A Net Characterization of Graphs Based on Interception Relations

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Abstract

While graphs are normally defined in terms of the 2-place relation of adjacency, we take the 3-place relation of interception as the basic primitive of the definition. The paper views graphs as an economic scheme for encoding interception relations, and establishes an axiomatic characterization of relations that lend themselves to representation in terms of graph interception, thus providing a new characterization of graphs.

1 Introduction

One of the main reasons that graphs offer useful representations for a wide variety of phenomena is that they display vividly the associations that exist among objects in the domain and they distinguish direct from indirect associations. Traditionally, graph theory takes the notion of adjacency (or neighborhood) as a basic primitive, on the basis of which more elaborate notions, such as connectivity and interception are defined and analyzed. In certain applications, the property of adjacency cannot be measured directly nor can it be defined uniquely in terms of the measured properties. Instead, adjacency can only be postulated as a convenient means for explaining associational phenomena resembling connectivity and interception for which the distinction between direct and indirect association has clear operational definition in the domain.

A typical example is the notion of dependence and conditional depen dence in probability theory. Given a probability function P on a collection of variables or events, it is straightforward to determine whether a pair of variables X and Y are dependent or independent, and whether X and Y are conditionally independent given a third variable Z. Yet P does not dictate which pairs of variables are considered adjacent. It is not even clear whether the notion of adjacency, hence graph theory, would be helpful in analyzing properties of conditional independence. While it is true that conditional independence bears similarity to interception in graphs, the similarity may not be complete, and it is not easy to determine what properties of conditional independence are mirrored by graph interception.

This paper takes the notion of interception as a basic primitive, and establishes necessary and sufficient conditions under which a relation of indirect associations can be faithfully represented by graph thcoretical interception. When no faithful mapping exists, we then establish sufficient conditions for finding a unique best approximate representation in graphs. Thus, the paper lays a logical basis for studies such as Markov random fields [Isham, I-81] and graphical models in statistics [Lauritzen, L-82], where graphs are used primarily as a language for encoding com plex patterns of mediated associations.

2 Definitions and notations

Let U be a finite set $U = \{u_1, \ldots, u_n\}$, and let X, Z, Y denote finite subsets of U. We consider ternary relations over U as sets of triplets of the form (X, Z, Y). In the sequel we shall assume, unless otherwise specified that all the relations R considered have the following properties:

(i) If $(X, Z, Y) \in R$ (notation: $R(X, Z, Y)$) then X, Z, Y are mutually disjoint sets.

(ii) $R(X, Z, \emptyset)$ and $R(\emptyset, Z, X)$ for all disjoint X, Z, Y, where $\emptyset$ denotes the empty set.

The elements of R will be called triplets, each triplet conveying the general notion of Z intercepting or mediating the indirect interaction between X and Y.

Undirected graphs (UG's) will be denoted be $G = (V, E)$ where V are the vertices and E are the edges of G. The graphs considered in this paper will be assumed to be simple and with no loops (i.e. if $(a, b) \in E$ then $a \neq b$).

Definition 1 *Let $G = (V, E)$ be a graph. The relation R_G over V, induced by G, is defined as follows. $(X, Z, Y) \in R_G$ iff either X is disconnected from Y in G or Z is a cutset between X and Y in G.* □

Notice that in the above definition Z is not required to be a minimal cutset between X and Y. For any set of vertices $X \subseteq V$, X is considered disconnected from $\emptyset$ by definition.

Definition 2 *Let t be a triplet over V where $G = (V, E)$ is a given graph. t is represented (or holds) in G if and only if $t \in RG$* □

Definition 3 *Let $G = (V, E)$ be a graph and let R be a relation over V. G is an* I-map *of R if $R_G \subseteq R$. G is a* D-map *of R if $R \subseteq R_G$. G is a perfect map of R and represents R if $R = R_G$.* □

Remark The I-map and D-map definitions are borrowed from the applications to probabilistic distribution representations by graphs where "I" stands for "Independency" and "D" stands for "Dependency".

