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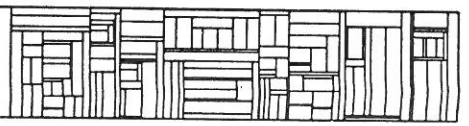

PB - 194

P. Wachtel: Generalizations of Liveness and Fairness Properties

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GENERALIZATIONS OF LIVENESS AND FAIRNESS PROPERTIES

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ABSTRACT

The definitions of many known properties of concurrent systems (e.g. liveness, fairness, impartiality, justice) require every action to satisfy some condition. Sometimes the level of actions is too fine and one would like to consider the corresponding conditions for sets of actions.

A framework for such treatment is proposed in this paper. The known liveness and fairness properties are generalized and investigated in accordance to the lattice structure of partitions of a set of actions. Partitions are used in order to group actions and form some cruder levels of abstraction.

INTRODUCTION

A generalization of well known behavioural properties of concurrent systems, such as liveness [LAU], impartiality, fairness, justice [LPS] is presented.

All the considered properties are defined in a similar way. It is required that every action should satisfy some condition (of liveness, impartiality etc.). But sometimes it is more appropriate to consider groups of actions, instead of simple actions. It may happen, that however the whole system does not satisfy a given property, large parts of it do. It would be nice then to carry this information in the behavioural analysis of the system.

The generalization proposed in this paper is done in the following way:

First of all we generalize the conditions defining properties to make sense for sets of actions, and not only for individual actions. This generalization is done in a natural way.

Secondly we check, whether for a given partition, all the equivalence classes satisfy this generalized condition. If so, then we say, that this partition satisfies the considered property.

Since partitions form lattice, a natural qu
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ability of these properties. Monotonicity resu
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(making the partitions cruder preserve these prop

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chapter 3. Two sets of actions are coordinated iff non
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infinitely often at least one of the actions of the other
It is shown that one can refine a partition along the lines
that separate coordinated sets, without losing any of the four
considered properties (antimonotonicity result).

The results of chapter 2 and 3 are applied then in the chapter 5,
where some subclasses of Petri Nets are considered, namely Marked
Graphs, State Machines and Free-Choice Petri Nets.

The whole paper is closed by chapter 6, in which some extensions
of these generalizations are proposed. A possibility of using
Covers
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results of the paper seem to hold also for vers.

Since partitions form lattice, a natural question to ask is whether the structure of this lattice has any impact on satisfiability of these properties. Monotonicity results are proved in chapter 2 for generalized liveness, impartiality and fairness (making the partitions cruder preserve these properties).

A coordination relation on sets of actions is introduced in chapter 3. Two sets of actions are coordinated iff none actions of any of them can be executed infinitely often without executing infinitely often at least one of the actions of the other set. It is shown that one can refine a partition along the lines that separate coordinated sets, without losing any of the four considered properties (antimonotonicity result).

The results of chapter 2 and 3 are applied then in the chapter 5, where some subclasses of Petri Nets are considered, namely Marked Graphs, State Machines and Free-Choice Petri Nets.

The whole paper is closed by chapter 6, in which some extensions of these generalizations are proposed. A possibility of using a cover of some subset of the set of actions is discussed. Covers on subsets give a chance, first of all to forget about some actions (the ones which find themselves outside the chosen subset) and, secondly, they permit some actions to be shared by several sets (sets of a cover may overlap). Most of the results of the paper seem to hold also for covers.

1. PRELIMINARY DEFINITIONS

Def. 1.1

$N = 0,1,2,\ldots$

$N_\omega = N \cup \{\omega\}$, where

$\forall n \in N: n < \omega,$

$n + \omega = \omega + n = \omega,$

$\omega + \omega = \omega$.

For every set A we denote its cardinality by $|A|$. If A is countable and finite, then we write $|A| = \omega$. □

Lemma 1.2 For any countable set A

$|A| = \omega \Leftrightarrow \forall n \in N : |A| > n,$

$|A| < \omega \Leftrightarrow \exists n \in N : |A| = n.$

If for $n,m \in N_\omega$ $n + m = \omega$ then either $n = \omega$ or $m = \omega$. □

We shall frequently use lemma 1.2 without mentioning it.

Let $\preccurlyeq$ denote a prefix order on T*, for some finite alphabet T.

Def. 1.3

$W \subseteq T^*$ is an infinite word iff $\forall w_1, w_2 \in W$:

(i) $w_1 \preccurlyeq w_2$ or $w_2 \preccurlyeq w_1$

(ii) $\forall w_3 \in T^*$: if $w_3 \preccurlyeq w_1$ then $w_3 \in W$

(iii) $|W| = \omega$ □

Lemma 1.4 If W is an infinite word, then

(i) $\preccurlyeq$ is a total order on W

(ii) for every $w \in W$ there exists exactly one $t \in T$ s.t. $wt \in W$. □

Def. 1.5

The set of infinite words over T will be denoted by T^{ω}. □

We shall use the notation

$W = t_1 t_2 t_3 \ldots$ for $W \in T^{\omega}$ such that

$W = \{t_1, t_1 t_2, t_1 t_2 t_3, \ldots\}$.

Def. 1.6

A language over the alphabet T is any subset of T*. □

Def. 1.7

For any $L \subseteq T^*$: $L_{\omega} = 2^L \cap T^{\omega}$.

L_{ω} is called a language of inifinite words of L. □

Def. 1.8

$L \subseteq T^*$ is called prefix-closed iff

$$\forall u \in L: \forall v \in T^*: v \preccurlyeq u \Rightarrow v \in L.$$ □

Corollary 1.9

Any infinite word is prefix-closed. □

Def. 1.10

For $L \subseteq T^*$: $A(L) = \{t \in T \mid \exists w \in T^*: wt \in L\}$.

$A(L)$ is called an alphabet of L. □

Def. 1.11

$\forall w \in T^*$: $w^0 = \varepsilon$

$w^{i+1} = ww^i$

By ε we denote an empty word. □

Def. 1.12

$\#: T \times (T^* \cup T^\omega) \to N_\omega$

$$\#(t,w) = \begin{cases} |\{u \in T^* \mid ut \preccurlyeq w\}| & \text{if } w \in T^* \\ |\{u \in T^* \mid ut \in w\}| & \text{if } w \in T^\omega. \end{cases}$$ □

Lemma 1.13

For $t \in T$, $W \in T^\omega$

$$\#(t,W) = \begin{cases} \max_{w \in W} (\#(t,w)) & \text{if it exists} \\ \omega & \text{otherwise} \end{cases}$$ □

Lemma 1.14

If for some $t \in T$, $W \in T^\omega$, $n \in N$: $\#(t,W) = n$, then there exists $w \in W$ s.t. $\#(t,w) = n$ and for every $u \in W$

$$w \preccurlyeq u \Rightarrow \#(t,u) = n$$ □

Lemma 1.15

If $v_1, v_2, \ldots$ is an infinite sequence of words of some language L such that for every i: $v_i \preccurlyeq v_{i+1}$, and $v_i \neq v_{i+1}$ then there exists an infinite word V from L_ω such that for every $i \in N$: $v_i \in V$. □

Def. 1.16

A property is any family of languages. □

Def. 1.17

For $T' \subseteq T$, $V \in T^* \cup T^\omega$.

$$\#(T',V) = \sum_{t \in T} \#(t,V).$$ □

Def. 1.18

For any $V \subseteq T^*$, $W \subseteq T^*$ and $t \in T$

$$Y_V(W,t) = \{w \in W \mid wt \in V\}$$ □

Lemma 1.19

For every $W \in T^* \cup T^\omega$, $t \in T$: $|Y_W(W,t)| = \#(t,W)$. □

Def. 1.20

$\pi = \{T_1, \ldots, T_n\}$ is called a partition of T iff

(i) $\bigcup_{i=1}^{n} T_i = T$

(ii) $\forall i,j = 1,\ldots,n: T_i \cap T_j = \emptyset$. □

Def. 1.21

Let $\pi_1 = \{T_1,\ldots,T_n\}$, $\pi_2 = \{T^1,\ldots,T^m\}$ be partitions of T. π_1 is finer than π_2 (π_2 is cruder than π_1) iff $\forall i = 1,\ldots,n: \exists j = 1,\ldots,m : T_i \subseteq T^j$.

