N is observable.

Proof

1) \Rightarrow 2) Assume that N is b-discrete and let x_0, x_1, \dots be a fixed enumeration of some countable subset $X' \subseteq X_N$. We want to prove the existence of an observer for $(\langle X' \rangle, \leq')$, where \leq' is $\leq = F^*$ restricted to $\langle X' \rangle$. The proof builds up such an observer by constructing an observer O_i for $\langle X_i \rangle = \langle \{x_0, x_1, \dots, x_i\} \rangle$ and extending this to an observer O_{i+1} for $\langle X_{i+1} \rangle = \langle \{x_0, \dots, x_{i+1}\} \rangle$ (i.e. preserving O_i -values for $\langle X_i \rangle$).

So the induction hypothesis is that we have an observer O_i for $\langle X_i \rangle$ taking values from some finite interval $[-k_i, k_i]$. This is trivially obtainable for $i = 0$ (choosing $O_0(x_0) = 0$, $k_0 = 0$).

Now for the induction step. If $x_{i+1} \in \langle X_i \rangle$ we just define $O_{i+1} = O_i$. If x_{i+1} is in the relation co to all elements of $\langle X_i \rangle$ we extend O_i to O_{i+1} by choosing $O_{i+1}(x_{i+1}) = 0$.

These were the easy cases. Two are left -

- a) x_{i+1} is greater than some elements of $\langle X_i \rangle$, but not smaller than any.
- b) x_{i+1} is smaller than some elements of $\langle X_i \rangle$, but not greater than any.

In the following we just treat case a), the treatment of b) being completely symmetrical.

Claim 1 The length of any chain from an element of $\langle X_i \rangle$ to x_{i+1} is finitely bounded by

$$c_i \stackrel{\text{def}}{=} \sup\{n \mid 0 \leq j \leq i \text{ and } x_j F^n x_{i+1}\}$$

This follows easily from the nature of $\langle X_i \rangle$ and b-discreteness.

Now define

$$O_{i+1}(x) = \begin{cases} O_i(x) & \text{if } x \in \langle X_i \rangle \\ k_i + \sup\{n \mid 0 \leq j \leq i \text{ and } x_j F^n x\} & \text{if } x \in \langle X_{i+1} \rangle \setminus \langle X_i \rangle \end{cases}$$

Claim 2 O_{i+1} is an observer for $\langle X_{i+1} \rangle$ extending O_i and taking values in the finite interval $[-k_i - c_i, k_i + c_i]$.

This follows by simple arguments using the induction hypothesis and Claim 1.

2) \Rightarrow 3) Trivial.

3) \Rightarrow 1) Take any two elements $x, y \in X_N$. Assuming that the convex closure $\langle \{x, y\} \rangle$ is observed by O , it is clear that the length of chains between x and y are bounded by $|O(x) - O(y)|$. □

Let us note that the observability of an occurrence net can be rephrased as

N is observable iff the convex closure of every subset of X_N is observable.

Then the last theorem brings out the difference between observability and b-discreteness, which makes Winskel's result - Theorem 4.2 - fit nicely into the framework.

Now, does a similar characterisation result hold for b-discreteness in terms of the notion of full slice (state) space? The following result gives a positive answer to this question.

Theorem 4.5 Let $N = (B, E; F)$ be an occurrence net. Then the following statements are equivalent.

- 1) N is b-discrete
- 2) Every countable subset of X is covered by some slice space
- 3) Every finite subset of X is covered by some slice space.

(where $X' (\subseteq X_N)$ is covered by a slice space iff there exists $s \in SL_N$ such that $X' \subseteq \cup\{s' \mid s' \in [s]\}$).

Proof

1) \Rightarrow 2) Let X' be a countable subset of X. According to Theorem 4.4 $\langle X' \rangle$ has an observer, and by arguments similar to the ones in the proof of Lemma 2.6, it follows that we can find a maximal co-set of $\langle X' \rangle$, call it s_1 , such that

$$\forall x \in \langle X' \rangle: d_F(x, s_1) < \infty$$

However, s_1 is not necessarily a slice of N, nor can it be extended in an arbitrary way to a slice of N with the desired property (that its slice space covers $\langle X' \rangle$), as indicated by the following example.

