FLOW ANALYSIS OF LAMBDA EXPRESSIONS

by

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0. INTRODUCTION

Overview

Program flow analysis determines properties of the computation(s) induced by a program without actually running it. The purpose is usually to extract information which may be used to optimize the program, compile code, certify the absence of certain runtime errors, etc. Excellent introductions to flow analysis may be found in [Hec77] and [Aho77].

We describe a method to analyze the data and control flow during mechanical evaluation of lambda expressions. The method produces a finite approximate description of the set of all states entered by a call-by-value $\lambda$-calculus interpreter; a similar approach can easily be seen to work for call-by-name. A proof is given that the approximation is "safe", i.e. that it includes descriptions of every intermediate $\lambda$-expression which occurs in the evaluation.

From a programming languages point of view the method extends previously developed interprocedural analysis methods to include both local and global variables, call-by-name or call-by-value parameter transmission and the use of procedures both as arguments to other procedures and as the results returned by them.

The main emphasis is on development of the flow analysis framework rather than on applications, although a few are given (termination, finiteness, dependence, constant propagation). Other familiar analyses easily fit into the same framework, e.g. available expressions and deciding whether a base function's arguments always have the right type.

The information gathered could be used to compile unusually efficient code for programming languages based on the $\lambda$-calculus such as LISP and SCHEME [Ste76]. Hopefully it will be possible to extend these methods to the flow analysis of denotational definitions. The use of such analyses in compiler generation was described in [JoS60] and provided the initial motivation for this study.

The methods developed here are not limited to the $\lambda$-calculus, but may be applied to any programming language whose semantics are specified by (or specifiable) by a definitional interpreter using recursively defined data structures. The $\lambda$-calculus was chosen because of the challenge of tracing control and data flow in
a computation; it provides a "worst-case" example of many problems encountered in interprocedural analysis.

Related Work

Lambda calculus machines include the SECD and CUCH machines of Landin and Böhm ([Lan64], [Böhm72]) and those due to Reynolds, Wegner and McGowan ([Rey72], [Weg66], [McG70]). McGowan and Plotkin ([McG70], [Plo75]) have proved correctness of their machines, and Plotkin further investigates a number of questions concerning call-by-name and call-by-value.

Interprocedural flow analysis has been investigated by (among others) Rosen, Cousot and Cousot, and Sharir and Pnueli ([Ros79], [Cou77], [Sha80]). Levy has developed sufficient conditions for termination of $\beta$-reduction sequences in [Lev75], and Mycroft ([Myc80]) developed sufficient conditions for the replacement of call-by-need by call-by-value in schemes of recursion equations using the flow analytic idea of "abstract interpretation". Pleban is currently doing a flow analysis of the SCHEME language, expressed using denotational semantics [Ple80].

Outline of the Paper

In the first section we introduce the OI interpreter, a nondeterministic machine which can perform an arbitrary sequence of outside-in $\beta$- and $\delta$-reductions on a closed lambda expression. The initial state of the OI interpreter for input $M_0$ will be $\text{Load}(M_0)$, and every OI state $\sigma$ will represent a $\lambda$-expression $\text{Unload}(\sigma)$. A computation $\text{Load}(M_0) = \sigma_0 \Rightarrow \sigma_1 \Rightarrow \sigma_2 \Rightarrow \cdots$ will correspond to a series of $\beta$-reductions, $\delta$-reductions or identity transformations.

There are two reasons for introducing yet another $\lambda$-calculus interpreter. First, the OI machine can naturally be restricted to yield deterministic call-by-name and call-by-value submachines CBN and CBV. While we analyze only CBV (because of its similarity with existing programming languages) it will be apparent that CBN can be analyzed by the same methods. Second, both CBV and CBN seem to be significantly simpler than other $\lambda$-calculus interpreters, which in turn simplifies our model-building process. The appendix* contains correctness proofs for these machines.

Section 2 develops analysis methods for a closed $\lambda$-expression $M_0$ without constants; this we call the control flow analysis of the call-by-value computation. The result of this is a safe description of

$$\text{States}(M_0) = \{ \sigma | \text{CBV enters state } \sigma \text{ during its computation on } M_0 \}$$

* Appendix omitted in this preliminary version.
A lattice \( D \) will be defined whose elements \( \delta \) will each describe a set of states. An effectively computable method will be described to obtain from \( M_0 \) a description \( \delta(M_0) \in D \). It will be proven that this description is "safe" in the sense that every state \( \sigma \in \text{States}(M_0) \) is represented in \( \delta(M_0) \). Safeness implies that answers to questions about the computation which can be answered by examination of \( \delta(M_0) \) can at worst err "on the safe side", since every state in \( \text{States}(M_0) \) is accounted for. However precise answers cannot be given to all such questions since some (such as the halting problem) are undecidable. Technically this occurs since \( D \) is finite with size recursively bounded in the size of \( M_0 \), so some information must be lost.

Section 2 concludes by constructing a context-free grammar \( G(M_0) \) which generates linear representations of all \( \sigma \) in \( \text{States}(M_0) \), followed by a proof that safe answers may be computably obtained to several questions about the computation.

Section 3 extends the method to handle \( \lambda \)-expressions with constants and \( \delta \)-reduction, using an approximation lattice to describe effectively sets of constant values. An example is given which is suitable for constant propagation, i.e. to find out which variables only receive constant values, and what those values are. Applications to error-checking and type-correctness of base functions are considered.

Section 4 ends with conclusions, future directions and acknowledgments.

Notational Conventions

The power set of \( X \), written \( \mathcal{P}(X) \) is the set of all subsets of \( X \).

Given sets \( X \) and \( Y \), \( X \mathcal{P} Y \) is the set of partial functions from \( X \) to \( Y \). Given \( f \in (X \mathcal{P} Y) \), \( \text{Domain}(f) \) is its domain. Two functions in \( X \mathcal{P} Y \) are equal iff their domains are equal and they have the same values on arguments for which they are defined. If \( x \in X \), \( y \in Y \) and \( f \in X \mathcal{P} Y \) then \( f[y/x] \) denotes the unique partial function \( f' \) such that \( f'(z) = f(z) \) if \( z \neq x \) and \( f'(x) = y \).

A function \( f \in X \mathcal{P} Y \) with \( \text{Domain}(f) = \{x_1, \ldots, x_n\} \) may be written as \( \{x_1 \mapsto f(x_1), \ldots, x_n \mapsto f(x_n)\} \). The totally undefined function is written \( \{\} \).

Given a relation \( \rightarrow \) (always in an infix notation), \( \mathcal{P} \) is its \( n \)th power (\( n \geq 0 \)), \( \mathcal{P} \) is its transitive closure and \( \mathcal{P} \) is its transitive reflexive closure.

Inductive definitions will be written in the style of the abstract syntax of McCarthy [McC63]; for example binary lists can be defined by \( \text{List} ::= \text{Atom} \mid \text{List} \text{List} \) where a list in \( \text{List} \) may be thought of as an abstract syntax tree. This notation will be extended to encompass sets of partial functions, e.g. \( E ::= \text{Var} \mathcal{P} \text{Cl} \) where \( \text{Var} \) is a countable set of variables. This may also be viewed as defining abstract syntax trees, with at most one subtree of type \( \text{Cl} \) for each variable. The notation is similar to VDM notation [Bjø78].
The Lambda Calculus

Given predefined disjoint sets \( \text{Var} = \{ x, y, z, \ldots \} \) and \( \text{Con} = \{ a, b, c, \ldots \} \) of variables and constants respectively, the set of \( \lambda \)-calculus terms \( \text{Lam} = \{ M, N, \ldots \} \) is the smallest set such that

1. Any variable or constant is in \( \text{Lam} \).
2. If \( x \) is a variable and \( M \) is in \( \text{Lam} \) then the abstraction \( \lambda xM \) is in \( \text{Lam} \).
3. If \( M \) and \( N \) are in \( \text{Lam} \) then the combination \( MN \) is also in \( \text{Lam} \). \( M \) is its rator and \( N \) is its rand.

A term is a value if it is not a combination.

This inductive definition may also be written as follows, using abstract syntax:

\[
\text{Lam} ::= \text{Var} \mid \text{Con} \mid \lambda \text{Var Lam} \mid \text{Lam Lam}
\]

The free and bound variables \( FV(M) \) and \( BV(M) \) of a term \( M \) are defined by

1. \( FV(a) = \emptyset \); \( FV(x) = \{ x \} \); \( FV(MN) = FV(M) \cup FV(N) \); \( FV(\lambda xM) = FV(M) \setminus \{ x \} \)
2. \( BV(a) = \emptyset \); \( BV(x) = \emptyset \); \( BV(MN) = BV(M) \cup BV(N) \); \( BV(\lambda xM) = BV(M) \cup \{ x \} \)

A term \( M \) is closed if \( FV(M) = \emptyset \). The substitution prefix \([M/x]\) defines the following operation on \( \text{Lam} \): \([M/X]N \) is the result of substituting \( M \) for all free occurrences of \( x \) in \( N \), renaming variables of \( N \) as necessary to avoid capturing bound variables as in \([\text{Cur}58]\). Plotkin calls a closed term a program.

The set of contexts \( C[ ] \) is defined by

\[
\text{Ctx} ::= [ ] \mid \text{Ctx Lam} \mid \text{Lam Ctx} \mid \lambda \text{Var Ctx}
\]

A context may be viewed as a lambda expression with a "hole" \([ ] \) in it. Noting that a context would be a \( \lambda \)-expression if \([ ] \) were regarded as a variable, we define \( C[M] \) to be the result of "filling the hole" in context \( C[ ] \) by term \( M \).

Now supposing we are given a partial function

\[
\text{Constapply} : \text{Con} \times \text{Con} \rightarrow \text{Closed Values}
\]

we define the reduction relation \( > \) on terms by

1. \( \lambda xM > \lambda y[y/x]M \) (if \( y \notin FV(M) \)) \( \alpha \) reduction
2. \( (\lambda xM)N > [N/x]M \) \( \beta \) reduction
3. \( ab > \text{Constapply}(a, b) \) (if this is defined) \( \delta \) reduction
4. \( M > N \) for any context \( C[ ] \) \( \gamma \) reduction in context
Note that if \( M \) is closed and \( M > N \) without \( \alpha \) reduction then no renaming occurs. Define an outside-in context to be one formed without the rule \( \text{Ctx} := \lambda \text{Var Ctx} \), so the "hole" is not in the scope of any \( \lambda \). We write \( M \triangleright N \) if \( M > N \) by \( \beta \) or \( \delta \) reduction, possibly in an outside-in context.