3 Properties of R_G

Let G be a graph and let R_G be the relation induced by G. The subscript G will be omitted from R_G in the sequel and when understood.
Lemma 1 *The relation R induced by G has the following properties:*

(1) $R(X,Z,Y) \to R(Y,Z,X)$ *: Symmetry.*

(2) $R(X,Z,Y \cup W) \to R(X,Z,Y) \wedge R(X,Z,W)$ *: Decomposition.*

(3) $R(X,Z,Y) \to R(X,Z \cup W,Y)$ *for all W disjoint* from $X \cup Z \cup Y$: *Strong Union.*

(4) $R(X,Z \cup W,Y) \wedge R(X,Z \cup Y,W) \to R(X,Z,Y \cup W)$ *: Intersection.*

(5) $R(X,Z,Y) \to R(\{a\},Z,Y) \vee R(X,Z,\{a\})$ *for any $a \in V, a \notin \{X \cup Z \cup Y\}$: Transitiuity.*

The above properties will be called the Graph Axioms in the sequel.

Proof: The proof of the first three properties is trivial and left to the reader.

Proof of the intersection property: Assume that the lefthand side of (4) holds for $R(= R_G)$. If X is disconnected in G from both Y and W then the righthand side of (4) holds by definition.

If X is disconnected from Y say, but is not disconnected from W then $R(X,Z \cup Y,W)$ on the lefthand side of (4) implies that all the paths between X and W intersect Z. Thus $R(X,Z,Y \cup W)$. The case where X is disconnected from W only is similar.

The remaining case is the case where X is connected to both Y and W. Assume that this is the case and that the righthand side of (4) does not hold. From $\neg R(X,Z,Y \cup W)$ we infer that there is a path from X to Y not intercepted by Z or there is a path from X to W not intercepted by Z. Assume the former w.l.o.g.. From $R(X,Z \cup W,Y)$ we infer that the path from X to Y not intercepted by Z must be intercepted by W. We conclude that there is a path from X to W not intercepted by $Z \cup Y$ contrary to $R(X, Z \cup Y, W)$. This contradiction completes the proof.

Proof of Transitivity: If $R(X, Z, Y)$ and $a \notin \{X \cup Z \cup Y\}$ then either Z disconnects X from a or Z disconnects Y from a since otherwise X is connected to Y via a path through a not intersecting Z. Thus the righthand side of (5) must hold. □

Definition 4 *Let R be a relation and let f be a boolean formula involving triplets. If f is an (atomic) triplet then f holds in R iff $f \in R$.*

If $f = g \vee h$ then f holds in R iff either g or h holds in R.

If $f = h \wedge g$ then f holds in R iff both h and g hold in R.

If $f = \neg g$ then f holds in R if and only if g does not hold in R.

The relation R is closed *under a set of axioms A iff whenever the lefthand side of an axiom holds in R then the righthand side of that axiom holds in R.* □

Corollary 1 *If R is a relation induced by a graph then R is closed under the graph axioms.*

Lemma 2 *Let R be a relation closed under the graph axioms. Then R also satisfies the following properties:*

(6) $R(X, Z, Y) \wedge R(X, Z, W) \rightarrow R(X, Z, Y \cup W)$ where X, Z, Y, W *are mutually disjoint.*

(7) $R(X, Z, Y) \leftrightarrow (\forall a \in X)(\forall b \in Y) R(a, Z, b)$

Proof: From $R(X, Z, Y) \wedge R(X, Z, W)$ we get, by axiom (3), $R(X, Z \cup W, Y) \wedge R(X, Z \cup Y, W)$ which imply, by axiom (4) $R(X, Z, Y \cup W)$. This proves property (6). Property (7) is a direct consequence of properties (1), (2) and (6). □

4 The Graph Characterization Theorem

We can now prove our main Theorem.

Theorem 3 *Let R be a ternary relation over V. Iff R is closed under the Graph Axioms then a graph G can be constructed such that G is a perfect map of R (i.e. $R = R_G$).*

Proof: Given a relation R over V construct the graph $G_0 = (E_0, V)$ such that for every pair $a, b \in V$, $a \neq b$; $(a, b) \in E_0$ iff $(a, V/\{a, b\}, b) \notin R$. (We use here and will use in the sequel the notation "a" for "$\{a\}$", "b" for "$\{b\}$", etc. where a, b etc. are vertices.)