We denote it by $\pi_1 \leqq \pi_2$, and say also, that π_1 is a refinement of π_2. □

Corollary 1.22

Partitions of T form a complete lattice with respect to the order "$\leqq$". □

Lemma 1.23

For two partitions π_1, π_2 of T: if $\pi_1 \leqq \pi_2$, then for every $P'' \in \pi_2$ there exists a set $\{P'_1,\ldots,P'_n\} \subseteq \pi_1$ such that $\bigcup_{i=1}^{n} P'_i = P''$. □

2. GENERALIZATION OF LIVENESS AND FAIRNESS PROPERTIES

Assume for this section, that L is a prefix-closed language, and $A(L) = T$. The elements of T will be called transitions.

Def. 2.1

For $t \in T$:

t is live in L iff $\forall w \in L: \exists u \in T^*: wut \in L$.

t occurs impartially in L iff $\forall W \in L_\omega: \#(t,W) = \omega$.

t occurs fairly in L iff

$$\forall W \in L_\omega: |Y_L(W,t)| = \omega \Rightarrow |Y_W(W,t)| = \omega .$$

t occurs justly in L iff

$$\forall W \in L_\omega: \forall w \in W: \exists u \in W: \quad w \preccurlyeq u \text{ and } (ut \in L \Rightarrow ut \in W). \quad \square$$

Def. 2.2

L is live iff every $t \in T$ is live in L.

L is impartial iff every $t \in T$ occurs impartially in L.

L is fair iff every $t \in T$ occurs fairly in L.

L is just iff every $t \in T$ occurs justly in L. $\square$

Def. 2.3

The class of live (resp. impartial, fair, just) languages will be denoted by LIV (resp. IMP, FAIR, JUST). $\square$

It is easy to observe, that the definitions (2.2) of properties are built in a similar way: we require some condition to be satisfied for every transition $t \in T$.

Our aim will be to generalize these properties so that sets of transitions instead of single transitions will be used in the new definitions of properties.

There exists a natural way of extending the concatenation operation to hold for sets of words:

Def. 2.4

For $L_1, L_2 \subseteq T^*$

$$L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1,\ w_2 \in L_2\}.$$ □

When it does not lead to ambiguities, let us not distinguish between a word which consists of one symbol and this symbol itself. Similarly if $L = \{w\}$ for some $w \in T^*$, we will omit the brackets and write w to identify L, whenever it is clear from the context, what is meant.

Now we can simply assign the meaning to the expression "wT'" for some $w \in T^*$ and $T' \subseteq T$:

Def. 2.5

$$wT' = \{wt' \mid t' \in T'\}.$$ □

We need also a generalization of the relation "$\in$". There exist two natural ways to do this:

Def. 2.6

For $V,L \subseteq T^*$:

$V \underline{\in} L$ iff $\exists v \in V: v \in L$ weak generalization

$V \overline{\in} L$ iff $\forall v \in V: v \in L$ strong generalization

□

Comment: $V \underline{\in} L$ iff $V \cap L \neq \emptyset$, $V \overline{\in} L$ iff $V \subseteq L$.

Now we can easily generalize two other important notions:

Def. 2.7

For $X,V \subseteq T^*$:

$\underline{\#}(X,V) = \{v \in V \mid vX \underline{\in} V\}$

$\overline{\#}(X,V) = \{v \in V \mid vX \overline{\in} V\}$ □

Note, that $\underline{\#}(X,V) = \#(X,V)$, as defined in 1.17.

Def. 2.8

For $X,V,L \subseteq T^*$:

$\underline{Y}_L(V,X) = \{v \in V \mid vX \underline{\in} L\}$

$\overline{Y}_L(V,X) = \{v \in V \mid vX \overline{\in} L\}$. □

We are almost ready to generalize all the considered properties. The only thing left is the choice between "$\underline{\in}$" and "$\overline{\in}$".

Let us choose the weak relation "$\underline{\in}$". The other choice is also possible, but it would lead to less interesting results. The discussion concerning this problem will be made at the end of this section.

Def. 2.9

A generalized property is any set of pairs (L,π), where L is some language and π is a partition of its alphabet. □

Def. 2.10

$(L,\pi) \in \underline{LIV}$ iff $\forall P \in \pi: \forall w \in L: \exists u \in T^*: wuP \underline{\in} L$

$(L,\pi) \in \underline{IMP}$ iff $\forall P \in \pi: \forall W \in L_\omega: \underline{\#}(P,W) = \omega$

$(L,\pi) \in \underline{FAIR}$ iff $\forall P \in \pi: \forall W \in L_\omega:$

$$(|\underline{Y}_L(W,P)| = \omega) \Rightarrow (|\underline{Y}_W(W,P)| = \omega)$$

$(L,\pi) \in \underline{JUST}$ iff $\forall P \in \pi: \forall W \in L_\omega: \forall w \in W: \exists u \in W:$

$$(w \preccurlyeq u) \ \& \ (uP \underline{\in} L \Rightarrow uP \underline{\in} W).$$ □

Remark

We underline the names of generalized properties to mark, that the relation "$\underline{\in}$" is used in the definitions. □

When $(L,\pi) \in \psi$ for some generalized property ψ, we say that π satisfies ψ, when it is clear from the context, which L we have in mind.

Def. 2.11

Let $\perp = \{\{t_1\},\ldots,\{t_n\}\}$

$\top = \{T\}$

$\perp$ is the finest and $\top$ the crudest partition of the set T.

□

Corollary 2.12

$(L,\perp) \in \underline{LIV} \Leftrightarrow L \in LIV$

$(L,\perp) \in \underline{IMP} \Leftrightarrow L \in IMP$

$(L,\perp) \in \underline{FAIR} \Leftrightarrow L \in FAIR$

$(L,\perp) \in \underline{JUST} \Leftrightarrow L \in JUST$ □

So the generalized properties are really generalizations of the normal ones.

Corollary 2.13

For any $L \in T^*$:

$(L,\top) \in \underline{IMP} \cap \underline{FAIR} \cap \underline{JUST}$. □

Corollary 2.14

$\underline{IMP} \subseteq \underline{FAIR} \subseteq \underline{JUST}$. □

Digression

There is another definition of liveness, by Lautenbach [LAU] called 1-liveness, by many others - deadlock-freeness (DFR).

Def. 2.15

$L \in DFR$ iff $\forall w \in L: \exists t \in T: wt \in L$. □

As a matter of fact deadlock-freeness is a property, which can be expressed in terms of generalized liveness:

Corollary 2.16

$L \in DFR \Leftrightarrow (L,T) \in \underline{LIV}$. □

So generalized liveness LIV really generalizes the two kinds of livenesses: deadlock-freeness and ordinary liveness. It is easy to prove, that liveness implies deadlock-freeness [LAU]:

$$LIV \subseteq DFR$$

We shall generalize this result in theorem 2.19 (end of digression).

Before we prove the main result of this section, we present two elementary lemmas, which will be used in the proofs:

Lemma 2.17

For any $A,B,L \subseteq T^*$, $W \in T^* \cup T^\omega$

(i) $\underline{Y}_L(W,A \cup B) = \underline{Y}_L(W,A) \cup \underline{Y}_L(W,B)$

(ii) $A \subseteq B \rightarrow \underline{Y}_L(W,A) \subseteq \underline{Y}_L(W,B)$ □

Lemma 2.18

$\forall W \in T^* \cup T^\omega$, $\forall A,B \subseteq T$

(i) $\underline{\#}(A \cup B,W) \leq \underline{\#}(A,W) + \underline{\#}(B,W)$

(ii) $(A \subseteq B) \Rightarrow \underline{\#}(A,W) \leq \underline{\#}(B,W)$ □

Def. 2.19

A generalized property ψ is monotonic (resp. antimonotonic) iff for any language L and any pair of partitions π_1,π_2 on A(L) such that $\pi_1 \leq \pi_2$

$$(L,\pi_1) \in \psi \Rightarrow (L,\pi_2) \in \psi$$

(resp.: $(L,\pi_2) \in \psi \Rightarrow (L,\pi_1) \in \psi$) □

Theorem 2.20

(i) LIV

(ii) IMP

(iii) FAIR are monotonic.