A machine-independent definition (from [Plo75]) of call-by-value evaluation is given by the partial function \( \text{eval}_V : \text{Programs} \not\rightarrow \text{Programs} \) defined recursively as follows:

\[
\text{eval}_V(a) = a; \quad \text{eval}_V(\lambda x M) = \lambda x M;
\]

\[
\text{eval}_V(MN) = \begin{cases} 
\text{eval}_V([N'/x]M') & \text{if } \text{eval}_V(M) = \lambda x M' \text{ and } \text{eval}_V(N) = N' \\
\text{eval}_V([N/x]M) & \text{if } \text{eval}_V(M) = a, \text{ eval}_V(N) = b \text{ and } \text{Constapply}(a, b) = a' \text{ is defined}
\end{cases}
\]

The call-by-name evaluation function \( \text{eval}_N : \text{Lam} \not\rightarrow \text{Lam} \) is similarly defined:

\[
\text{eval}_N(a) = a; \quad \text{eval}_N(\lambda x M) = \lambda x M;
\]

\[
\text{eval}_N(MN) = \begin{cases} 
\text{eval}_N([N'/x]M') & \text{if } \text{eval}_N(M) = \lambda x M' \\
\text{eval}_N([N/x]M) & \text{if } \text{eval}_N(M) = a, \text{ eval}_N(N) = b \text{ and } \text{Constapply}(a, b) = a' \text{ is defined}
\end{cases}
\]

It is shown in [Plo75] that these are good definitions of partial functions. Note that both could have been restricted to Programs \( \not\rightarrow \) Closed values. The following is easily shown.

**Lemma 0.1** If \( M \) is closed and \( \text{eval}_V(M) \) is defined then \( M \phantom{\triangleright}^{*} \text{eval}_V(M) \); and similarly for \( \text{eval}_N(M) \).

1. **LAMBDA CALCULUS INTERPRETERS**

   We first introduce a nondeterministic interpreter which can do arbitrary outside-in reduction sequences and some lemmas about its behaviour. This machine is then restricted to yield two deterministic interpreters CBV and CBN which are proved to perform correctly call-by-value and call-by-name reductions. The terminology and methods of this section owe much to [Plo75].

   **The OI interpreter**

   The OI interpreter is given by a set \( \Sigma \) of states and a binary transition relation \( \rightarrow_{OI} \) on \( \Sigma \); its data structures and transition rules are summarized in Figure 1. Auxilliary functions Load: Closed terms \( \not\rightarrow \Sigma \) and Unload: \( \Sigma \not\rightarrow \text{Lam} \) are used to initia-
lize the machine and to read out the λ-expression denoted by a state.

**Data Structures**

To avoid explicit substitutions into λ-expressions (in the interest of efficiency) an expression will be represented in an interpreter state by a closure of the form \((M, e)\) where \(M\) is a term and \(e \in E\) is an environment binding its free variables (if any) to other closures. Further, a closure may take the form \(cl_{1} \cdot cl_{2}\), i.e. a combination of two closures. The function \(\text{Real}: Cl \rightarrow \text{Lam}\) mapping the set \(Cl\) of all closures into the λ-expressions they denote is given as follows.

In this paper every closure \((M, e)\) will satisfy \(FV(M) \subseteq \text{Domain}(e)\), so \(\text{Real}(cl)\) will always be closed.

\[
\begin{align*}
\text{Real}(cl_{1} \cdot cl_{2}) &= \text{Real}(cl_{1}) \cdot \text{Real}(cl_{2}) \\
\text{Real}((M, e)) &= \left[\text{Real}(e(x_{1})/x_{1})/x_{1}\right] \ldots \left[\text{Real}(e(x_{n})/x_{n})/x_{n}\right] \cdot M \\
\text{where Domain}(e) &= \{x_{1}, \ldots, x_{n}\}
\end{align*}
\]

To extend the idea of outside-in contexts to apply to closures we define the set of Cl-contexts by \(C ::= [ ] \mid C \cdot C \mid C \cdot C\). The notations \(c[[ ]]\) and \(c[cl[ ]]\) will have many uses; they denote the Cl-contexts obtained by replacing the single occurrence of \([ ]\) in \(c\) by the Cl-contexts \([ ]\) and \(cl[ ]\), respectively.

A state \(\sigma\) in \(\Sigma\) may be viewed intuitively as a closure in which a particular subclosure has been identified to be processed next by the interpreter. Formally \(\sigma\) is a pair consisting of a Cl-context \(c\) and a closure \(cl\), written in the suggestive notation \(c[cl]\) (which does not indicate substitution). If \(c = c_{1}[cl[ ]]\) then \(c[cl]\) may also be written \(c_{1}[cl_{1}[cl_{2}]]\), and similarly \(c_{1}[[ ]][cl_{2}[cl_{1}]]\) may be written \(c_{1}[[cl_{1}][cl_{2}]]\).

**Load**: Programs \(\rightarrow \Sigma\) and Unload: \(\Sigma \rightarrow \text{Lam}\) are now defined, using \([ ]\) for the initial environment with empty domain. Note that Unload\((c[cl])\) may be seen as the result of substituting \(\text{Real}(cl)\) into the context naturally obtained from \(c\).

\[
\begin{align*}
\text{Load}(M_{0}) &= [(M_{0}, \{ \})] \\
\text{Unload}([cl]) &= \text{Real}(cl) \\
\text{Unload}(c[cl_{1}[cl_{2}]]) &= \text{Unload}(c[[cl_{1}][cl_{2}]] = \text{Unload}(c[cl_{1}cl_{2}])
\end{align*}
\]

**Transition Rules**

Figure 1 contains the transition rules defining \(\Rightarrow\) and repeats the definitions of Cl, E etc. in more compact form.
Data Structures

\[
\begin{align*}
\text{cl} & : \text{Cl} ::= \text{Lam } E \mid \text{Cl Cl} & \text{Closures} \\
\text{e} & : E ::= \text{Var } P \mid \text{Cl} & \text{Environments} \\
\text{c[ ]} & : C ::= \text{c[ ]} \mid \text{Cl Cl} \mid \text{Cl C} & \text{Clcontexts} \\
\sigma & : \Sigma ::= \text{C[Cl]} & \text{States}
\end{align*}
\]

Transition Rules

1. \( \text{c[[}\lambda xM, e\text{]cl]} \rightarrow_0 c[(M, e\{c/\lambda\text{]cl}\}] \) \( \beta \) reduction
2. \( \text{c[[}\text{a, e}](\text{b, e'})]} \rightarrow_0 c[(\text{a'}, \{ \})] \) \( \delta \) reduction
   (if \( \text{a'} = \text{Constapply}(\text{a, b}) \) is defined)
3. \( \text{c[(x e)]} \rightarrow_0 c[e(x)] \) variable expansion
   (if \( e(x) \) is defined)
4. \( \text{c[(MN, e)]} \rightarrow_0 c[(M, e)(N, e)] \) combination
5. \( \text{c[cl_1 cl_2]} \rightarrow_0 c[[\text{cl_1]}\text{cl_2]} \) scan rator
6. \( \text{c[[cl_1]}\text{cl_2]} \rightarrow_0 c[\text{cl_1 cl_2]} \) return from rator
7. \( \text{c[cl_1 cl_2]} \rightarrow_0 c[\text{cl_1 [cl_2]}] \) scan rand
8. \( \text{c[cl_1 [cl_2]} \rightarrow_0 c[\text{cl_1 cl_2]} \) return from rand

**Figure 1. OI Interpreter.**

**Example Computation**

Following is an example OI computation on \( M_0 = (\lambda \text{ff7)\text{square}} \):

\[
\begin{align*}
\text{Load}(M_0) & = [(\lambda \text{ff7)\text{square, } }\})] & \text{combination} \\
& \rightarrow_0 [(\text{ff7, }\})\text{\text{(square, }\})] & \text{scan rator} \\
& \rightarrow_0 [[[\text{ff7, }\})\text{\text{(square, }\})] & \beta \text{ reduce} \\
& \rightarrow_0 [([f7, f \rightarrow (\text{square, }\emptyset)])] & \text{scan rator} \\
& \rightarrow_0 [(f, e)](7, e)] & \text{call this } e \\
& \rightarrow_0 [[(\text{square, }\})](7, e)] & \text{expand "f"} \\
& \rightarrow_0 [(49, \{ \})] & \delta \text{ reduce}
\end{align*}
\]
Mathematical Justification

The following provides the mathematical justification of the OI interpreter; its proof is straightforward but detailed and so appears in the appendix.

Theorem 1.1

a) If $\sigma_1 \overset{*}{\rightarrow}_I \sigma_2$ and $\text{Unload}(\sigma_1)$ is closed then $\text{Unload}(\sigma_1) \overset{*}{\rightarrow}_I \text{Unload}(\sigma_2)$

b) If $M \overset{*}{\rightarrow}_I N$ and $M$ is closed then $\text{Load}(M) \overset{*}{\rightarrow}_I \sigma$ for some $\sigma$ with $\text{Unload}(\sigma) = N$.

A similar interpreter has been constructed and proven correct which can do arbitrary reduction sequences (not just outside-in), at the expense of more complexity in handling environments. A deterministic restriction of it could provide an alternative to the CUCH machine [Böh72], but is omitted due to the difficulties encountered in approximating a renaming interpreter.

Call-by-Value and Call-by-Name Interpreters

The OI interpreter may be simplified and made deterministic by imposing a consistent ordering on operator and operand evaluation. If the operand is always left unevaluated we have the usual implementation of call-by-name. Figure 2 contains the CBN interpreter; it was obtained from OI by combining transition rules 4 and 5, dropping 2, 6, 7, 8 and simplifying closures by omitting $C_1 ::= C_1 C_1$. This machine has been studied by Schmidt [Sch81].