We split the proof into two parts.

First we prove that if R is closed under the axioms (1), (2) and (4) then the graph G_0, defined above, is an I-map of R. In the second part of the proof we will show that if R is closed under all the graph axioms then G_0 is a D-map of R thus showing that G_0 is a perfect map of R.

Proof of I-mapness. Assume that R is closed under the axioms (1), (2) and (4). We show, by finite descending induction on the size of the middle set $\mid Z \mid$ that $R_{G_0} \subseteq R$ ($\mid S \mid$ denotes the number of elements in the set S).

Basis: $t = (a, V/\{a, b\}, b), \mid V/\{a, b\} \mid = n-2$. t is represented in G_0 iff (a, b) is not an edge of G_0, iff $t \in R$, by the construction of G_0.

Step: Assume that all $t = (X, Z, Y) \in R_{G_0}$ with $\mid Z \mid = k$, for some $k(\leq n-2)$, are in R, and let $t' = (X', Z', Y') \in R_{G_0}$ be a triplet such that $\mid Z' \mid = k - 1(< n-2)$. To show that $t \in R$, we distinguish between 2 subcases.

Subcase 1: $X' \cup Y' \cup Z' = V$. From $\mid Z' \mid = k-1(< n-2)$ we infer that either $\mid X' \mid \geq 2$ or $\mid Y' \mid \geq 2$ and we may assume w.l.o.g. that $\mid Y' \mid \geq 2$ with $Y' = Y'' \cup_c$, where c is a vertex. Then $R_{G_O}(X', Z', Y'' \cup_c) \rightarrow R_{G_0}(X', Z', c) \wedge R_{G_0}(X', Z', Y'')$ by decomposition (which holds for graph relations). By strong union we get from the above that $R_{G_0}(X', Z' \cup Y'', c) \wedge R_{G_0}(X', Z' \cup_c, Y'')$. By the induction hypothesis we get $R(X', Z' \cup Y'', c) \wedge R(X', Z' \cup c, Y'')$ since $\mid Z' \cup Y'' \mid, \mid Z' \cup c \mid \geq k$. By intersection, which holds for R, we get $R(X', Z', Y'' \cup c) = R(X', Z', Y')$ as required.

Subcase 2: $X' \cup Y' \cup Z' \overset{\subset}{\neq} V$. Let c be a vertex $c \notin X' \cup Y' \cup Z'$. From $R_{G_0}(X', Z', Y')$ we get, by transitivity, that $R_{G_0}(c, Z', Y') \vee R_{G_0}(X', Z', c)$. By strong union we get, from the above that $R_{G_0}(X', Z' \cup c, Y')$ and $R_{G_0}(c, Z' \cup X', Y') \vee R_{G_0}(X', Z' \cup Y', c)$ by induction, since now the size of the middle sets is at least k, we get

$$R(X', Z' \cup c, Y') \wedge [R(c, Z' \cup X', Y') \vee R(X', Z' \cup Y', c)]$$

By intersection and symmetry, which holds for R we get from the above that

$$R(c \cup X', Z', Y') \vee R(X', Z', Y' \cup c)$$

Finally, by decomposition and symmetry, which holds for R we get $R(X', Z', Y')$. as required. We have thus shown that G_0 is an I-map of R.

Proof of D-mapness. Based on lemma 2 it is enough to prove D-mapness (i.e., that $R(X, Z, Y)$ implies $R_{G_0}(X, Z, Y)$) for triplets of the form (a, Z, b) where a and b are single vertices. The proof is again by descending induction on the size of $\mid Z \mid$.

Basis: For $\mid Z \mid = n - 2$ we know that $t = (a, V/\{a, b\}, b) \in G_0$ iff (a, b) is not an edge of G_0, iff $t \in R$, by the construction of G_0.