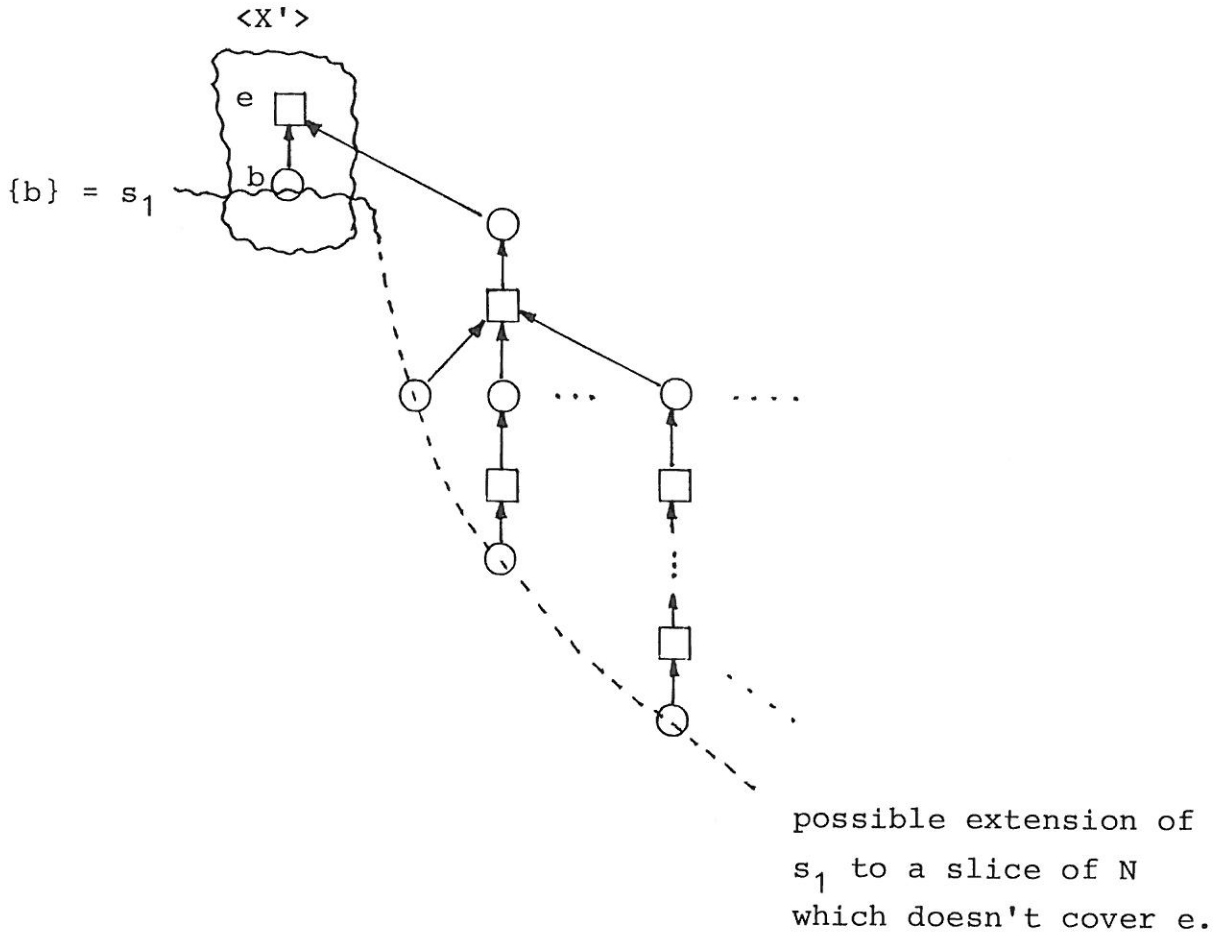


Figure 4.1

For this reason we introduce the notion of border elements of a subset Y of X_N .

$$\text{Border}(Y) \stackrel{\text{def}}{=} \{x \in X_N \setminus Y \mid (x^* \cap Y \neq \emptyset) \vee (x \cap Y \neq \emptyset)\}$$