Define $\text{Last}_N : \Sigma \rightarrow \Sigma$ and $\text{Eval}_N : \text{Lam} \rightarrow \text{Lam}$ by:

$$\text{Last}_N(\sigma) = \begin{cases} 
\text{Last}_N(\sigma') & \text{if } \sigma \Rightarrow \sigma' \text{ for some } \sigma'; \\
\sigma & \text{otherwise}
\end{cases}$$

$$\text{Eval}_N(M_0) = \text{Unload}(\text{Last}_N(\text{Load}(M_0)))$$

Theorem 1.2

$\text{Eval}_N(M_0) = \text{eval}_N(M_0)$ for all closed constant-free terms $M_0$.

Thus CBN correctly performs call-by-name evaluation. Proof is omitted; a very similar proof for call-by-value is found in the appendix. Constants are omitted for simplicity, and because CBN will not be studied in detail. They could be handled by adding transition rules to evaluate the operand in case the operator value is a constant, plus a $\delta$-reduction rule. A simpler alternative (from [Sch81]) is to require that constant functions be applied in postfix order instead of prefix.
Data Structures

\[
\begin{align*}
\text{cl} &: \text{Cl} :::= \text{Lam} \ E \\
\text{e} &: \text{E} :::= \text{Var} \ \overset{P}{\to} \ \text{Cl} \\
\text{c}[ \ ] &: \text{C} :::= [ \ ] | \text{C Cl} | \text{Cl C} \\
\sigma &: \Sigma :::= \text{C[Cl]}
\end{align*}
\]

Closures

Environments

Cl-contexts

States

Transition Rules

1. \[
\text{c}\left[\left(\lambda x \cdot M, e\right)\right]\text{Cl} \quad \overset{\text{cbn}}{\Rightarrow} \quad \text{c}\left[M, e\mid \text{Cl}\mid x\right]
\]
\(\beta\) reduction

2. \[
\text{c}\left[(x, e)\right] \quad \overset{\text{cbn}}{\Rightarrow} \quad \text{c}\left[e(x)\right]
\]
variable expansion
(if \(e(x)\) is defined)

3. \[
\text{c}\left[(M, N, e)\right] \quad \overset{\text{cbn}}{\Rightarrow} \quad \text{c}\left[[M, e][N, e]\right]
\]
combination

Figure 2. CBN Interpreter.

Data Structures

\[
\begin{align*}
\text{cl} &: \text{Cl} :::= \text{Lam} \ E \\
\text{e} &: \text{E} :::= \text{Var} \ \overset{P}{\to} \ \text{Cl} \\
\text{c}[ \ ] &: \text{C} :::= [ \ ] | \text{C Cl} | \text{Cl C} \\
\sigma &: \Sigma :::= \text{C[Cl]}
\end{align*}
\]

Closures

Environments

Cl-contexts

States

Transition Rules

1. \[
\text{c}\left[\left(\lambda x \cdot M, e\right)\right]\text{Cl} \quad \Rightarrow \quad \text{c}\left[M, e\mid \text{Cl}\mid x\right]
\]
\(\beta\) reduction

2. \[
\text{c}\left[(a, e)(b, e)\right] \quad \Rightarrow \quad \text{c}\left[(a', \{ \} \right]
\]
\(\delta\) reduction
(if \(a' = \text{Constapply}(a, b)\) is defined)

3. \[
\text{c}\left[(x, e)\right] \quad \Rightarrow \quad \text{c}\left[e(x)\right]
\]
variable expansion
(if \(e(x)\) is defined)

4. \[
\text{c}\left[(M, N, e)\right] \quad \Rightarrow \quad \text{c}\left[[M, e][N, e]\right]
\]
combination

5. \[
\text{c}\left[\text{cl}_1[\text{cl}_2]\right] \quad \Rightarrow \quad \text{c}\left[\text{cl}_1\text{cl}_2\right]
\]
scan operator
(if \(\text{cl}_2 = (M, e)\) where \(M\) is a closed value)

Figure 3. CBV Interpreter.
For call-by-value we evaluate both operator and operand before $\beta$ or $\delta$ reduction. The result is in Figure 3; it was obtained from OI by combining transition rules 4 and 7, and applying 8 followed by 5 if operand evaluation produced a value. Rule 6 is omitted since either $\beta$ or $\delta$ reduction must occur after operator evaluation. Define $\text{eval}_\vee(M_0) = \text{Unload}(\text{Last}(\text{Load}(M_0)))$ where
\[
\text{Last}(\sigma) = \begin{cases} 
\text{Last}(\sigma') & \text{if } \sigma = \sigma' \\
\sigma & \text{if } \sigma \neq \sigma' \text{ for all } \sigma' \text{ and } \sigma = [c_l] \text{ for some } c_l
\end{cases}
\]

Proof of the following is found in an appendix.

**Theorem 1.3**

$\text{Eval}_\vee(M_0) = \text{eval}_\vee(M_0)$ for all closed terms $M_0$.

**Corollary 1.4**

CBV is computationally equivalent to the SECD machine.

**Proof** Plotkin has shown that SECD computes $\text{eval}_\vee$ in [Plo75].

**A Useful Property**

In every closure $(M, e)$ which was obtained in the example computation, $M$ was a subexpression of $M_0$ or a constant. This is in fact always true.

**Lemma 1.5** Suppose $M$ is closed and $\text{Load}(M_0) \overset{*}{\text{O}_I} c[(M, e)]$. Then

a) $\text{Domain}(e) \subseteq \text{BV}(M_0)$ and

b) $M$ is a subexpression either of $M_0$ or of $\text{Constapply}(a, b)$ for some $a, b \in \text{Con}$.

**Proof** Define "p appears in $c_l$" for closures $c_l$ and $\lambda$-expressions or environments $p$ as follows:

i) $M$ and $e$ appear in $(M, e)$

ii) if $p$ appears in $e(x)$ for some $x \in \text{Domain}(e)$ then $p$ appears in $(M, e)$

iii) if $p$ appears in $c_{l_1}$ or $c_{l_2}$ then it appears in $c_{l_1}c_{l_2}$.

An easy induction on $n$ now verifies that if $\text{Load}(M_0) \overset{n}{\text{O}_I} c[c_l]$ and $p$ appears in $c_l$ or any closure in $c$, then $p$ satisfies a) or b) above.

Lemma 1.5 implies that for each fixed input $M_0$ we may regard OI as operating on occurrences of expressions rather than on arbitrary expressions. This useful property follows from the fact that we only do outside-in reductions. It implies that a computer implementation of an OI $\lambda$-calculus machine can manipulate pointers instead of arbitrary $\lambda$-expressions. Incidentally, the SECD machine also has this property.
The approximations to be developed later will trace occurrences (so all \( x \)'s are not treated alike, for instance), so we introduce some terminology.

\[
\text{Sub}(M_0) = \{ M \mid M \text{ is an occurrence of a subexpression in } M_0 \}
\]

\[
\text{Subcon} = \{ M \mid M \text{ is an occurrence of a subexpression in } N = \text{Constapply}(a, b), \text{ where } a, b \in \text{Con} \text{ and } N \notin \text{Con} \}
\]

\[
\text{Lam}(M_0) = \text{Sub}(M_0) \cup \text{Subcon} \cup \text{Con}
\]

The specialization of the OI interpreter to \( M_0 \) is written \( \text{OI}(M_0) \) and defined in Figure 4. The same concept will also be applied to the call-by-value interpreter CBV yielding its specialized form \( \text{CBV}(M_0) \).

Clearly \( \text{OI}(M_0) \) has a computation \( \sigma_1 \Rightarrow \sigma_2 \Rightarrow \ldots \Rightarrow \sigma_n \) with \( \text{Unload}(\sigma_n) = M \) if and only if \( \text{OI} \) has a corresponding computation \( \text{Load}(M_0) = \sigma_1 \Downarrow \Rightarrow \sigma_2 \Downarrow \Rightarrow \ldots \Rightarrow \sigma_n \Downarrow \) with \( \text{Unload}(\sigma_n \Downarrow) = M \).

\[
\begin{align*}
\text{Data Structures} \\
\text{cI} : \text{Cl} & : = \text{Lam}(M_0) \cup \text{Cl} \cup \text{Cl} & \text{Closures} \\
\text{e} : \text{E} & : = \text{BV}(M_0) \downarrow \uparrow \text{Cl} & \text{Environments} \\
\text{c} \left[ \_ \right] : \text{C} & : = \left[ \_ \right] \cup \text{Cl} \cup \text{Cl} \cup \text{Cl} & \text{Cl-contexts} \\
\sigma : \Sigma & : = \text{C}[\text{Cl}] & \text{States}
\end{align*}
\]

\[
\text{Transition Rules}
\]

Identical to those of OI, but regarded as operating on elements of \( \text{Lam}(M_0) \).

\[
\text{Figure 4. OI}(M_0) \text{ Interpreter.}
\]

2. **ANALYSIS OF CONTROL FLOW**

Let \( M_0 \) be a given closed \( \lambda \)-expression without constants, and let \( \text{CBV}(M_0) \) be the CBV interpreter specialized to \( M_0 \). Define

\[
\text{States}(M_0) = \{ \sigma \mid \text{Load}(M_0) \Downarrow \Rightarrow \sigma \text{ by } \text{CBV}(M_0) \}
\]

Note that \( \text{CBV}(M_0) \) can only halt by entering a state of the form \( \left[ (\lambda x M, e) \right] \), so no "error halts" are possible. It will be shown that a safe description \( \delta(M_0) \) may be effectively obtained as follows.
1. A method will be developed to represent finitely the data structures of CBV($M_0$), yielding a finite lattice $D$ containing computation descriptions $\delta$.

2. For each $\delta \in D$ a representation relation $\overset{\delta}{\sigma} \in \Sigma \times \Sigma'$ will be defined, where $\Sigma'$ models the states of CBV($M_0$). The relation $\sigma \overset{\delta}{\sim} \sigma'$ will mean that state $\sigma$ is represented by $\sigma' \in \Sigma'$ in the computation description $\delta$.

3. A continuous simulation function $f : D \rightarrow D$ will be defined satisfying

**Lemma 2.1** If $\sigma_1 \Rightarrow \sigma_2$ and $\sigma_1 \overset{\delta}{\sim} \sigma_1'$ then $\sigma_2 \overset{f(\delta)}{\sim} \sigma_2'$ for some $\sigma_2' \in \Sigma'$.