Step. Assume that all $t = (a, Z, b) \in R$ with $\mid Z \mid = k$, for some $k (\leq n - 2)$, are in R_{G_0}. Let $t' = (a', Z', b') \in R$ be a triplet such that $\mid Z' \mid = k - 1, (< n - 2)$. Then there is some $c \notin Z' \cup \{a, b\}$. From $t' \in R$ we get, by transitivity that $R(c, Z', b') \vee R(a', Z', c)$ and from this and t' we get by strong union that

$$R(a', Z' \cup c, b') \wedge [R(c, Z' \cup a', b') \vee R(a', Z' \cup b', c)]$$

by induction (since the middle sets have now size $\geq k$) we get

$$R_{G_0}(a', Z' \cup c, b') \wedge [R_{G_0}(c, Z' \cup a', b') \vee R_{G_0}(a', Z' \cup b', c)]$$

By intersection and symmetry we get $R_{G_0}(c \cup a', Z', b') \vee R_{G_0}(a', Z', c \cup b')$.

Finally by decomposition and symmetry we get from the above that $R_{G_0}(a', Z', b')$ holds, as required. This completes the proof of the D-mapness and of the "if" part of the theorem. The "only if" part follows from lemma 1. □

Corollary 2 *Let R be a relation over V, closed under the axioms (1), (2) and (4). Then a unique graph $G = (V, E)$ can be constructed such that $R_G \subseteq R$ (i.e. G is an I-map of R) and such that G is edge minimal (i.e. if an edge is removed from G then its I-mapness is violated).*

Proof The first part of the corollary follows from the first part of the proof of Theorem 3, showing that G_0 is an I-map of R, under the condition of the corollary. If an edge is added to G_0 it will still be an I-map of R since addition of edges to a graph G can only remove triplets from R_G but cannot add triplets to R_G. On the other hand if an edge (a, b) is removed from G_0 then (at least) the triplet $t = (a, V/\{a, b\}, b)$ is added to R_{G_0}, but this triplet is not in R since, by construction $t \in R$ iff (a, b) is not an edge of G_0. Thus any edge minimal I-map of R must be equal to G_0. □

Not all the graph axioms are needed to guarantee the existence of a unique minimal I-map G_0. In Section 7 we give a weaker set of axioms which is sufficient to provide this guarantee.

5 Extensions

Definition 5 *Let $\sum$ be a set of triplets over a set V. A relation R, over V, is an extension of $\sum$ (notation $R_{\sum}$) if it satisfies the following conditions*

1. $\sum \subseteq R$
2. *R is closed under the graph axioms.*

R is a minimal extension of $\sum$ if no proper subset R is an extension of $\sum$. R is a minimum extension of $\sum$ if any other extension R' of $\sum$ satisfies $\mid R' \mid \geq \mid R \mid$. □

Example: Let $\sum = \{(a, c, b), (a, d, b)\}$ over $\{a, b, c, d\}$. The relations shown below are minimal extensions of $\sum$ and both are at the same time minimum extensions too.

$$
\begin{aligned}
R_1 = \{&(a,\{b,d\},c),(a,\{c,d\},b),(b,\{a,c\},d),(a,d,\{b,c\}),(b,c,\{a,d\}),\\
&(a,c,b),(a,d,c),(a,d,b),(b,c,d) + \text{ symmetric triplets}\}\\
R_2 = \{&(a,\{b,c\},d),(a,\{c,d\},b),(b,\{a,d\},c),(a,c,\{b,d\}),(b,d,\{a,c\}),\\
&(a,c,b),(a,c,d),(a,d,b),(b,d,c) + \text{ symmetric triplets}\}
\end{aligned}
$$

There are additional extensions which are minimal but not minimum. The extension including all the possible triplets over V is neither minimal nor minimum.

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An algorithm is shown below which provides minimal extensions of a given set $\sum$, whose time complexity is polynomial in the size of $\sum$. Finding an extension which is minimum, in polynomial time, is an open problem.