Thus $\text{Border}(\langle X' \rangle)$ are the events and conditions "sticking out" from $\langle X' \rangle$:

The idea is now to extend s_1 to a maximal co-set of $\langle X' \rangle \cup \text{Border}(\langle X' \rangle)$, $s_1 \cup s_2$. This may still not be a slice of N , but let us start by proving

Claim 1 $\forall x \in \langle X' \rangle. d_F(x, s_2) < \infty.$

The first observation is that $s_2 \subseteq B$. Assume $b F e$, where $b \in \langle X' \rangle$ and $e \in \text{Border}(\langle X' \rangle)$ ($e F b$ handled symmetrically). We want to prove a contradiction. Since $b \in \langle X' \rangle$ there exists $y \in s_1$ such that $b \text{ li } y$. $b < y$ implies $e \leq y$ (since $|b'| \leq 1$) and hence $e \in \langle X' \rangle$ (since $\langle X' \rangle$ is convex) - a contradiction. $y \leq b$ implies $y < e$ - a contradiction to $s_1 \cup s_2$ being a co-set. Thus $s_2 \subseteq B$.

Claim 1 follows now by a proof of the following

$$\forall x \in \langle X' \rangle \quad \forall b \in s_2. (x F^n b \vee b F^n x) \Rightarrow n \leq d_F(x, s_1)$$

Let us just take the case $x = x_1 F x_2 F \dots F x_n F b$. From the definition of $\text{Border}(\langle X' \rangle)$, the convexity of $\langle X' \rangle$, and the fact that $|b'| \leq 1$ it follows that all the x_i 's belong to $\langle X' \rangle$, and hence, there exists $z \in s_1$ such that $x_n \text{ li } z$. $z \leq x_n$ leads to a contradiction to $s_1 \cup s_2$ being a co-set, and therefore $x_n < z$. Hence, there exists a chain from x to s_1 (z) of length at least n , and $n \leq d_F(x, s_1)$ follows.

This finishes the arguments for Claim 1.

Now we extend $s_1 \cup s_2$ arbitrarily to a slice of N ,
 $s = s_1 \cup s_2 \cup s_3$.

Claim 2 $\forall x \in \langle X' \rangle. d_F(x, s_3) = 0.$

To see this assume w.l.o.g. that $x = x_0 F x_1 F \dots F x_n = y$ where $y \in s_3$. We want to prove that no such chain can exist. Since $x \in \langle X' \rangle$ and $y \notin \langle X' \rangle$, there must exist an i such that $x_i \in \langle X' \rangle$ and $x_{i+1} \notin \langle X' \rangle$, in which case $x_{i+1} \in \text{Border}(\langle X' \rangle)$. Since $s_1 \cup s_2$ is a maximal co-set in $\langle X' \rangle \cup \text{Border}(\langle X' \rangle)$, we must have for some $z \in s_1 \cup s_2$, $x_{i+1} \text{ li } z$. But $z < x_{i+1}$ leads to the contradiction $z < y$ (s being a co-set).

Suppose $x_{i+1} < z$. Then this leads to the contradiction $x_{i+1} \in \langle X' \rangle$. To see this first consider $z \in s_1 \subseteq \langle X' \rangle$. Then $x_i \in \langle X' \rangle$ and $x_i < x_{i+1} < z$ so that $x_{i+1} \in \langle X' \rangle$. Next consider $z \in s_2 \subseteq B$ (as proved in establishing claim 1). Then $x_{i+1} < z$ would imply that $x_{i+1} \leq e$ where $\{e\} = z$. Since $z \in \text{Border}(\langle X' \rangle)$ we must have $e \in \langle X' \rangle$. (Otherwise $z \cap \langle X' \rangle \neq \emptyset$ and from $x_i \in \langle X' \rangle$ and $x_i < x_{i+1} \leq e < z$ we would obtain the contradiction that $z \in \langle X' \rangle$.) But we now have $x_i < x_{i+1} \leq e$ with $x_i, e \in \langle X' \rangle$ so that $x_{i+1} \in \langle X' \rangle$.