4. Safeness will be shown by the following, where $\delta(M_0)$ is the least element of $D$ which describes Load($M_0$) and is a fixpoint of $f$.

**Theorem 2.2**

If $\sigma \in \text{States}(M_0)$ then $\sigma \overset{\delta(M_0)}{\sim} \sigma'$ for some $\sigma' \in \Sigma'$.

5. A linear encoding le of states will be introduced, and a context-free grammar $G(M_0)$ will be constructed from $\delta(M_0)$ will be constructed from $\delta(M_0)$ such that

$L(G(M_0)) \supseteq \{ \text{le}(\sigma) \mid \sigma \in \text{States}(M_0) \}$

6. Effective methods will be developed to give "safe" positive answers to the following questions

- Is a subexpression of $M$ never evaluated?
- Will the computation terminate?
- Is $\text{States}(M_0)$ finite?
- Is $M$ independent of $N$, for subexpressions $M, N$ of $M_0$?

**The Description Lattice $D$**

The sets of closures, contexts etc. of CBV($M_0$) are infinite, so for effective approximation it is desirable to represent them finitely. We first develop informal methods for finite representation and then give a mathematical definition of what a representation is. The data structure representations (and more) are summarized in Figure 5.

The sets $E$ and $\text{Cl}$ are infinite, since defined by mutual recursion. However, notice that every value $e(x) \in \text{Cl}$ comes from a $\text{Cl}$-context as the result of $\beta$-reduction; in fact CBV could easily be modified to work with $E ::= \text{Var} \; \text{P} \; \text{C}$ instead of $E ::= \text{Var} \; \text{P} \; \text{Cl}$. We thus approximate $E$ by $E' ::= \text{Var} \; \text{P} \; \text{Cl}'$ where $\text{Cl}'$ is an as-yet-undefined representation of $\text{Cl}$. 
Since $M_0$ is constant-free, the closures computed by CBV($M_0$) must all lie in $\text{Sub}(M_0) \times E$. We thus approximate closures by $C^l := \text{Sub}(M_0)E$. State $\sigma = c_{[c]}$ will be represented by a pair $\sigma^l = (c^l, c_1^l)$, so we define $\Sigma^l := C^l C^l$.

$C^l$-context is also infinite due to its recursive definition. During the CBV($M_0$) computation, contexts are manipulated "inside out", i.e. from the vicinity of $[\ ]$. The reduction rules remove the innermost closure, and rule 3 has no effect on the closure. Rule 4 "deepens" the $C^l$-context, changing $c_{[\ ]}$ to $c_{[(M, e)[\ ]]}$ when processing a combination $(M, e)$, and rule 5 changes $c_{[c_1[\ ]]}$ to $c_{[[\ ]]}c_2$. Let $\text{Com}(M_0)$ denote the set of occurrences of combinations within $M_0$; these will be used for local representations of $C$.

The $C^l$-contexts appearing in a computation will be represented by two data structures: a set $C^l$ of local representations, plus a single global retrieval function $cc : C^l \to P(C^l C^l)$ which is used to extract the structure of any $C^l$-context if given its local representation. A containment $(c_1^l, c_1^r) \in cc(c^l)$ indicates that $c^l$ locally represents a context $c_1^l[c_{[\ ]}]$ or $c_1^r[c_{[\ ]}]$. More specifically,

- $[\ ]$ locally represents the empty $C^l$-context.
- $MN[\ ]$ locally represents any $C^l$-context created by a transition $c_{[(M, e)[\ ]]} \to c_{[(M, e)[(N, e)][\ ]]}$. The remaining structure is represented globally by the containment $(c_{[\ ]}, c^l) \in cc(MN[\ ])$, where $e', c'$ represent $e$, $c$. $(M, e')$ represents the operator of the combination $(MN, e)$. Further, if $MN[\ ]$ represents $c_{[c_{[\ ]}]}$ locally, then
- $[\ ]MN$ similarly represents (in the local sense) the context created by $c_{[[c_1[\ ]]]} \to [[c_1][\ ]].$ This is globally represented by $(c_1^l, c_1^r) \in cc[[\ ]MN]$ where $c_1^r$ locally represents $c_{[\ ]}$ and $cl_1^l$ represents the reduced form of operand $(N, e)$.

Consequently we set $C^l ::= [\ ] | [\ ]\text{Com}(M_0) | \text{Com}(M_0)[\ ]$ and $CC ::= C^l \leftrightarrow P(C^l C^l)$.

An entire computation $\sigma_1 \Rightarrow \sigma_2 \Rightarrow \ldots$ will be represented by a pair $\delta = (S, cc)$ where $S \in \Sigma^l$ contains representations of the states (using local context representations) and cc globally represents the structures of all the $C^l$-contexts.

Let $D ::= P(\Sigma) CC$ with the ordering: $(S_1, cc_1) \leq (S_2, cc_2)$ iff $S_1 \subseteq S_2$ and $\forall c^l \in C^l cc_1(c^l) \subseteq cc_2(c^l)$. $D$ is clearly a complete finite lattice, and is called the description lattice. We write $\bot, T, U, \lor$ for the least element, greatest element, least upper bound and greatest lower bound, respectively.

The way in which elements of $D$ represent closures, states, etc. is now made precise:
Definition 2.3 Let $\delta = (S, cc) \in D$. The representation relation

$\tilde{\sim} \subseteq (C \times C') \cup (E \times E') \cup (C \times C') \cup (\Sigma \times \Sigma')$ is defined inductively by:

a) $(M, e) \tilde{\sim} (M, e')$ if $e \sim e'$

b) $e \sim e'$ if $\forall x \in \text{Domain}(e) \exists (c, c') \in cc(e'(x))$ such that $e(x) \tilde{\sim} c'$

c) $[\ ] \tilde{\sim} [\ ]$

$[c][c][\tilde{\sim} \text{MN}[\ ]$ if $\exists (c, c') \in cc(MN[\ ])$ such that $c \tilde{\sim} c'$ and $c' \tilde{\sim} c$

$c'[\ ] \tilde{\sim} [\ ] \text{MN}$ if $\exists (c, c') \in cc([\ ] \text{MN})$ such that $c' \tilde{\sim} c'$ and $c \tilde{\sim} c'$

d) $c[\ ] \tilde{\sim} (c', c'')$ if $c \tilde{\sim} c'$, $c' \tilde{\sim} c''$ and $(c', c'') \in S$.

Note that $\tilde{\sim}$ is monotonic with respect to $\delta$: $\delta_1 \subseteq \delta_2$ and $\sigma \tilde{\sim} \sigma'$ implies $\delta \tilde{\circ} \sigma'$.

This technique may be used to approximate the behavior of many algorithms using recursively defined data types, providing an alternative for example to the methods of Jones and Muchnick [JoM81] and Reynolds [Rey68] for simple LISP-like programs. The underlying idea is to represent a recursively-defined structure by the set of program points which contain constructor operations, plus a retrieval function on this set which can be used to retrieve the components put together by a construction. These program points are propagated as descriptions of variables' values along control paths during flow analysis, and selector operations are performed by consulting the retrieval function.

The CBV($M_0$) Simulation Function

It is well known that most forward flow analysis methods essentially carry out an abstract interpretation of the simulated algorithm over a lattice of approximations to states or sets of states. Descriptions of this approach may be found in [Sin72] and [Cou79].

This approach was used to construct the simulation function $f: D \rightarrow D$ which is defined in Figure 5; the clauses defining $f$ in essence apply the CBV($M_0$) transition rules to representations of states. Clearly $f$ is monotonic; $D$ is finite, so $f$ is also continuous and so $\delta(M_0)$ is well-defined. Lemma 2.1 will be proved after an example; we now prove that Theorem 2.2 follows from Lemma 2.1.

Proof of Theorem 2.2

Suppose $\text{Load}(M_0) = \sigma_0 \Rightarrow \sigma_1 \Rightarrow \ldots \Rightarrow \sigma_n$. For $n = 0$,

$\text{Load}(M_0) = ([\ ], (M_0, \{\ }) \delta_0 ([\ ], (M_0, \{\ })) = \sigma_0'$ where $\delta_0 = (\{\sigma_0'\}, \{\ })$ and $\{\ }$ is the empty $C$-context retrieval function. Now $\sigma_0 \tilde{\delta}(M_0) \sigma_0'$ holds since $\delta_0 \subseteq \delta(M_0)$. 

Now suppose inductively that $\sigma_i \sim^{\delta(M_0)} \sigma_{i+1}^1$. By Lemma 2.1 there exists $\sigma_{i+1}^1$ such that $\sigma_{i+1}^1 \sim^{\delta} \sigma_{i+1}^1$ where $\delta = f(\delta(M_0)) \cup \delta(M_0)$, so $\sigma_i \sim^{\delta(M_0)} \sigma_{i+1}^1$.

Data Structure Descriptions

| cl'  | ::= | Sub(M_0)^E' | Closures |
| e'   | ::= | BV(M_0) P \rightarrow C' | Environments |
| c'   | ::= | [ ] | [ ] Com(M_0) | Com(M_0) | Contexts - local descriptions |
| cc   | ::= | C' \rightarrow P(C'| C') | Contexts - global descriptions |
| S \subseteq \Sigma' | ::= | C' \subset C'| C' | States |
| \delta : D | ::= | P(\Sigma'| CC) | Descriptions of states and sequences |

Simulation Function $f: D \rightarrow D$

Let $\delta = (S, cc) \in D$. Then $f(\delta)$ is the least pair $(S_1, cc_1) \not\subseteq (S, cc)$ such that the following hold:

1. $\beta$ reduction: if $([ ] NP, (\lambda x M, e')) \in S$ and $(c', c') \in cc([ ] NP)$ then $(c', (M, e'[ ] NP/x]) \in S_1$
2. Variable expansion: if $(c', (x, e')) \in S$ and $(c_1', c_1') \in cc(e'(x))$ then $(c', c_1') \in S_1$.
3. Combination: if $(c', (MN, e')) \in S$ then $(MN[ ], (N, e')) \in S_1$ and $((M, e'), c') \in cc_1(MN[ ])$
4. Scan operator: if $(MN[ ], c_2') \in S$ where $c_2' = (P, e')$ for some closed value $P$ and $(c_1', c') \in cc(MN[ ])$ then $([ ] MN, c_1') \in S_1$ and $(c_2', c') \in cc_1([ ] MN)$

Control Flow Description $\delta(M_0) \in D$

This is the least solution to the equation $\delta = f(\delta) \cup Load'(M_0)$, where

$Load'(M_0) = ([ ] M_0, [ ]) \cup [ ]$ describes $\{ Load(M_0) \}$.