An algorithm for finding minimal extensions of a given set of triplets $\sum$ over a set V

1. Start with the complete graph over V and remove all edges (a, b) such that $a \in X$ and $b \in Y$ for some $(X, Z, Y) \in \sum$. Denote the resulting graph by $G_{\sum}$.
2. If $\sum$ (i.e. all the triplets in $\sum$) is represented in $G_{\sum}$ then return $G_{\sum}$.
3. Let $\sigma = (X, Z, Y)$ be the first triplet in $\sum$ not represented in $G_{\sum}$. This implies that there are vertices a, b, c in V such that $a \in X, b \in Y$ and $c \notin X \cup Y \cup Z$ and such that there is a path from a to b in $G_{\sum}$ passing through c and not passing through Z (this follows from the fact that as a result of step 1, all the vertices in X are not directly connected to the vertices in Y in $G_{\sum}$). Choose c as above to be a vertex with at least one neighbour in X (i.e. c is chosen to be the first vertex outside of $X \cup Y \cup Z$ on a path between a and b outside of Z). Reset $G_{\sum}$ by removing from it all edges connecting c to a vertex in X. Go to 2.

End of algorithm.

The number of iterations of the algorithm is $O(n^2)$, since at every iteration at least one edge is removed from $G_{\sum}$, and, as the number of operations at every iteration is polynomial in the size of $\sum$, the algorithm is polynomial in the size of $\sum$.

The graph $G_{\sum}$ output by the algorithm defines the relation $R_{G_{\sum}}$ which includes $\sum$ as a subset and is closed under the graph axioms. Thus $R_{G_{\sum}}$ is an extension of $\sum$. That the extension is minimal can be shown as follows: Every extension of $\sum$ is a relation closed under the graph axioms. By (the characterization) theorem 3, every relation closed under the graph axioms has a unique graph which is a perfect map of it. The algorithm directly constructs the graph representing such an extension and the edges removed at steps 1 and 3 are a minimal set of edges whose removal is necessary in order to enable the representation of $\sum$ in the graph.

6 Soundness and Completness

Denote by A the set of graph axioms and by $\mathcal{G}$ the farnily of simple undirected graphs with no loops.

Definition 6 *Let $\sum$ be a set of triplets and σ a single triplet. $\sum$ A-derives σ (notation: $\sum \vdash_A \sigma$) iff σ is an element of every extension of $\sum$.* □

The relation of A-derivation to the usual concept of deductive derivation will be given in Definitlon 9.

Definition 7 *Let $\sum$ be a set of triplets and σ a single triplet. $\sum \mathcal{G}$ implies σ (notation: $\sum \vdash_{\mathcal{G}} \sigma$) iff for any graph $G \subset \mathcal{G}, \sum \subset R_G$ implies that $\sigma \in R_G$.* □

Definition 8 *A set B of axioms is sound for $\mathcal{G}$ if $\sum \vdash_B \sigma$ implies that $\sum \models_{\mathcal{G}} \sigma$. The set of axioms is complete for $\mathcal{G}$ if $\sum \models_{\mathcal{G}} \sigma$ implies $\sum \vdash_B \sigma$ for any set of triplets $\sum$ and single triplet σ.* □

Theorem 4 *The graph axioms A are sound and complete for $\mathcal{G}$.*

Proof of soundness. Let G be any graph in $\mathcal{G}$, assume that $\sum$ is represented in G and assume that $\sum \vdash_A \sigma$. R_G is an extension of $\sum$ and therefore, from $\sum \vdash_A \sigma$, we get that σ is represented in R_G as required.

Proof of completeness. Given $\sum$ and σ, let $R_{\sum}$ be an extension of $\sum$. By theorem 3, there is a graph G which is a perfect map of $R_{\sum}$. Thus $\sum \subset R_G = R_{\sum}$. From $\sum \models_{\mathcal{G}} \sigma$ we get that $\sigma \in R_G$. Thus $\sigma \in R_{\sum} = R_G$. $\Box$ The concept of an A-derivation, as defined in Definition 6 depends on the concept of an extension. The usual (and stronger) concept of A-derivation is defined below.