Proof

Let us take any language L and two partitions π_1,π_2 on A(L) such that $\pi_1 \leq \pi_2$.

(i) (LIV is monotonic)

Assume, that $(L,\pi_2) \notin$ LIV.

Let $P'' \in \pi_2$ and $w \in L$ be such that $\forall u \in T^*$: $\forall t \in P''$: $wut \notin L$.

Let $P' \in \pi_1$ be such that $P' \subseteq P''$. For this P' and for the same $w \in L$ none of $u \in T^*$ and $t \in P'$ satisfy $wut \in L$. Hence $(L,\pi_1) \notin$ LIV.

(ii) (IMP is monotonic)

Assume, that $(L,\pi_2) \notin \underline{IMP}$.

Let $P'' \in \pi_2$ and $W \in L_\omega$ be such that $\underline{\#}(P'',W) < \omega$. Let $P' \in \pi_1$ be such that $P' \subseteq P''$.

From Lemma 2.18 we get $\underline{\#}(P',W) \leq \underline{\#}(P'',W) < \omega$, so $\underline{\#}(P',W) < \omega$.

So $(L,\pi_1) \notin \underline{IMP}$.

(iii) (FAIR is monotonic)

Assume, that $(L,\pi_2) \notin \underline{FAIR}$

Let $P'' \in \pi_2$ and $W \in L_\omega$ be such that $|\underline{Y}_L(W,P'')| = \omega$ and $|Y_W(W,P'')| < \omega$.

Let $\{P'_1,\ldots P'_n\} \subseteq \pi_1$ be such, that $\bigcup_{i=1}^{n} P'_i = P''$. (Lemma 1.23)

From Lemma 2.17 and definition of $\underline{Y}$ we deduce the following:

$$\omega = |\underline{Y}_L(W,P'')| = |\underline{Y}_L(W, \bigcup_{i=1}^{n} P'_i)| = |\bigcup_{i=1}^{n} \underline{Y}_L(W,P'_i)| \leq \sum_{i=1}^{n} |\underline{Y}_L(W,P'_i)|.$$

Hence there must exist such $i = 1,\ldots,n$ that $|\underline{Y}_L(W,P'_i)| = \omega$.

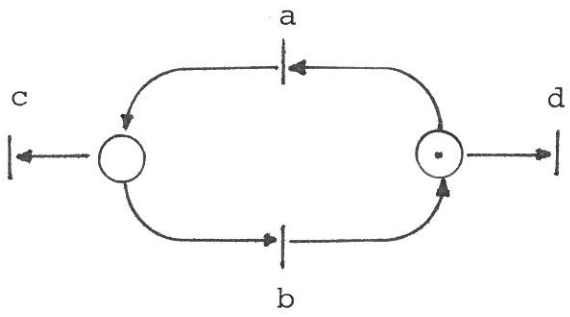
But again from Lemma 2.17 (ii) we have

$|\underline{Y}_W(W,P'_i)| \leq |\underline{Y}_W(W,P'')| < \omega$. So we have found such $P'_i \in \pi_1$ and $W \in L_\omega$, that $|\underline{Y}_L(W,P'_i)| = \omega$ and $|\underline{Y}_W(W,P'_i| < \omega$, which means, that $(L,\pi_1) \notin \underline{FAIR}$.

In order to complete the investigations on monotonicities, one should remark, that JUST is not monotonic.

Counterexample

Consider the following net (the definitions of net theory are given in chapter 4):



Let $\pi_1 = \bot$

$\pi_2 = \{\{a\},\{b\},\{c,d\}\}$

$\pi_1 \leqq \pi_2$

Figure 1

Let L be a language generated by this net. The only infinite word W can arise by firing alternately the transitions a and b. $(L,\pi_1) \in$ JUST, because for every $w \in W$ and $P' \in \pi_1$, the condition $uP' \underline{\in} L$ is false for some $u \in W$ such that $w \leqslant u$.

At the same time for $P'' = \{c,d\} \in \pi_2$ and for every $u \in W$, the condition $uP'' \underline{\in} L$ is true and $uP'' \underline{\in} W$ is false, so $(L,\pi_2) \notin$ JUST.

Since partitions of T form a lattice, we can deduce the following facts:

Corollary 2.21

For every generalized property $\psi \in \{$IMP,FAIR$\}$ and every language L there always exists a set of partitions $\pi = \{\pi_1,\ldots,\pi_k\}$ such that

(i) $\forall j = 1,\ldots,k: (L,\pi_j) \in \psi$

(ii) for every partition π of A(L):

$$(L,\pi) \in \psi \Leftrightarrow \exists j = 1,\ldots,k: \pi_j \leq \pi$$ □

Proof is a consequence of Corollary 2.13 and the monotonocity theorem. □

Corollary 2.22

For every $L \in DFR$ there exist a set of partitions $\pi = \{\pi_1,\ldots,\pi_m\}$ such that:

(i) $\forall i = 1,\ldots,m: (L,\pi_i) \in \underline{LIV}$

(ii) for every partition π of A(L)

$$(L,\pi) \in \underline{LIV} \Leftrightarrow \exists i = 1,\ldots,m: \pi_i \leq \pi$$ □

Proof is a consequence of Corollary 2.16 and the monotonicity theorem. □

The sets of partitions mentioned in the Corollaries 2.21 and 2.22 are the sets of minimal partitions satisfying the considered properties. Each of them cuts the lattice of partitions of T into two parts. In the upper part one can find all the partitions, which satisfy the property. In the lower part - all that do not satisfy it.

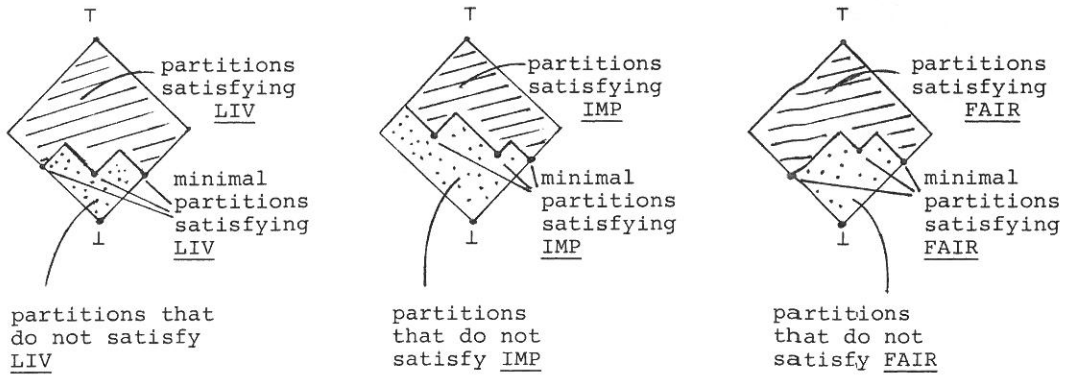


Figure 2

For a given L one can find the sets of minimal partitions satisfying LIV, IMP and FAIR.

Certainly for every L the cut line for IMP must lie above the cut line for FAIR. since IMP $\subseteq$ FAIR (Corollary 2.14).

With the help of generalized properties one can precisely specify "how much" a certain property is (or is not) satisfied, for a given language. Informally speaking, the higher the cut line is, the "less" a given peoperty is satisfied. The strongest level is of course the bottom cut, consisting of $\perp$, in which case a given language just belongs to an ungeneralized property.

In order to complete this section, we shall discuss our choice of the "$\underline{\epsilon}$" relation instead of "$\overline{\epsilon}$".

Lemma 2.23

For every $W \in L_\omega$, $w \in W$ and $\hat{T} \subseteq T$ $(|\hat{T}| > 1) \Rightarrow w\hat{T} \overline{\notin} W$.

Proof

The thesis comes from lemma 1.4: for every $w \in W$ there exists a unique $t \in T$ such that $wt \in W$. □

We can easily observe, that for any interesting (different from $\perp$) partition, the conditions defining $\overline{IMP}$, $\overline{FAIR}$ and $\overline{JUST}$, with the help of the relation "$\overline{\in}$", degenerate a lot and give no satisfactory results. As a matter of fact neither of them is monotonic nor antimonotonic.