Figure 5. Simulation Function and Description Lattice.

An Example

Let $M_0 = AB = (\lambda xxx)(\lambda y y)$. The reduction sequence $M_0 > (\lambda y y)(\lambda y y) \geq \lambda y y$ as computed by CBV(M_0) and as represented by $\delta(M_0) = (S, cc)$ are as in Figure 6.
### CBV(M₀) Computation

<table>
<thead>
<tr>
<th>State</th>
<th>Action</th>
<th>( S )</th>
<th>cc</th>
</tr>
</thead>
<tbody>
<tr>
<td>([(A, {}),(B,{})])</td>
<td>scan operand</td>
<td>([(A,{}),(B,{})])</td>
<td>({}) cc((A,{}),({}))</td>
</tr>
<tr>
<td>([(A,{}),(B,{})])</td>
<td>scan operator</td>
<td>((A,B,{}))</td>
<td>({}) cc((A,B,{}))</td>
</tr>
<tr>
<td>([(A,{}),(B,{})])</td>
<td>(\beta) reduce</td>
<td>({}) cc((A,B,{}))</td>
<td>({}) cc(({}A{}))</td>
</tr>
<tr>
<td>([(x,{}),(x\rightarrow(B,{})])</td>
<td>scan operand</td>
<td>({}) cc((x{}x\rightarrow(B,{})))</td>
<td>call this (e)</td>
</tr>
<tr>
<td>([(a,e),(x,e))]</td>
<td>expand (x)</td>
<td>({}) cc((x{}x\rightarrow(B,{})))</td>
<td>call this (e)</td>
</tr>
<tr>
<td>([(x,e),(B,{})])</td>
<td>scan operator</td>
<td>({}) cc((x{}x\rightarrow(B,{})))</td>
<td>({}) cc(({})x(x))</td>
</tr>
<tr>
<td>([(x,e),(B,{})])</td>
<td>expand (x)</td>
<td>({}) cc((x{}x\rightarrow(B,{})))</td>
<td>({}) cc(({})x(x))</td>
</tr>
<tr>
<td>([(B,\emptyset),(B,{})])</td>
<td>(\beta) reduce</td>
<td>({}) cc((x{}x\rightarrow(B,{})))</td>
<td>({}) cc(({})x(x))</td>
</tr>
<tr>
<td>([(y,{y\rightarrow(B,{})})]</td>
<td>expand (y)</td>
<td>({}) cc((x{}x\rightarrow(B,{})))</td>
<td>({}) cc(({})x(x))</td>
</tr>
<tr>
<td>([(B,{})])</td>
<td></td>
<td>({}) cc((B,{})))</td>
<td>({}) cc(({})x(x))</td>
</tr>
</tbody>
</table>

**Figure 6. CBV(M₀) Computation and Approximation.**

**Proof of Lemma 2.1**

Suppose \(\sigma_1 \Rightarrow \sigma_2\) and \(\sigma_1 \sim \sigma_1\) where \(\Delta = (S,cc)\).

**Case 1.** \(c[(\lambda e M,e)] = \sigma_1 \Rightarrow \sigma_2 = c[(M,e)\{c1/x\}]\). \(\sim\)

Let \(\sigma_1 \sim (\{\}NP,\{\lambda e M,e\})\). Then \(\exists c1, c1\) such that \(e \sim e\), \(\sim c1, c1\) and \((c1,e')\) cc(\(\{\}NP\)). By definition of \(f\), \(\sigma_2 = (c1,M,e\{\}NP/x\})\) cc(\(\{\}NP\)). To show \(\sigma_2 \sim \sigma_2\) we need only establish \(e\{\}c1/x\} \sim \{\}NP/x\}. This follows from the definition of \(\sim\) and the facts that \(e \sim e\) and \((c1,e')\) cc(\(\{\}NP\)).

**Case 2.** \(c[(x,e)] = \sigma_1 \Rightarrow \sigma_2 = c[e(x)]\).

If \(\sigma_1 \sim (c1, (x,e'))\) then \(\sigma_2 \sim (c1, e'(x))\) by the definitions of \(e \sim e\) and \(f\).

**Case 3.** \(c[(MN,e)] = \sigma_1 \Rightarrow \sigma_2 = c[(M,e)[(N,e)]]\).

Let \(\sigma_1 \sim (c1, (MN,e'))\). By definition of \(f\), \(\sigma_2 = (MN, (N,e'))\) cc(\(\{\}NP\)). Now \((M,e) \sim (M,e')\) so \(c[(M,e)[(N,e)]] \sim MN\}. This and \((N,e) \sim (N,e')\) implies \(\sigma_2 \sim \sigma_2\).
Case 4. \[ c[cl_1[cl_2]] = \sigma_1 \Rightarrow \sigma_2 = c[cl_1 cl_2] \] with \( cl_2 \) suitably restricted.

Let \( \sigma_1 \sim (MN[ \ ]), cl_2' \). Then \( \exists c', cl_1' \), such that \( cl_2' \sim cl_2 \), \( c \sim c' \), \( cl_1 \sim cl_1' \) and \( (cl_1', cl') \in cc(MN[ \ ]) \). By definition of \( \tilde{f} \), \( \sigma_2' = ([ ]MN, cl_1') \in S_1 \) and \( (cl_2', cl') \in cc([ ]MN) \). Clearly \( c[cl_1 cl_2] \sim [ ]MN \) so \( \sigma_2 \sim \sigma_2' \) is immediate.

\[ \square \]

**A Context-free Approximation to States(\( M_0 \))**

First, we define a way to encode states, etc. as linear strings of symbols. A Cl-context will be written linearly as \( (i_{11}, cl_1) \ldots (i_{nk}, cl_n) \) where each \( i_{ij} \) is 1 if \( cl_j \) is in operator position and 2 if in operand position. For example \( c = (cl_1([ ]cl_2))cl_3 \) becomes \( (2, cl_2)(1, cl_1)(2, cl_3) \). This would be suitable for computer implementation since the Ot and CBV transition rules in effect treat such a string as a stack with the top at the left end. An environment \( e \) will be encoded as \( \{ x_1 \rightarrow e(x_1), \ldots, x_n \rightarrow e(x_n) \} \) where \( \text{Domain}(e) = \{ x_1, \ldots, x_n \} \) and \( x_1 < x_2 < \ldots < x_n \) relative to some arbitrary fixed order relation on \( \text{Var} \).

The linear encoding function \( le : Cl \cup E \cup C \cup \Sigma \rightarrow A^* \) is defined as follows, where \( A \) is the alphabet \( A = \text{Var} \cup \{ ( ), \{ \}, [ ] \}, \rightarrow, 1, 2 \} \cup \{ , \} \).

\[
\begin{align*}
le((M, e)) & = (M, le(e)) \\
le(e) & = \{ x_1 \rightarrow le(e(x_1)), \ldots, x_n \rightarrow le(e(x_n)) \} \\
& \quad \text{where Domain}(e) = \{ x_1, \ldots, x_n \} \text{ and } x_1 < \ldots < x_n \\
le([ ]) & = \varepsilon \text{ (the empty string)} \\
le(cl) & = le(cl)(1, le(cl)) \\
le(c cl) & = le(c)(2, le(cl)) \\
le(c[cl]) & = le(c)[le(cl)]
\end{align*}
\]

**Definition 2.4** Let \( \delta = (S, cc) \) be the control flow description of \( CBV(M_0) \). The context-free grammar \( G(M_0) \) is defined in Figure 7.

An example derivation from Figure 5 is

\[
S \Rightarrow [ ] \Rightarrow [((xx, e(1)))] \Rightarrow [((xx, e(1)))] \\
\Rightarrow [((xx, \{ x \rightarrow (B, \{ \}) \})] \Rightarrow [((xx, \{ x \rightarrow (\lambda y, \{ \}) \})]
\]
Nonterminals: \{ S_0 \} \cup \{ \overline{X} \mid X \in C_1 \cup E_1 \cup C_1 \cup \Sigma \}

Terminals: A; Start symbol: S_0

Productions:

Closures: (M, e^1) ::= (M, e^1) \quad \text{for each } M \in \text{Sub}(M_0), e^1 \in E^1

Environments: e^1 ::= \{ x_1 \rightarrow \overline{c_1}, \ldots, x_n \rightarrow \overline{c_n} \}

where \text{Domain}(e^1) = \{ x_1, \ldots, x_n \}, x_1 \leq \ldots \leq x_n

and \forall i \in \overline{1}, n \exists c_i^1 \quad (c_i^1, c_i^1) \in \text{cc}(e^1(x_i))

CI-contexts: \epsilon ::= [ ]

MN[ ] ::= (1, c_1^1) c^1 \quad \text{for each } (c_1^1, c^1) \in \text{cc}(MN[ ])

[ ]MN ::= (2, c_1^1) c^1 \quad \text{for each } (c_1^1, c^1) \in \text{cc}( [ ] MN)

States: (c^1, c_1^1) ::= \overline{c^1} [ \overline{c_1^1} ] \quad \text{for each } (c^1, c_1^1) \in S

Initial: S_0 ::= \sigma^1 \quad \text{for each } \sigma^1 \in S

Figure 7. Context-free Grammar G(M_0) Approximating States(M_0).

For any nonterminal \overline{X} let L(\overline{X}) = \{ x \in A^* \mid \overline{X} \Rightarrow x \}.

Theorem 2.5

Let M_0 be a closed constant-free \lambda-expression and \delta its control flow description. For each x^1 \in X^1 where X = C_1, E, C or \Sigma:

L(\overline{x^1}) = \{ Ie(x) \mid x \overline{\sim} x^1 \text{ and } x \in X \}

Proof is a straightforward induction on definition 2.3 of x \overline{\sim} x^1.