So we have ^{shown} proven the existence of a slice s in N for which $d_F(x, s) < \infty$ for all $x \in \langle X' \rangle$, and the desired result follows from Lemma 2.6.

2) \Rightarrow 3) Trivial.

3) \Rightarrow 1) Assume the convex closure of every finite subset is covered by a slice space. By arguments similar to the ones in the proof of Lemma 2.6, it follows that these closures also have observers, and hence from Theorem 4.4 that N is b-discrete. \square

Note that parts of Theorems 4.4 and 4.5 can be read as "compactness results" for countable occurrence nets. Let us just state such an interpretation for the slice space case. Note that the result doesn't hold for occurrence nets in general (Example 4.3).

Corollary 4.6 Let N be a countable occurrence net. Then the following statements are equivalent.

- 1) Any subset of X_N is covered by a slice space.
- 2) Any finite subset of X_N is covered by a slice space.

\square

As mentioned in the introduction, this paper has been an investigation into four restrictions on occurrence nets suggested in

the literature: Observable, slice space covered, b-discrete and finitely approximable. As we have seen these give rise to three different classes. Let us finish off by stating some quite liberal restrictions under which these three classes collapse into one. Before stating the result let us agree that an occurrence net $N = (B, E, F)$ is degree-finite if for every event e , $|e^{\bullet}|, |e^{\circ}| < \infty$.

Theorem 4.7 Let N be a countable, degree finite occurrence net. Then the following statements are equivalent.

- 1) N is observable
- 2) N is covered by a slice space
- 3) N is b-discrete
- 4) N is finitely approximable.

Proof Since N is countable, 1), 2) and 3) are equivalent by previous results in this paper. 4) implies 1) by Theorem 3.10. And finally b-discreteness and degree-finiteness together imply interval-finiteness so that 3) implies 4) by Theorem 3.9. □

5. CONCLUSIONS

In this paper we have addressed the question: In what sense is the causality relation associated with a non-sequential process "realizable"?

Using occurrence nets as a model of non-sequential processes we have examined a number of fairly natural proposals to formalise the notion of realizable processes. We have shown that the notions of observability and full state space coincide (Theorem 2.11). We have shown that these two notions are also equivalent to approximability by chain-bounded occurrence nets (Theorem 3.6). Using these results we have also obtained two characterisations of those occurrence nets that can be approximated by finite occurrence nets; they are precisely the occurrence nets that are interval-finite and countable on the one hand (Theorem 3.9) and which are injectively observable on the other hand (Theorem 3.10).

As for the density property called b-discreteness we have obtained two characterisations; one in terms of observability (Theorem 4.4) and the other in terms of full slice (state) space (Theorem 4.5). Our results further strenghten the insight due to Winskel (Theorem 4.2) that countability is the essential difference between observability and the weaker notion of b-discreteness. Finally by imposing the restrictions of countability - justified by our wish to consider the only "discrete" processes - and finite-degree - to capture the demand that each event should have only "finite" causes and effects - we have identified a class of occurrence nets for which the four appealing notions - observability, full state space, finite approximability and b-discreteness - considered in the paper turn out to be equivalent (Theorem 4.7). What is lacking at present is the way to approximate b-discrete occurrence nets.

C.A. Petri has proposed a number of other restrictions one might impose on a model of non-sequential processes [P]. In the context of the present paper two of the restrictions proposed by Petri are highly relevant. One is the demand that every slice and

every line should have an intersection - the so-called K-density property - and the other is a generalisation of the classical Dedekind countinuity of the reals - the so-called D-continuity property. K-density has been studied in [B] and D-continuity in [FT].

Here we have restricted our attention to deterministic processes. A more general model called event structures using which one can model non-deterministic non-sequential processes have been proposed in [NPW] and extensively investigated in [W]. Most of our results however can be - we believe - smoothly carried over to this more general model. Thus we feel justified in claiming that under reasonable circumstances (countability and degree-finiteness!), the various proposals made so far to capture the intuitive demand that one should consider only "realisable" processes lead to the same class of objects. And to demonstrate this has been our major goal here.

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