However the condition for $\overline{LIV}$ does not degenerate:

Def. 2.24

$(L,\pi) \in \overline{LIV}$ iff $\forall w \in L: \forall P \in \pi: \exists u \in T^*: wuP \overline{\in} L$. □

This is in some sense a dual notion to LIV and one can prove the following antimonotonicity result for $\overline{LIV}$:

Theorem 2.25

For every language L and a pair of partitions of A(L): π_1 and π_2

$$(\pi_1 \leq \pi_2) \Rightarrow ((L,\pi_2) \in \overline{LIV} \Rightarrow (L,\pi_1) \in \overline{LIV}$$ □

Proof omitted.

One can assign some useful interpretation to this strong generalization of liveness. With some additional restrictions it could be interpreted, as a "liveness of concurrency" - which would mean, that the transitions grouped in the classes by some partition cannot lose the ability of concurrent occurrence (the restriction should exclude conflicts between them).

We leave this subject, however, and due to uniformity concentrate on weak generalization of liveness and fairness properties.

To illustrate the results of this section, let us present a graphical representation of different situations in which none, some or all the partitions satisfy the considered properties.

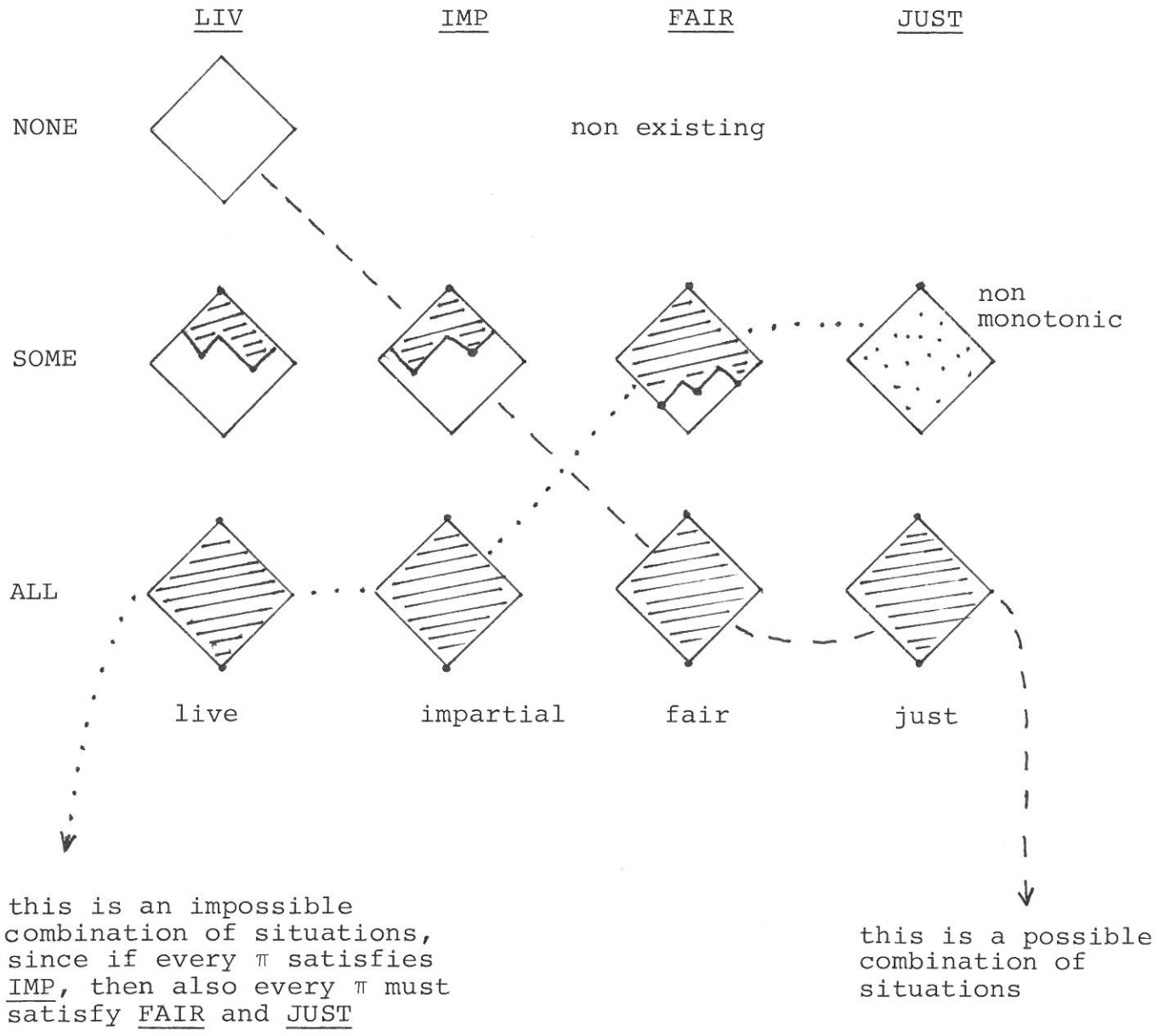


Figure 3

Only 12 out of 24 combinations can reflect a real situation.

3. ANTIMONOTONICITIES OF LIVENESS AND FAIRNESS PROPERTIES WITH RESPECT TO THE COORDINATION RELATION

In the previous chapter we have proved that three of the four considered properties are monotonic.

The following question arises: under which conditions do these properties become antimonotonic? It is rather clear, that at least the monotonic ones cannot be just antimonotonic without any additional conditions. If a property is monotonic and antimonotonic at the same time, then varying the partitions does not change anything - all of them are equivalent with respect to the answer to the question whether $(L,\perp)$ belongs to this property or not.

We introduce first the coordination relation on sets of transitions (which is in fact an equivalence relation), and then show, that one can safely make refinements along the lines that separate coordinated sets, without losing the considered properties.

Def. 3.1

$\sigma_L \subseteq T \times T$

$(t_1,t_2) \in \sigma_L$ iff $\forall W \in L_\omega : (\#(t_1,W) = \omega) \Leftrightarrow (\#(t_2,W) = \omega)$.

σ_L is called a coordination relation. □

We extend this relation to hold for sets of transitions.

Def. 3.2

$\Sigma_L \subseteq 2^T \times 2^T$

$(T_1,T_2) \in \Sigma_L$ iff $\forall W \in L_\omega : (\underline{\#}(T_1,W) = \omega) \Leftrightarrow (\underline{\#}(T_2,W) = \omega)$.

When $(T_1,T_2) \in \Sigma_L$, we call them coordinated. □

Due to the result of Lemma 2.23 we shall not consider the dual definition with the function "$\overline{\#}$" - it does not look very appealing.

Def. 3.3

For the two partitions of A(L): π_1 and π_2, π_1 is called a <u>coordinated refinement</u> of π_2 with respect to L iff

(i) π_1 is a refinment of π_2 $\quad (\pi_1 \leqq \pi_2)$

(ii) $\forall P'' \in \pi_2 \quad \forall P_1', P_2' \in \pi_1$

$$(P_1' \subseteq P'' \ \& \ P_2' \subseteq P'') \Rightarrow (P_1', P_2') \in \Sigma_L$$

π_1 is called a <u>split</u> of π_2 iff $\exists P_1', P_2' \in \pi_1$:

$$\pi_2 = (\pi_1 \setminus \{P_1', P_2'\}) \cup \{P_1' \cup P_2'\}.$$

Obviously every split is a refinement. □

Lemma 3.4

For π_1, π_2 being partitions of T: π_1 is a coordinated refinement of π_2 iff there exists a sequence of partitions of T: $\pi^0, \pi^1, \ldots, \pi^n$ such that

(i) $\pi^0 = \pi_2$, $\pi^n = \pi_1$

(ii) π^{i+1} is a coordinated split of π^i for $i = 0,\ldots,n-1$

We omit an easy and tedious proof here. □

Def. 3.5

A generalized property ψ is antimonotonic with respect to coordination relation, or simply Σ_L-antimonotonic iff for every language L and two partitions π_1 and π_2 on A(L) such that π_1 is a coordinated refinement of π_2 w.r.t. L:

$$(L,\pi_2) \in \psi \Rightarrow (L,\pi_1) \in \psi$$ □

We present here one technical lemma, which will be used in the proof of the main theorem of this section (3.7).