Corollary 2.6

L(G(M_0)) = \{ Ie(\sigma) \mid \sigma \in \text{States}(M_0) \}.
Applications

Let a "safe positive reply" P to a question whose answer is Q be one such that P logically implies Q. Given \( N \in \text{Sub}(M_0) \) and \( (M, e) \in C1 \) define \((M, e)\) to depend on \( N \) if either \( M = N \) or for some \( x \in \text{FV}(M) \), \( e(x) \) depends on \( N \). We say that \( M \) depends on \( N \) for \( M, N \in \text{Sub}(M_0) \) if \( S \) contains a state \( c[(M, e)] \) with \((M, e)\) dependent on \( N \).

Theorem 2.7

There is a decidable method to obtain nontrivial safe positive replies to the following questions about a closed constant-free \( \lambda \)-expression \( M_0 \):

1. Is evaluation of \( M \in \text{Sub}(M_0) \) never attempted? (\( \text{Meaning: does States}(M_0) \) contain no state \( c[(M, e)] \) ?).

2. Will the computation terminate?

3. Will the computation fail to terminate?

4. Is \( \text{States}(M_0) \) finite?

5. Is \( M \) independent of \( N \) (given \( M, N \in \text{Sub}(M_0) \))?

Proof: Is by showing how to analyze the structure of \( \delta(M) = \delta = (S, cc) \). Question 1 is simple: if \( S \) contains no pair \((c', (M, e'))\) then \( \text{States}(M_0) \) contains no state \( c[(M, e)] \) by Theorem 2.2.

Define the flowchart of \( M \) to have nodes in \( \Sigma I \) and an edge \( \sigma_1 \rightarrow \sigma_2 \) just in case \( \sigma_2 \in S \) where \( f(\sigma_1, cc) = (S, cc) \). It is easy to see that if \( \sigma_1 \sim \sigma_1' \) then \( \sigma_1 \rightarrow \sigma_2 \) implies \( \sigma_1 \rightarrow \sigma_2 \) in the flowchart for some \( \sigma_2 \) with \( \sigma_2 \sim \sigma_2' \). Letting the \( \text{CBV}(M_0) \) computation be \( \text{Load}(M_0) = \sigma_0 \rightarrow \sigma_1 \rightarrow \ldots \) there exists a path \( \sigma_0' \rightarrow \sigma_1' \rightarrow \ldots \) where each \( \sigma_i \sim \sigma_i' \) for \( n \geq 0 \) and \( \sigma_0' = ([]), (M_0, []) \).

If the computation is infinite there must exist \( \sigma' \) such that \( \sigma_0' \rightarrow \sigma_1' \rightarrow \ldots \), since \( \Sigma I \) is finite. This condition is certainly decidable, and its falsity implies the computation is finite. Question 3 may be answered "yes" if there is no path \( \sigma_0' \rightarrow \sigma_1' \rightarrow \ldots \) where \( \sigma_i \neq \sigma_i' \) for all \( \sigma_1' \). Note that safe answers to questions 2 and 3 can both be "no".

Question 4 can be answered simply by constructing \( G(M_0) \) and testing \( L(G(M_0)) \) for finiteness; by Corollary 2.6 a positive answer implies \( \text{States}(M_0) \) is finite. It is well known that finiteness of context-free languages is decidable.

For question 5, define "cl" depends on "N" for \( c' \in \text{Cl} \) by
a) \((M, e')\) depends on \(M\) for any \(e' \in E'\)

b) \((M, e')\) depends on \(N\) if there exist \(x \in \text{FV}(M)\) and \((c^l, c^l') \in cc(e'(x))\) such that \(c^l\) depends on \(N\).

It is easily seen that if \((M, e)\) depends on \(N\) and \((M, e) \preceq (M, e')\) then \((M, e')\) depends on \(N\). The set of all pairs \((M, c^l)\) such that \(c^l\) depends on \(M\) is clearly effectively computable. Finally, \(M\) is independent of \(N\) if there exists no \((c, (M, e')) \in S\) with \((M, e')\) dependent on \(N\).

3. **ANALYSIS OF DATA AND CONTROL FLOW**

A method will now be given to obtain a safe description of \(\text{States}(M_0)\) where \(M_0\) is an arbitrary \(\lambda\)-expression. The method involves the use of an auxiliary lattice to approximate sets of constants. The development is otherwise quite parallel to the previous one, so only the essential details are given. For the method to succeed finitely, however, we need the following reasonable assumptions:

**Assumption** Subcon is finite.

**Approximation of Constants**

Since the set of constants is infinite and our descriptions of \(\text{States}(M_0)\) are finite, it is necessary to find a way to finitely represent unbounded sets of constants. This is traditionally done in flow analysis by use of a complete approximation lattice \(L\) whose elements represent sets of constants via an abstraction function \(\text{abs}: P(\text{Con}) \rightarrow L\). For example a suitable \(L\) for "constant propagation" would be the one in Figure 8. Constant propagation allows the recognition of subexpressions which always evaluate to the same value, and determination of that value.

![Diagram](image)

\[
\text{abs}(\{0, 1, 2, \ldots\}) \rightarrow L
\]

\[
\text{abs}(A) = \begin{cases} 
\bot & \text{if } A = \emptyset \\
\text{if } A = \{a\} \text{ then } \text{else} & \\
\text{if } A = \{a\} \text{ then } \text{else} & \\
T & \end{cases}
\]

**Figure 8. A Simple Approximation Lattice L.**
Not all abstraction functions and lattices are suitable for flow analysis. An appropriate and useful restriction due to Cousot is that there exist a concretization function \( \text{conc}: L \rightarrow \mathcal{P}(\text{Con}) \) such that \((\text{abs},\text{conc})\) are a pair of adjoined functions. Letting \( \sqsubseteq \) be the ordering on \( L \) this means that

\[
\forall A \subseteq \text{Con} \quad \forall I \in L \quad A \subseteq \text{conc}(I) \iff \text{abs}(A) \sqsubseteq I
\]

Motivation of this definition and useful mathematical properties may be found in [Cou79]. A suitable concretization function for constant propagation is: \( \text{conc}(\bot) = \emptyset \), \( \text{conc}(a) = \{a\} \), \( \text{conc}(T) = \text{Con} \).

In order to effectively obtain finite descriptions we also require that \( L \) have the finite chain property, i.e. that there exist no infinite properly increasing chains. Note that the example is infinite but has the finite chain property.

The Description Lattice Dcon

This extends the D lattice used before. Figure 9 contains Dcon and the simulation function \( f_{\text{con}} \). Note that \( E', C' \) and \( \Sigma' \) are as in Figure 5 but that \( C1', C'C \) and \( Dcon \) are more elaborate.

By Lemma 1.5 an achievable closure \( (M, e) \) must have \( M \in \text{Lam}(M_0) = \text{Sub}(M_0) \cup \text{Subcon} \cup \text{Con} \). In Dcon we represent a closure by either \( I \in L \) describing a set of constant values, or a pair \((M, e')\) with \( M \in \text{Sub}(M_0) \cup \text{Subcon} \).

We cannot use Dcon := \( \mathcal{P}(\Sigma')\text{CC} \) since \( \mathcal{P}(\Sigma') = \mathcal{P}(C' \cup C1') \) may not possess the finite chain property (as in Figure 8). This problem is resolved by merging \( \{ (c', I_1), \ldots, (c', I_n) \} \in C' \times C1' \) with \( I_1, \ldots, I_n \in L \) into the single pair \( (c', \bigcup_{i=1}^{n} I_i) \).

\( \mathcal{P}(\Sigma') \) is replaced by

\[
C' \otimes C1' = \{ X \subseteq C' \times C1' \mid (c', I_1),(c', I_2) \in C' \times L \text{ implies } I_1 = I_2 \}
\]

Clearly \( C' \otimes C1' \) has no infinite ascending chains. We now define

\[
\text{Dcon} := (C' \otimes C1')\text{CC}.
\]

An order relation \( \sqsubseteq \) can now be imposed on \( C' \otimes C1' \) to make Dcon a complete lattice, as follows. The following function is easily seen to be an isomorphism:

\[
g : C' \otimes C1' \rightarrow \mathcal{P}(C' \times ((\text{Sub}(M_0) \cup \text{Subcon}) \times E)) \times (C' \rightarrow L)
\]

\[
g(X) = (X \cap (C' \times ((\text{Sub}(M_0) \cup \text{Subcon}) \times E)), \lambda c. \text{ if } (c, I) \in X \text{ then } I \text{ else undefined})
\]

Informally, if \( g(X) = (Y, h) \) then \( Y \) consists of those pairs \((c', c1')\) where \( c1' \) is not in \( L \), and \( h \) maps \( c' \) to \( I \) iff \((c', I) \in X \) and \( I \in L \). The range of \( g \) is certainly a complete lattice with \((Y, h) \leq (Y', h') \) iff \( Y \subseteq Y' \) and \( \forall c \in C' \ (h(c) \leq h(c')) \). Now define \( X_1 \sqsubseteq X_2 \) iff \( g(X_1) \leq g(X_2) \), making \( C' \otimes C1' \) into a complete lattice.
Note that \((c',c_1') \in X\) and \(\{(c',c_1')\} \subseteq X\) are both meaningful. Both are used in Figure 9.

Similarly the set of global context descriptions \(CC := C' \rightarrow CL \otimes C'\) is a complete lattice, with \(CL \otimes C'\) defined symmetrically to \(C' \otimes CL\). It should now be evident that Dcon is a complete lattice with the finite chain property.