Definition 9 *Let $\sum$ be a set of triplets and σ a single triplet. $\sum$ strongly A-derives σ (notation: $\sum \mid\!\vdash_A \sigma$) if a can be derived from $\sum$ by a deductive chain of formulas $f_1, f_2, \ldots, f_k$ such that $f_k = \sigma$ and every $f_i, i < k$, is a boolean formula of triplets such that either $f_i \in \sum$ or f_i is derived from previous f_j's in the chain as a derivation of the propositional calculus extended by the A-axioms.* $\Box$

Example: Let $\sum = \{(3, 2, \{1, 4\}), (1, 2, \{3, 4\}), (1, 4, 3)\}$ and let $\sigma = (1, \emptyset, 3)$, over $V = \{1, 2, 3, 4\}$. Below is a derivation chain for σ.

f_1 : $(1, 4, 3) \in \sum$
f_2 : $(2, 4, 3) \vee (1, 4, 2)$ by transitivity
f_3 : $(2, \{1, 4\}, 3) \vee (1, \{3, 4\}, 2)$ from f_2 by strong union and propositional calculus
f_4 : $(\{1, 4\}, 2, 3)$ from $\sum$ by symmetry
f_5 : $(1, 2, \{3, 4\}) \in \sum$
f_6 : $(1, 2, 4, \emptyset, 3) \vee (1, \emptyset, \{2, 3, 4\})$ from f_3, f_4 and f_5 by symmetry, intersection and propositional calculus
f_7 : $(1, \emptyset, 3) \vee (1, \emptyset, 3)$ from f_6 by decomposition and propositional calculus
f_8 : $(1, \emptyset, 3)$ from f_7 by propositional calculus.

It is easy to see that strong A-derivation implies A-derivation: Let $f_1, \ldots, f_k$ be a strong derlvation of σ from $\sum$, and let $R_{\sum}$ be an extension of $\sum$. Then $f_i, 1 < i < k$, holds in $R_{\sum}$, since $\sum \subset R_{\sum}$, and $R_{\sum}$ is closed under A. It follows that $\sigma = f_k \in R_{\sum}$, so that $\sum$ A-derives σ. We have thus proved

Lemma 5 *For any set of triplets $\sum$ and single triplet σ,$\sum \mid\!\vdash_A \sigma$ implies* *that* $\sum \vdash_A \sigma$.

The question whether $\sum \vdash_A \sigma$ implies $\sum \mid\vdash_A \sigma$ is open. A positive answer to this question will assert (based on the fact that every extension of $\sum$ has a perfect graph representation) that every valid cut-set graph property of the form: "For any graph G, if $\sum$ holds in R_G then σ holds in R_G" can be proved in propositional calculus when extended by the graph axioms. Consider e.g. again the example above. The example can be extended to the following: Let $G(V, E)$ be a graph and let X, Y, Z, W be a partition of V such that Y is a connected set of vertices. If $(Z, Y, X \cup W), (X, Y, Z \cup W)$ and (X, W, Z) hold in R_G then G has at least 2 components with X in one component and Z in the other (which is equivalent to $(X, \emptyset, Z) \in R_G$). The example shows that this particular property can be proved in the propositional calculus when extended by the graph axioms. The question whether every valid property of graph separation can be decided by these means depends on whether $\vdash$ implies $\mid\vdash$.

7 NP-completeness of weak independence

Let $\sum$ be a set of triplets and σ a triplet over V. We shall say that σ is weakly independent on $\sum$ if $\sum \vdash_a \sigma$ does not hold. σ is strongly independent on $\sum$ if $\sum \not\mid\vdash_A \sigma$ does not hold. It follows from lemma 5 that weak independence implies strong independence. It follows from theorem 4 that σ is weakly independent on $\sum$ (notation: $\sum \not\vdash_A \sigma$) if and only if there exists a graph G such that $\sum$ is represented in G and σ is not represented in G. We will show now that the problem of ascertaining whether a graph G as above (representing $\sum$ and not representing σ) exists for any given $\sum$ and σ is NP-complete.