Lemma 3.6

If $(L,\pi) \in$ JUST, then for every $P \in \pi$, $W \in L_\omega$: if there exist infinitely many $u \in W$ such that ($w \leqslant u$ and $uP \in L$) then $\#(P,W) = \omega$.

Proof straight from the definitions. □

Theorem 3.7 (Σ_L-antimonoticities)

(i) LIV

(ii) IMP

(iii) FAIR

(iv) JUST are Σ_L-antimonotonic.

Proof

According to Lemma 3.4 it is sufficient to prove that the implication holds for π_1 being a coordinated split of π_2.

Let L be any language, $\pi_1 = \{P_1, P_2, P_3, \ldots, P_n\}$, $\pi_2 = \{P_{12}, P_3, \ldots, P_n\}$ be partitions on $A(L) = T$ such that π_1 is a coordinated split of π_2.

Obviously $(P_1, P_2) \in \Sigma_L$.

(i) (LIV is Σ_L-antomonotonic).

Assume that $(L, \pi_2) \in$ LIV.

(*) $\forall w \in L : \forall P'' \in \pi_2 : \exists u \in T^* : wuP'' \subseteq L$.

Assume that $(L, \pi_1) \notin$ LIV.

Let $P' \in \pi_1$, $w' \in L$ be such that

(**) $\forall u \in T^* : wuP' \not\subseteq L$.

P' must be either P_1 or P_2, otherwise (*) would be immediately violated. Let us assume, for symmetry, that $P' = P_1$.

Let us construct the following sequence of words:

$$v_0 = w'$$

$$v_{i+1} = v_i u_i t_i \quad \text{for some } t_i \in P_{12} \text{ and } u_i \in T^*.$$

The existence of such a sequence is assured by (*) - at each step we take $w = v_i$, $P'' = P_{12}$.

For every $i \in N$: $t_i \in P_1 \cup P_2$ and $t_i \notin P_1$ (according to our choice of w' and P'), so $t_i \in P_2$ for every $i = 0, 1, \ldots$.
Due to Lemma 1.15 there exists an infinite word $W \in L_\omega$ such that $v_i \in W$ for every $i = 0, 1, \ldots$.

Clearly $\underline{\#}(P_2,v_{i+1}) > \underline{\#}(P_2,v_i)$ for every i, so (Lemma 1.13) $\underline{\#}(P_2,W) = \omega$.

At the same time w' was chosen in such a way that $\underline{\#}(P_1,v_i) = \underline{\#}(P_1,v_{i+1})$, so $\underline{\#}(P_1,W) < \omega$, which contradicts the assumption that $(P_1,P_2) \in \Sigma_L$.

(ii) ($\underline{IMP}$ is Σ_L-antimonotonic).

Let $(L,\pi_2) \in \underline{IMP}$:

(*) $\forall W \in L_\omega$: $\forall P'' \in \pi_2$: $\underline{\#}(P'',W) = \omega$.

Assume that $(L,\pi_1) \notin \underline{IMP}$. Let $P' \in \pi_1$ and $W' \in L_\omega$ be such that $\underline{\#}(P',W) < \omega$. P' can be only one of the sets P_1 or P_2 (otherwise (*) is immediately violated).

Assume for symmetry, that $P' = P_1$.

For $W = W'$ and $P'' = P_{12}$ we have, from (*),

$$\underline{\#}(P'',W) = \omega.$$

But $\underline{\#}(P'',W) = \underline{\#}(P_1 \cup P_2,W) \leq \underline{\#}(P_1,W) + \underline{\#}(P_2,W)$, (we have used Lemma 2.18) so $\underline{\#}(P_2,W) = \omega$, since $\underline{\#}(P_1,W) < \omega$, which is a contradiction to the assumption $(P_1,P_2) \in \Sigma_L$.

(iii) ($\underline{FAIR}$ is Σ_L-antimonotonic)

Let $(L,\pi_2) \in \underline{FAIR}$:

(*) $\forall W \in L_\omega$: $\forall P'' \in \pi_2$: $(|\underline{Y}_L(W,P'')| = \omega) \Rightarrow (|\underline{Y}_W(W,P'')| = \omega)$.

Assume that $(L,\pi_1) \notin \underline{FAIR}$:

Let $W' \in L_\omega$: $P' \in \pi_1$ be such that

$|\underline{Y}_L(W',P')| = \omega$ and $|\underline{Y}_{W'}(W',P')| < \omega$.

Again $P' \in \{P_1,P_2\}$. Assume that $P' = P_1$. From Lemma 2.17 we get

$$\underline{Y}_L(W',P_1) \subseteq \underline{Y}_L(W',P_1 \cup P_2) = \underline{Y}_L(W',P_{12}).$$

But $|\underline{Y}_L(W',P_1)| = \omega$, so $|\underline{Y}_L(W',P_{12})| = \omega$. Using Lemma 1.19 and (*) we deduce that $\underline{\#}(P_{12},W') = |\underline{Y}_{W'}(W',P_{12})| = \omega$.
So $\underline{\#}(P_2,W') = \omega$, since $P_{12} = P_1 \cup P_2$ and $\underline{\#}(P_1,W') < \omega$, which is a contradiction to the assumption $(P_1,P_2) \in \Sigma_L$.

(iv) (JUST is Σ_L-antimonotonic)

Let $(L,\pi_2) \in$ JUST:

(*) $\forall P'' \in \pi_2$: $\forall W \in L_\omega$: $\forall w \in W$: $\exists u \in W$:

$$(w \preccurlyeq u) \ \& \ (uP'' \underline{\in} L \Rightarrow uP'' \underline{\in} W)$$

Assume that $(L,\pi_1) \notin$ JUST:

Let $P' \in \pi_1$, $W' \in L_\omega$, $w' \in W'$ be such, that for every $u' \in W'$:

(**) $(w' \preccurlyeq u') \Rightarrow (u'P' \underline{\in} L \ \& \ u'P' \underline{\notin} W')$

Again $P' \in \{P_1,P_2\}$ - otherwise (*) is violated.

Let $P' = P_1$.

W' is infinite, so there is infinitely many $u' \in W'$ such that (**) is satisfied. This implies, that for infinitely many $u \in W'$: $uP' \underline{\in} L$. Moreover for none of $u \in W'$ such that $w' \preccurlyeq u$: $u'P' \underline{\in} W'$. Hence $\underline{\#}(P_1,W') < \omega$.

Clearly for $P_{12} \in \pi_2$, which includes P_1, there also exist infinitely many $u \in W'$ such that $uP_{12} \underline{\in} L$, and, according to Lemma 3.6, $\underline{\#}(P_{12},W') = \omega$. So $\underline{\#}(P_2,W') = \omega$, since $P_{12} = P_1 \in P_2$ and $\underline{\#}(P_1,W') < \omega$. Again we have got a contradiction to the assumption $(P_1,P_2) \in \Sigma_L$. □

The above theorem can be applied in a nontrivial way only when there exist coordinated pairs of transitions in the language L. The greater the coordination relation is, the more useful is this theorem.

Let us close this section with some conclusions for the case when the coordination relation is the full relation on T.

Def. 3.8

L is totally coordinated iff $\sigma_L = T^2$. □

Corollary 3.9

$\sigma_L = T^2$ iff $\Sigma_L = (2^T)^2$.

So L is totally coordinated iff any pair of subsets of A(L) is coordinated.

Corollary 3.10

$\sigma_L = T^2$ iff $(L,\perp) \in \underline{IMP}$.

Proof - direct from the definitions. □

Lemma 3.11

If L is totally coordinated, then for every partition π on A(L):

(i) $(L,\pi) \in$ IMP

(ii) $(L,\pi) \in$ FAIR

(iii) $(L,\pi) \in$ JUST

(iv) $(L,\pi) \in$ LIV iff $(\bar{L},T) \in$ LIV

Proof

(i) is a consequence of 3.10 and 2.20.

(ii) & (iii) are consequences of (i) and 2.14.

(iii) is a consequence of 2.20 and 3.7. □

From Lemma 3.11 one can easily deduce, that in the case of totally coordinated languages all the considered fairness properties collapse, (and hold), and all the partitions are equivalent with respect to liveness (in particular deadlock-freeness is equivalent to liveness).

The following corollary forms a bridge between the coordination relation and sets of minimal partitions satisfying LIV, IMP and FAIR.