The Simulation Function

This function \(f_{con}\) is constructed in the same way as \(f\), to simulate the effect of a \(CBV(M_0)\) transition \(\sigma \rightarrow \sigma_2\) in terms of the data structure \(\delta\) representing \(\sigma_1\). Constants can either be computed or original subexpressions of \(M_0\); thus we add a rule to convert closure (con, e) into its representation \(\text{abs}\{\text{con}\}\) in \(L\). For \(\delta\) reduction we must map backwards from \(L\) to \(Con\) and then forward into \(L\) again. This is done with the aid of Conap: \(L \times L \rightarrow \mathcal{P}(Con)\) and Lamap: \(L \times L \rightarrow \mathcal{P}(Lam)\) defined by:

\[
\text{Conap}(l_1, l_2) = Con \cap \text{Back}, \quad \text{Lamap}(l_1, l_2) = \text{Subcon} \cap \text{Back}
\]

\[
\text{Back}(l_1, l_2) = \{a^1 | \exists a \in \text{conc}(l_1), b \in \text{conc}(l_2) \text{ such that } a^1 = \text{Constapply}(a, b) \text{ is defined}\}
\]

Further Development

It seems clear that the steps taken earlier for control flow analysis can now be paralleled: a representation relation \(\delta \rightarrow \delta'\) could be defined, safeness proved, and a context-free grammar generating a superset of \(\{\text{le}(\sigma) | \sigma \in \text{States}(M_0)\}\) could be constructed. We do not do this for brevity and because no new ideas are involved.

The \(CBV(M_0)\) interpreter may perform an "error halt" if \(M_0\) contains constants; a state \(c[(a,e)](b,e_1)\) with \(a \in \text{Con}\) causes nonstandard termination unless \(b \in \text{Con}\) and \(\text{Constapply}(a, b)\) is defined. Taking this into account it appears straightforward to extend the methods of Theorem 2.7 to obtain effectively safe positive replies to the five questions stated there, plus:

6. Is the computation free of error halts?

7. Is a given variable occurrence bound only to a single constant value? (If so, its value can be obtained.)
Data Structure Descriptions

cl' : Cl' ::= (Sub(M_0) | Subcon)E | L  \quad \text{Closures}

e' : E' ::= BV(M_0) P C'  \quad \text{Environments}

c' : C' ::= [] | [ ] Com(M_0) | Com(M_0)[ ]  \quad \text{Contexts - local}

cc' : CC ::= C' \rightarrow (Cl' \otimes C')  \quad \text{Contexts - global}

\Sigma' ::= C' Cl'  \quad \text{State sets}

\delta : Dcon ::= (C' \otimes Cl')CC  \quad \text{States and sequences}

Simulation Function \( f_{con} : Dcon \rightarrow Dcon \)

Let \( \delta = (\Sigma, cc) \in Dcon \). Then \( f_{con}(\delta) \) is the least pair \((\Sigma_1, cc)\) satisfying:

1. \( \beta \) reduction: if \(([] NP, (\lambda x M, e')) \in \Sigma \) and \((c', c') \in cc([[] NP])\)
   then \((c', (M, e'([] NP/x'))) \in \Sigma_1\)

2. \( \delta \) reduction: if \(([] NP, l_1) \in \Sigma_1 \) and \((l_2, c') \in cc_1([[] NP])\)
   then
   a) \((c', \text{abs}(\text{Conap}(l_1, l_2))) \in \Sigma_1\)
   b) \((c', (M, l_1)) \in \Sigma_1\) for each \( M \in \text{Lamap}(l_1, l_2) \)

3. Constant representation: if \((c', (\text{con}, e)) \in \Sigma \)
   then \((c', \text{abs}[\text{con}]) \in \Sigma_1\)

4. Variable expansion: if \((c', (x, e')) \in \Sigma \) and \((c', c_1') \in cc(e'(x))\)
   then \((c', c_1') \in \Sigma_1\)

5. Combination: if \((c', (MN, e')) \in \Sigma \)
   then \((MN[[]], (N, e')) \in \Sigma_1 \) and \((M, e'), c') \in cc_1(MN[[]])\)

6. Scan operator: if \((MN[[]], cl_2') \in \Sigma \) where \(cl_2' = (P, e')\) for some closed value \( P \),
   and \((cl_1', c') \in cc(MN[[]])\)
   then \(([] MN, cl_1') \in \Sigma_1 \) and \((cl_2', c') \in cc_1([[]] MN)\)

Data and Control Flow Description

\( \delta(M_0) \) is the least solution to the equation \( \delta = f(\delta) \cup \text{Load}'(M_0) \)

\textbf{Figure 9.} Approximate Description of Data and Control Flow.
CONCLUSIONS AND ACKNOWLEDGEMENTS

It has been shown that safe answers may be effectively obtained to a variety of questions about call-by-value reduction sequences including finiteness, termination, freedom from errors, and independence of subexpressions. The methods used are clearly applicable to call-by-name; further since abstract interpretation does not depend on determinism it seems likely that the OI interpreter could be similarly analyzed, giving information about the set of all outside-in reduction sequences. One application would be to determine from the flow analytic information a combination of call-by-value and call-by-need which have the same termination properties as call-by-name but allow a more efficient implementation. This would extend the results of Mycroft [Myc80].

The analysis method applied the classical flow-analytic idea of abstract interpretation to a new call-by-value interpreter CBV which appears to be somewhat simpler than the SECD machine. This application required a new description technique involving both local and global data representations due to the recursiveness of CBV's data structures. The technique is applicable to many programs which manipulate tree-like data structures; it is anticipated that it can be used to develop practical interprocedural flow analysis methods for more conventional imperative programming languages. Another application would be the development of compiling methods for applicative languages capable of producing highly efficient object code.

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Appendix I

Proof of Theorem 1.1

Lemma 1 \hspace{1cm} \text{Real}(\text{cl}_1) \overset{*}{\Rightarrow}_o \text{Real}(\text{cl}_2) \text{ implies } \text{Unload}(c[\text{cl}_1]) \overset{*}{\Rightarrow}_o \text{Unload}(c[\text{cl}_2]).

Proof is by a simple induction on the size of $c[\ ]$.

Lemma 2 \hspace{1cm} \text{Suppose } \text{Real}((\lambda x M, e)) = \lambda x M', \, \text{Real}(cl) = N \text{ and } N \text{ is closed. Then } \text{Real}((M, e[ cl/x ])) = [N/x]M'.

Proof is straightforward and so omitted.

Lemma 3 \hspace{1cm} \sigma_1 \overset{o}{\Rightarrow}_i \sigma_2 \text{ and } \text{Unload(}\tau_1\text{) closed implies } \text{Unload}(\sigma_1) \overset{*}{\Rightarrow}_o \text{Unload}(\sigma_2).

Proof \hspace{1cm} \text{If } \sigma_1 \overset{o}{\Rightarrow}_i \sigma_2 \text{ by transition rule 5, 6, 7 on 8 then } \text{Unload}(\sigma_1) = \text{Unload}(\sigma_2) \text{ by definition of Unload. If } \sigma_1 = c[\text{cl}_1] \Rightarrow \sigma_2 = c[\text{cl}_2] \text{ by rule 3 or 4 then } \text{Real}(\text{cl}_1) = \text{Real}(\text{cl}_2) \text{ so the result holds by Lemma 1.}

\text{If } \sigma_1 = c[\{a, e\}(b, e')] \Rightarrow \sigma_2 = c[\{a', e\}] \text{ by rule 2 then } \text{Real}((a, e)(b, e')) = \text{Real}((a', e)). \text{ By Lemma 1, } \text{Unload}(\sigma_1) = \text{Unload}(\sigma_2) \text{ as in rule 1, and let } \text{Real}((\lambda x M, e)) = \lambda x M' \text{ and } \text{Real}(cl) = N. \text{ Unload}(\sigma_1) \text{ is closed so } N \text{ must also be closed. By } \beta\text{-reduction and Lemma 2,}

\text{Real}((\lambda x M, e)cl) = \lambda x M'[N/x]M' = \text{Real}((M, e[ cl/x ]))

Thus Unload(\sigma_1) = Unload(c[(\lambda x M, e)cl]) \overset{*}{\Rightarrow}_o Unload(\sigma_2) \text{ by Lemma 1.} \hfill \square

This completes the proof of part a) of Theorem 1.1 For part b), first define a closure $(M, e)$ to be simple if $M$ is not a variable. Note that any state $\sigma = c[cl]$ yields a state $\sigma' = c[cl']$ with simple $cl'$ by a finite number of applications of transition rule 3 (rule 3 cannot be applied infinitely often since environments are defined inductively). A consequence is that if $\text{Real}(cl)$ is a combination $M_1M_2$ then there exist $\text{cl}_1, \text{cl}_2$ such that $c[cl] \overset{*}{\Rightarrow}_o c[cl_1cl_2]$ and $\text{Real}(\text{cl}_i) = M_i$ for $i = 1, 2$ (since the $cl'$ mentioned above must be a combination by the definition of Real).

Similar results hold if $\text{Real}(cl)$ is an abstraction or a constant.

Part b) of Theorem 1.1 is an immediate consequence of the following.

Lemma 4 \hspace{1cm} \text{If } \text{Unload}(\sigma_1) > N \text{ and } \text{Unload}(\sigma_1) \text{ is closed then } \sigma_1 \overset{*}{\Rightarrow}_o \sigma_2 \text{ for some } \sigma_2 \text{ satisfying } N = \text{Unload}(\sigma_2).
Proof  It is easily seen that the inference rule for reduction in context can be replaced by the two simpler rules

\[
\begin{align*}
4'. & \quad M_1 > M_2 \\
M_1 N > M_2 N
\end{align*}
\]

\[
\begin{align*}
4'''. & \quad M_1 > M_2 \\
NM_1 > NM_2
\end{align*}
\]

Proof  of the lemma is by induction on the number \( k \) of times inference rules \( 4' \) or \( 4'' \) are used to establish \( \text{Unload}(\sigma_1) \triangleright N \).

Basis \( k = 0 \): Suppose \( \text{Unload}(\sigma_1) = (\lambda x M) N \overset{\sigma_1}{\triangleright} [N/x] M \). The following machine computation occurs:

\[
\begin{align*}
\sigma_1 & \overset{*}{\triangleright} \text{[cl]} \\
& \overset{*}{\triangleright} \text{[cl}_1\text{cl}_2] \\
& \overset{\text{Oi rule 5}}{\overset{*}{\triangleright}} \text{[[cl}_1\text{]cl}_2] \\
& \overset{*}{\triangleright} \text{[[\lambda x M', e]|cl}_2\text{]} \\
& \overset{\text{Oi rule 1}}{\overset{*}{\triangleright}} \text{[[M', e|cl}_2/x\{]} \\
& = \sigma_2
\end{align*}
\]

where \( \text{Unload}(\sigma_1) = \text{Real(cl)} \) by OI transition rules 6, 8

\[
\begin{align*}
\text{where } \text{Real(cl}_1\text{)} = \lambda x M, \text{Real(cl}_2\text{)} = N \text{ by rule 3}
\end{align*}
\]

Now \( N \) must be closed so by Lemma 2 \( \text{Unload}(\sigma_2) = \text{Real}((M', e|\text{cl}_2/x\}) = [N/x] M \) as required. Delta reduction is similar but a bit simpler.