Corollary 3.12

If π is a minimal partition (Corollary 2.20) satisfying LIV or IMP or FAIR then for every $P \in \pi$, and for every partition $\{P_1,P_2\}$ of P

$$(P_1,P_2) \notin \Sigma_L$$

Proof immediate from 3.7. □

4. DEFINITIONS FROM THE PETRI NET THEORY

Def. 4.1

A net is a triple PN = <S,T,F> where S and T are disjoint, nonempty and finite sets, called sets of places and transitions respectively, $F \subseteq (S \times T) \cup (T \times S)$ is a flow relation such that $\forall x \in S \cup T: \exists y \in S \cup T: (x,y) \in F$. □

Let PN = <S,T,F> be an arbitrary, but fixed net. Let $X = S \cup T$.

Def. 4.2

$$\forall x \in X: \quad {}^\bullet x = \{y \in X \mid (y,x) \in F\}$$
$$x^\bullet = \{y \in X \mid (x,y) \in F\}$$

For $X_1 \subseteq X$ $\quad {}^\bullet X_1 = \bigcup_{x \in X_1} {}^\bullet x$

$$X_1{}^\bullet = \bigcup_{x \in X_1} x^\bullet$$ □

Def. 4.3

An underlying graph of PN is a directed graph <V,E>, where

$$V = X$$
$$E = \{(x,y) \in X \times X \mid (x,y) \in F\}$$ □

Def. 4.4

PN is strongly connected iff its underlying graph is strongly connected. □

Def. 4.5

PN is a T-graph iff $\forall s \in S: |{}^\bullet s| = |s^\bullet| = 1$

PN is a S-graph iff $\forall t \in T: |{}^\bullet t| = |t^\bullet| = 1$ □

Def. 4.6

$PN_1 = \langle S_1, T_1, F_1 \rangle$ is a subnet of N iff $S_1 \subseteq S$, $T_1 \subseteq T$, $F_1 = F \cap (S_1 \times T_1 \cup T_1 \times S_1)$. □

Let $PN_1 = \langle S_1, T_1, F_1 \rangle$ be a subnet of PN.

Def. 4.7

Let $Y \subseteq X$.

PN_1 is said to be a subnet generated by Y iff

$S_1 = S \cap ({}^\bullet Y \cup Y^\bullet \cup Y)$, $T_1 = T \cap ({}^\bullet Y \cup Y^\bullet \cup Y)$

$F_1 = F \cap (S_1 \times T_1 \cup T_1 \times S_1)$. □

Def. 4.8

PN_1 is a T-component (S-component) of PN iff it is a strongly connected T-graph (S-graph) generated by T_1 (S_1). □

Def. 4.9

A marking of PN is a function $M: S \to N$. □

Def. 4.10

Given a net PN and its marking M_0, we call a pair $\langle PN, M_0 \rangle$ a marked Petri net. □

Def. 4.11

By a marked graph we mean any marked net $\langle S,T,F;M_0\rangle$, where $\langle S,T,F\rangle$ is a T-graph.

By a state-machine we mean any marked net $\langle S,T,F;M_0\rangle$, where $\langle S,T,F\rangle$ is an S-graph, and $\sum_{s \in S} M_0(s) = 1$. □

Def. 4.12

PN is a free-choice net iff

$$\forall s \in S \quad |s^\bullet| > 1 \Rightarrow {}^\bullet(s^\bullet) = \{s\}$$ □

Let $MN = \langle PN, M_0\rangle$ be a marked Petri net.

Def. 4.13

A transition $t \in T$ is active at the marking M iff

$$\forall s \in {}^\bullet t: M(s) \geq 1$$

Def. 4.14

$Next \subseteq N^S \times T \times N^S$ is a relation satisfying $(M_1,t,M_2) \in Next$ iff

(i) t is active at M_1

(ii)
$$\forall s \in S \quad M_2(s) = \begin{cases} M_1(s) - 1 & \text{if } s \in {}^\bullet t \setminus t^\bullet \\ M_1(s) + 1 & \text{if } s \in t^\bullet \setminus {}^\bullet t \\ M_1(s) & \text{otherwise} \end{cases}$$

If $(M_1,t,M_2) \in$ Next, then we denote it by

$$M_1[t>M_2,$$

and say that M_2 is <u>reachable</u> from M_1 by <u>firing</u> a transition t.

□

<u>Def. 4.15</u>

Next* $\subseteq N^S \times T^* \times N^S$ is the least relation satisfying

(i) $(M_1,\varepsilon,M_1) \in$ Next*

(ii) if $(M_1,t,M_2) \in$Next and $(M_2,w,M_3) \in$Next*, then $(M_1,tw,M_3) \in$Next*.

If $(M_1,w,M_2) \in$ Next*, then we denote it by $M_1[w>M_2$ and say that M_2 is reachable from M_1 by firing w. □

<u>Def. 4.16</u>

The word $w \in (2^S \cup T)^*$, $w = M_0t_1M_1t_2 \ldots t_nM_n$ is called an <u>augmented firing sequence</u> of MN iff $\forall i = 1,\ldots,n-1$

$M_i[t_i>M_{i+1}$.

Let the set of <u>augmented firing sequences</u> of MN be denoted by AFS_{MN}. □

<u>Def. 4.17</u>

$IAFS_{MN} = 2^{AFS_{MN}} \cap ([M_0] \cup T)^{\omega}$.

$IAFS_{MN}$ is the set of <u>infinite augmented firing sequences</u> of MN. □

<u>Def. 4.18</u>

$[M_0] = \{M \in N^S \mid \exists w \in T^*: M_0[w>M\}$.

$[M_0]$ is called the <u>reachability set</u> of MN. □

Def. 4.19

$L_{MN} = \{w \in T^* \mid \exists M \in N^S: M_0[w>M\}$.

L_{MN} is called a language of MN. □

Corollary 4.20

For every marked Petri net MN the language L_{MN} is prefix-closed. □

Def. 4.21

MN is safe iff $\forall M \in [M_0]: \forall s \in S: M(s) \leq 1$. □

Def. 4.22

PN is called pure iff $F \cap F^{-1} = \emptyset$.

5. GENERALIZED LIVENESS AND FAIRNESS PROPERTIES IN MARKED GRAPHS, STATE MACHINES AND IN LIVE AND SAFE FREE-CHOICE NETS (LSFC NETS)

However, it is difficult to get some significant results concerning behavioural properties of Petri nets in general, many of them can be achieved, when we concentrate on some subclasses of Petri nets.

The main theorem of this chapter will be strongly based on the results obtained in [ThV], where the reader can find a concise and clear presentation of LSFC nets.

We start our investigation from the class of marked graphs. Since they allow no nondeterminacy, the infinite words, they produce, are very similar to each other.

Theorem 5.1

Marked graph languages are totally synchronized.

Proof - based on results of [CHEP], omitted. □

Corollary 5.2

If L is a language of a marked graph, then it is impartial, fair and just, and, moreover, deadlock-freeness is equivalent to liveness of L.

State machines are dual in some sense to marked graphs. They provide nondeterministic and sequential behaviour.

Corollary 5.3

If L is a state machine language, then for every π being a partition of A(L):

(i) $(L,\pi) \in \underline{LIV}$

(ii) $(L,\pi) \in \underline{IMP} \Leftrightarrow (L,\pi) \in \underline{FAIR}$

(iii) $(L,\pi) \in \underline{JUST}$ if the net is pure.

Proof

(i) State machine languages are live, so the thesis comes from the monotonicity theorem.

(ii) The proof can follow the proof of Theorem 6.2 from [ThV] with some minor changes: first of all relativization to partitions is needed, and secondly an observation, that fairness, as defined in this paper, implies local fairness defined in [ThV].

(iii) comes from the observation, that there is only one token flowing in state machines. □

We now turn to LSFC nets. From the definition, LSFC nets are live, so if L is an LSFC net language, then, obviously, for every partition π of A(L), $(L,\pi) \in \underline{LIV}$ (monotonicity).

The main result of this chapter will be a structural characterization of π-impartiality in LSFC nets. As in the case of state machines $(L,\pi) \in \underline{IMP}$ iff $(L,\pi) \in \underline{FAIR}$ in LSFC nets.

Lemma 5.4

If $PN = \langle S,T,F;M_0 \rangle$ is an LSFC net, $M \in [M_0]$, $\sigma \in T^+$ such that

$$M[\sigma>M$$

then there exists a T-component $PN_i = \langle S_i,T_i,F_i;M_{0_i} \rangle$ such that $\forall t \in T_i$: $\#(t,\sigma) > 0$. □

The following result, however, proved for LSFC nets is valid as well for state machines, since every state machine is also an LSFC net.

Theorem 5.5

Let PN = $<S,T,F;M_0>$ be an LSFC net, L be a language of PN,
TC = $\{TC_1,\ldots,TC_m\}$ be a set of T-components of PN,
$TC_i = <S_i,T_i,F_i;M_{0_i}>$
$\pi = \{P_1,\ldots,P_n\}$ be a partition of T.
Then

$(L,\pi) \in \underline{IMP}$ iff $\forall i = 1,\ldots,m;\ \forall j = 1,\ldots,n$
$T_i \cap P_j \neq \emptyset$.

Proof (outline)

"$\Rightarrow$"

Assume that $T_i \cap P_j = \emptyset$ for some i and j. A subnet generated by T_i is a strongly connected T-component, so $\forall s \in S_i$
$s^\bullet \cap T_i \neq \emptyset$.

From Lemma 3.2 of [ThV] we deduce the existence of such $w \in L$, $\sigma \in T_i^+$, and $M \in [M_0]$, that $M[\sigma> M$.
It is easy to see now that $\forall k \in N$: $v_k = w\sigma^k \in L$.

Let $V \in L_\omega$ be such that $v_i \in V$ for every $i \in N$ (existence of such V is assured by Lemma 1.18).
$\underline{\#}(P_j,V) = \underline{\#}(P_j,w) < \omega$, because $\sigma \in T_i^{\neq}$ and $T_i \cap P_j = \emptyset$.
So $(L,\pi) \notin \underline{IMP}$.

"⇐"

Assume that for every $i = 1,\ldots,m$ and $j = 1,\ldots,n$

$T_i \cap P_j \neq \emptyset$.

Let us take any $W \in L$, $W = t_{i_1} t_{i_2} \ldots$ and a corresponding infinite augmented firing sequence

$$\tau = M_0\ t_{i_1}\ M_1\ t_{i_2}\ M_2 \ldots \qquad \tau \in IAFS_{MN}$$

PN is safe, so the number of reachable markings is finite.

From König's lemma we conclude that there exists $\hat{M} \in [M_0]$ such that $\#(\hat{M},\tau) = \omega$.

From Lemma 5.4 we conclude that for every word

$\tau_k = M_{i_k}\ t_{i_{k+1}} \ldots t_{i_{k+l}}\ M_{i_{k+l}}$ such that $M_{i_k} = \hat{M} = M_{i_{k+l}}$ $l > 0$, there exists a T-component $TC_i \in TC$ such that $\forall t \in T_i$ $\#(t, \tau_k) > 0$.

The number of such τ_k is infinite, so there must exist a T-component TC_{i_0} such that for infinitely many $k \in N$

$$\#(t,\tau_k) > 0 \text{ for every } t \in T_{i_0}.$$

Hence $\forall t \in T_{i_0} : \#(t,\tau) = \omega$.

Let $t_{i_0 j} \in T_i \cap P_j$ for every $j = 1,\ldots,n$.

$\underline{\#}(P_j,W) = \underline{\#}(P_j,\tau) = \sum_{t \in P_j} \#(t,\tau) \geq \#(t_{i_0 j},\tau) = \omega$.

So $(L,\pi) \in \underline{IMP}$, since P_j and W were chosen arbitrarily.

□

Till now no satisfactory results concerning justice of LSFC nets can be presented. There is a conjecture [Thi] that fairness and justice coincide, iff every al-reduction [Hack], [ThV] contains one T-component.

6. FURTHER EXTENSIONS

According to the aim of greater clarity and simplicity of presentation, the whole theory was based on the idea of sets of transitions defined by some partition of A(L).

This brings two limitations: first of all we require every transition to belong somewhere, secondly the number of sets in which a transition is included cannot be greater than one.

To illustrate that such limitations can be awkward, let us consider the following situation:

Example 6.1

Assume that we use Petri nets to model some system, which posesses an "abort" action. If we want to model this action by a separate transition, then if an infinite execution is possible, then by no means the net can be impartial (in the classical sense). The "abort" action stops every execution and it should not occur even twice in any word, especially infinite.

However the rest of the net can be "impartial". We would like to express this fact, and carry this information.

Within our framework one should define a partition in which the "abort" action could be found in an equivalence class together with some other action which occurs impartially. But then we would be forced to choose one of the equally important actions, from the impartiality point of view, making it an

unhappy host forced to be a companion of an ugly guest.

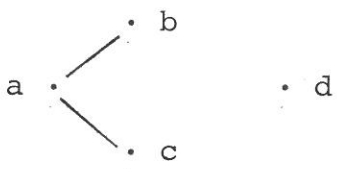
It would break down the symmetry of the system. There exist at least two ways to solve this problem.

One of them is to allow overlapping of sets, in which case we should consider covers rather than partitions, so that we would be allowed for instance to attach an "abort" action to every transition.

The other possibility is just to forget about the "abort" action, and concentrate only on some subset of transitions which interests us.

From the mathematical point of view the first situation requires covers instead of partitions, to be considered; the second one - partitions on a subset of T. We can also allow both of them at the same time (i.e. covers on a subset of T).

In order to make it more precise, let us assume that a symmetrical relation R is given on T. This relation will indicate, which elements we want to join in groups. For instance if $T = \{a,b,c,d\}$, $R = \{(a,b),(b,a),(a,c),(c,a)\}$, then we precisely want a and b to be joined, as well as a and c, but no others



As a matter of fact, we should also add pairs (a,a), (b,b) and (c,c) but not necessarily (d,d), in order to leave us a chance to express that we would like to "forget" about d. Hence we shall require that R is also reflexive on dom(R). ($dom(R) = \{t \in T \mid \exists t' \in T: (t,t') \in R\}$.)

Def. 6.2

A set $T' \subseteq T$ is a clique of R iff

(i) $\forall t_1, t_2 \in T': (t_1, t_2) \in R$

(ii) $\forall t_3 \in T \setminus T': \exists t \in T': (t, t_3) \notin R$. □

So every clique is a set of transitions, each two of them are in the relation R (condition (i)), and it is maximal such a set (condition (ii)).

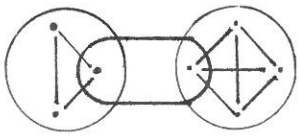


Figure 4

The relation presented by the above graph determinates three cliques.

Def. 6.3

Let $\mathcal{R}$ be a set of relations on T, which are symmetric and reflexive on their domains.

Let $\mathcal{R}_\rho \subseteq \mathcal{R}$ be a set of these relations of $\mathcal{R}$, which are reflexive on T.

$\mathcal{R}_\tau \subseteq \mathcal{R}$ is a set of these relations of $\mathcal{R}$ which are transitive.

$$\mathcal{R}_{\rho\tau} = \mathcal{R}_\rho \cap \mathcal{R}_\tau .$$ □

Def. 6.4

For $R \in \mathcal{R}$, $\mathcal{R}' \subseteq \mathcal{R}$.

$C(R) = \{T' \subseteq T \mid T' \text{ is a clique of } R\}$.