Inductive step  Suppose the result holds for fewer than \( k \) uses of \( 4' \) and \( 4'' \) where \( k > 0 \), and \( \text{Unload}(\sigma_1) = M_1 M_2 \overset{\sigma_1}{\triangleright} NM_2 \) by \( k \) uses. As in the basis case

\[
\begin{align*}
\sigma_1 & \overset{*}{\triangleright} \text{[[cl}_1\text{]cl}_2] \text{ where } \text{Real(cl}_i\text{)} = M_i \text{ (i = 1, 2), by a computation involving OI transition rules 6, 8, 3 and 5. By induction } [\text{cl}_1]\overset{*}{\triangleright} \sigma_1' \text{ for some } \sigma_1' \text{ with } \\
\text{Unload}(\sigma_1') = N. \text{ By OI rules 6, 8 } \sigma_1' \overset{*}{\triangleright} [\text{cl}_1'] \text{ for some cl}_1' \text{ with } \text{Real(cl}_1') = N.
\end{align*}
\]

Finally \( [\text{cl}_1]\overset{*}{\triangleright} [\text{cl}_1'] \text{ implies } [\text{cl}_1\text{]cl}_2]\overset{*}{\triangleright} [[\text{cl}_1']\text{cl}_2]. \text{ Let } \sigma_2 = [[\text{cl}_1']\text{cl}_2]. \text{ Clearly } \text{Unload}(\sigma_2) = \text{Real(cl}_1')\text{Real(cl}_2\text{)} = NM_2 \text{ as required. A symmetric argument applied if } \text{Unload}(\sigma_1) = M_1 M_2 \overset{\sigma_1}{\triangleright} M_1 N. \]
APPENDIX II

PROOF OF THEOREM 1.3

We first define \text{eval}_{\sqrt{\_}} more formally just as in [Plo75]. Let "M has value N at time t" mean \( T(M, t) = N \) where \( T: \text{Programs} \times \{0, 1, \ldots\} \xrightarrow{\_} \text{Programs} \) is defined as follows.

1. \( T(M, 0) = M \) if M is a constant or an abstraction

2. \( T(M_1, t_1) = a, \ T(M_2, t_2) = b \) and \( \text{Constapply}(a, b) \) exists\n   \[ T(M_1M_2, t_1+t_2+1) = \text{Constapply}(a, b) \]

3. \( T(M_1, t_1) = \lambda x M_1', \ T(M_2, t_2) = M_2', \ T([M_2'/x]M_1', t_3) = N \)
   \[ T(M_1M_2, t_1+t_2+t_3+1) = N \]

Now let \( \text{eval}_{\sqrt{\_}}(M) = N \) iff \( T(M, t) = N \) for some \( t \geq 0 \), i.e. iff M has value N at some time. Note that N cannot be a combination.

Lemma 1 If \( \text{Real}(\text{cl}) \) has closed value N at some time then \( \text{CBV} \) has a computation \( [\text{cl}] \xrightarrow{*} [\text{cl}!] \) such that \( N = \text{Real}(\text{cl}!) \).

Corollary 2 \( \text{eval}_{\sqrt{\_}}(M) = \text{Eval}_{\sqrt{\_}}(M) \) if \( \text{eval}_{\sqrt{\_}}(M) \) exists.

Proof Let \( T(M, t) = N \) where \( \text{Real}(\text{cl}) = M \). Proof is by induction on t. If \( t = 0 \) it follows immediately with \( \text{cl} = \text{cl}! \). Now suppose \( t > 0 \) and the result holds for all times less than t. By definition of \( T \text{Real}(\text{cl}) = M_1M_2 \) for some \( M_1, M_2 \). It is easy to see that \( [\text{cl}] \xrightarrow{*} [\text{cl}![\text{cl}_2]] \) by \( \text{CBV} \) transition rules 2, 4 where \( \text{Real}(\text{cl}_i) = M_i \) \( (i = 1, 2) \).

Case 1 \( T(M_1, t_1) = a, \ T(M_2, t_2) = b, \ N = \text{Constapply}(a, b) \) exists and \( t = t_1+t_2+1 \).

Then \( t_1, t_2 < t \) so by induction \( [\text{cl}_1] \xrightarrow{*} [(a, e_1)] \) and \( [\text{cl}_2] \xrightarrow{*} [(b, e_2)] \) for some \( e_1, e_2 \). Consequently \( \text{CBV} \) has the computation

\[
[\text{cl}] \xrightarrow{*} [\text{cl}![\text{cl}_2]] \xrightarrow{*} [\text{cl}![\text{cl}[(b, e_2)]]] \xrightarrow{*} [[[\text{cl}](b, e_2)]] \xrightarrow{*} [[(\text{Constapply}(a, b), \{\} \}]]
\]

Now let \( \text{cl}! = (\text{Constapply}(a, b), \{\}) \).
Case 2 \( T(M_1, t_1) = \lambda x M_1', T(M_2, t_2) = M_2', T([M_2'/x]M_1', t_3) = N \) and \( t = t_1 + t_2 + t_3 + 1 \).

Then \( t_1, t_2 < t \) so by induction \( [cl_1] \Rightarrow [cl_1'] \) and \( [cl_2] \Rightarrow [cl_2'] \) where \( \text{Real}(cl_1') = \lambda x M_1' \) and \( \text{Real}(cl_2') = M_2' \). Without loss of generality \( cl_1' = (\lambda x M_1'', e) \) for some \( e, M_1'' \). Let \( cl_3 = (M_1'', e \{ cl_2'/x \}) \). By Lemma 2 of Appendix 1, \( \text{Real}(cl_3) = [M_2'/x]M_1' \). Using induction again, \( [cl_3] \Rightarrow [cl'''] \) where \( N = \text{Real}(cl') \). Putting these together the required CBV computation is

\[
[cl] \Rightarrow [cl_1[cl_2]] \Rightarrow [cl_1[cl_2']] \Rightarrow [[cl_1]cl_2']
\]

\[
\Rightarrow [[[\lambda x M_1'', e]cl_2']] \Rightarrow [(M_1'', e \{ cl_2'/x \})] = [cl_3] \Rightarrow [cl''']
\]

Lemma 3 If \( [cl] \Rightarrow [\sigma] \), \( \text{Last}(\sigma) = \sigma \) and \( \text{Real}(\sigma) \) is closed then \( T(\text{Real}(cl), t) = \text{Unload}(\sigma) \) for some \( t \geq 0 \).

This result and Corollary 2 yield Theorem 1.3.

**Proof** Let \( [cl] \Rightarrow [\sigma] \) where \( \text{Last}(\sigma) = \sigma \), \( \text{Real}(cl) \) is closed and \( cl = (M, e) \). The result is immediate if \( M \) is a constant or an abstraction, which must hold if \( n = 0 \).

Assume inductively that the result holds for all computations of length less than \( n \), where \( n > 0 \). If \( M \) is a variable then \( [cl] \Rightarrow [e(M)] \Rightarrow [\sigma] \) and \( \text{Real}(cl) = \text{Real}(e(M)) \) so the result holds by induction.

The remaining case if \( M = M_1M_2 \), so \( [cl] \Rightarrow [cl_1[cl_2]] \Rightarrow [\sigma] \) where \( cl_1 = (M_1, e) \) for \( i = 1, 2 \). Consider the computation \( [cl_2] = \sigma_1 \Rightarrow \sigma_2 \Rightarrow \ldots \). This must terminate in some \( \sigma_m \) without an "error stop," else \( [cl] \Rightarrow [\sigma] \) is violated. It is easy to see that \( \sigma_m = [cl_2'] \) where \( cl_2' = (M_2', e_2) \) for some value \( M_2' \). Thus the computation may be refined further:

\[
[cl] \Rightarrow [cl_1[cl_2]] \Rightarrow [cl_1[cl_2']] \Rightarrow [[[cl_1]cl_2']] \Rightarrow [\sigma]
\]

Since \( n_2 < n \) and \( \text{Last}([cl_2']) = [cl_2'] \) we may use induction to show that \( T(\text{Real}(cl_2), t_2) = \text{Real}(cl_2') \) for some \( t_2 \).

By exactly the same reasoning applied to \( cl_1 \) there must exist \( cl_1' = (M_1', e_1), t_1 \) such that \( [cl_1] \Rightarrow [cl_1'] \) and \( T(\text{Real}(cl_1), t_1) = \text{Real}(cl_1') \). The computation may be re-expressed:

\[
[cl] \Rightarrow [[[cl_1']cl_2']] \Rightarrow [\sigma]
\]
Case 1. \(M_1'\) is a constant \(a\).

Since the computation terminates \(M_2'\) must be a constant \(b\) such that \(\text{Constapply}(a,b)\) exists. By inference rule 2 of the definition of \(T\)

\[
T(\text{Real}(cl), t_1+t_2+1) = T(\text{Real}(cl_1)\text{Real}(cl_2), t_1+t_2+1) = \text{Constapply}(a,b)
\]

Case 2. \(M_1'\) is an abstraction \(\lambda xM_1''\).

By CBV transition rule 1 \([cl] \xrightarrow{\ast} [\text{[cl_1']}\text{[cl_2']}]) \xrightarrow{\ast} ([M_1'', e\{cl_2'/x\}]) = [cl_3] \Rightarrow_\sigma^n \sigma\)

where \(n_3 < n\). By induction \(T(\text{Real}(cl_3), t_3) = \text{Unload}(\sigma)\) for some \(t_3\). Let \(\text{Real}(cl_1') = \lambda xM_1'''\). By Lemma 2 of Appendix 1

\[
\text{Real}(cl_3) = [\text{Real}(cl_2')/x]M_1'''
\]

By inference rule 3 of the definition of \(T, T(\text{Real}(cl), t_1+t_2+t_3+1) = \text{Unload}(\sigma)\).