$C(\mathcal{R}') = \{C(R) \mid R \in \mathcal{R}'\}$. □

Def. 6.5

Let Π be a set of partitions of T

$\mathcal{P}$ be a set of covers of T

Π_s be a set of partitions of subsets of T

$\mathcal{P}_s$ be a set of covers of subsets of T. □

Theorem 6.6

$\Pi = C(\mathcal{R}_{\rho\tau})$

$\mathcal{P} = C(\mathcal{R}_\rho)$

$\Pi_s = C(\mathcal{R}_\tau)$

$\mathcal{P}_s = C(\mathcal{R})$ □

Proof - omitted. □

To illustrate Theorem 6.6 we present the following diagram:

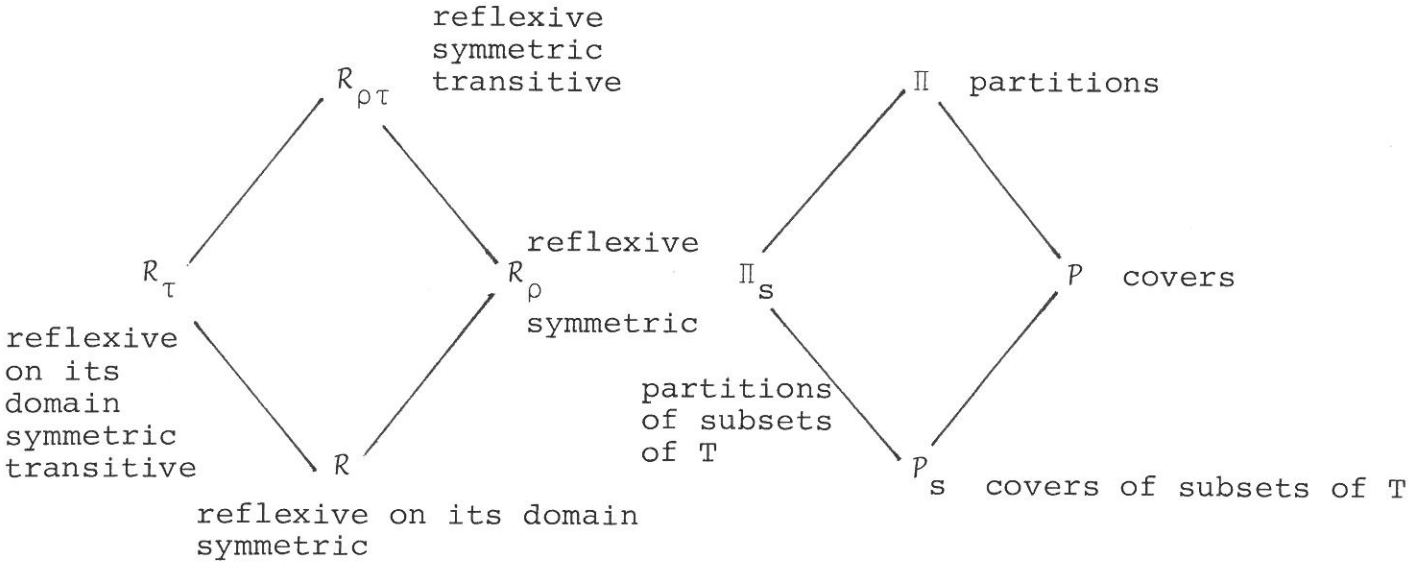


Figure 5

Vertices of these two graphs correspond to each other.

Hence every

- symmetric+reflexive+ transitive relation determines a unique partition
- symmetric + transitive relation determines a unique partition on a subset of T (reflexivity on the domain is implied by symmetry and transitivity)
- symmetric + reflexive relation determines a unique cover on a subset of T
- symmetric + reflexive on its domain relation determines a unique cover of a subset of T.

and vice versa (i.e. every partition determines a unique symmetric, reflexive and transitive relation etc.)

Remark 6.7

When we want to talk about the considered properties related to some cover p of some $T' \subseteq A(L)$, we should consider this generalized property to be a set of pairs

$$(L,p)$$

where L is a language, p is a cover of some $T' \subseteq A(L)$.
For example generalized in this way the definition of Liveness looks like the following :

$$(L,p) \in \underline{LIV} \quad \text{iff} \quad \forall P \in p \; \forall w \in L \; \exists u \in T^*: wut \in L \qquad \square$$

In order to consider the monotonicities of properties one should first of all define an ordering on covers. In order to obtain the wanted results it is necessary to allow ordering only between covers which cover the same subset of T. So if we say that cover p' is greater than p_2 then we certainly mean that

$$\bigcup_{P' \in p'} P' = \bigcup_{P'' \in p''} P''$$

Now, when we have two covers that cover the same subset of T, we must make a choice between one of the three known orderings of covers:

Def. 6.8

Let p', p" be two covers of the same set.

$p' \leq_1 p''$ iff $\forall P' \in p' : \exists P'' \in p'' : P' \subseteq P''$ (Smyth ordering)

$p' \leq_2 p''$ iff $\forall P'' \in p'' : \exists P' \in p' : P' \subseteq P''$ (Hoare ordering)

$p' \leq_3 p''$ iff ($p' \leq_1 p''$ and $p' \leq_2 p''$) (Egli-Milner ordering)

All three orderings differ from each other and build up three different lattices of covers. However for the lattice of partitions all three of them coincide.

For the monotonicity results we need $\leq_2$ or $\leq_3$ to be used ($\leq_1$ is not sufficient). □

The Σ_L-antimonotonicity results do not look very nice for covers, since the notion of split is violated. But certainly some further work can be done in this direction.

7. CONCLUSIONS

A generalization of liveness and fairness properties has been done. Liveness, impartiality, fairness and justice of sets of actions has been considered. The first three properties has turned out to be monotonic (they are preserved when the sets are being enlarged).

A coordination relation on sets of actions has been introduced and its relevance to the antimonotonicity results has been shown (all the considered properties are antimonotonic with respect to this relation).

The theory has been illustrated on some simple subclasses of Petri nets, however, the formalism is suited for much larger class of models.

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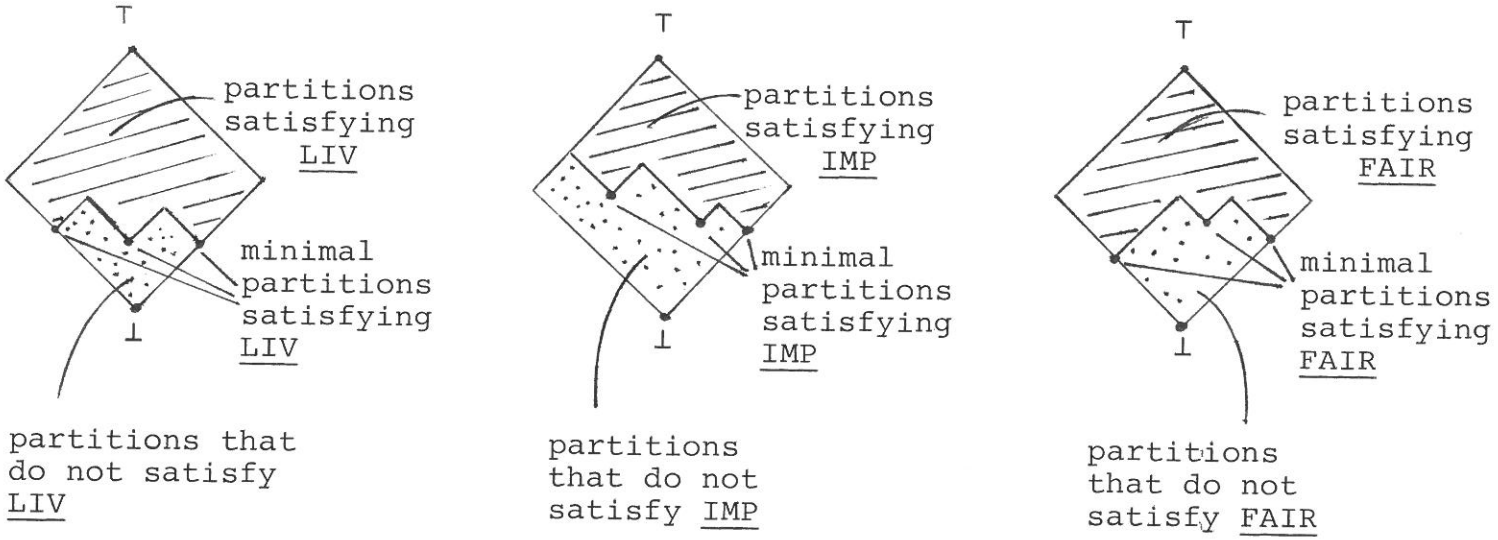
T
partitions satisfying LIV
minimal partitions satisfying LIV
⊥
partitions that do not satisfy LIV
T
partitions satisfying IMP
minimal partitions satisfying IMP
⊥
partitions that do not satisfy IMP
T
partitions satisfying FAIR
minimal partitions satisfying FAIR
⊥
partitions that do not satisfy FAIR

Figure 2