The fact that the problem is in NP is trivial since for any given $\sum$ and σ we can guess, in polynomial time, a graph and then check, in polynomial time, whether it has the required property. To show NP-completeness we will present a polynomial reduction from the Hamiltonian problem, a well known NP-complete problem (see [Garrey, GJ-79]), to the weak independence problem. We set first the definitions in standard form:

Hamiltonian. Input: a graph $G(V, E)$ and a pair of vertices $a, b \in V$.

Problem: Does there exist a path in G from a to b passing through every vertex in V exactly once?

Independence. Input: a set of triplets $\sum$ over V and a triplet σ over V. Problem: Does there exist a graph $G = (V, E)$ such that $\sum$ is represented in G and σ is not represented in G?

Theorem 6 *The independence problem is NP-complete.*

Proof: We have already shown that Independence is in NP. Consider now the following reduction. Given $G = (V, E)$ and $a, b \in V$, an input for the Hamiltonian problem, set:

$$\begin{aligned}
\textstyle\sum_1 &= \{(u, V/\{u, v\}, v) : (u, v) \notin E\} \\
\textstyle\sum_2 &= \{(a, v, b) : v \in v/\{a, b\}\} \\
\textstyle\sum &= \textstyle\sum_1 \cup \sum_2 \\
\sigma &= (a, \emptyset, b).
\end{aligned}$$

It is clear that $\sum$ and σ can be set in polynomial time. To complete the proof of the Theorem it suffices to prove the following claim: The Hamiltonian problem with input G and a, b has a solution if and only if the independence problem has a solution with input $\sum$ and σ.

To prove this claim we notice first that every graph G' that satisfies $\sum_1$ is a subgraph of G, by the definition of $\sum_1$. If in addition G' does not satisfy σ then there must be a path in G' between a and b. Finally, if G' satisfies $\sum_2$ then the path in G' between a and b must be intercepted by every vertex in V exactly once (every vertex in V must disconnect between a and b, as required by $\sum_2$. The path cannot have a loop since otherwise the vertices on the loop will not disconnect a from b). Thus, if G' satisfies $\sum$ and does not satisfy σ then it has a Hamiltonian path between a and b and this Hamiltonian path exists in G, since G' is a subgraph of G. On the other hand, if G has a Hamiltonian subgraph G' between a and b then G' is a subgraph of G satisfying $\sum$ and not satisfying σ, as is easy to see. $\square$

8 Neighborhoods

Given a relation R over a set V, the proof of theorem 3 provides a method for constructing a graph G_0 such that $R_{G_0} = R$ if and only if R is closed under the graph axioms. If R is closed under the symmetry, decomposition and intersection axioms then G_0 is only an edge minimal I-map of R. An intermediary situation will be considered in this section. We will show that if R is closed under the axioms of symmetry, decomposition, intersection and weak union - an axiom to be defined below - then the approximation provided by G_0 for R is not only an edge minimal I-map of R but stronger in the sense that it encodes and unifies two diverse notions of neighborhood in R.

Since each (X, Z, Y) triplet in R conveys the informal notion of broken interaction (Z breaks the interaction between X and Y), there are two natural ways of defining neighborhood. One is to proclaim a pair of elements a and b neighbors iff their interaction cannot be broken by all other elements in U, namely, $(a, U\{a, b\}, b) \notin R$. Alternatively, we may wish to define the neighbors of a as a minimal set of elements needed to break the interaction between a and all other elements of U. We will show that under certain conditions these two notions of neighborhood will become identical, and will coincide with ordinary adjacency in G_0.

The following property, for a relation R over a set V will be called the axiom of *weak union*

$$R(X, Z, Y \cup W) \rightarrow R(X, Z \cup Y, W) \wedge R(X, Z \cup W, Y).$$

Notice that a relation which satisfies strong union and decomposition also satisfies weak union; since we can remove first, by decomposition, W or Y from the lefthand side of $R(X, Z, Y \cup W)$ and then reinsert the removed set in the middle, by strong union.

Let R be a relation over V, let a be a vertex in V and let G_0 be the graph defined in the proof of theorem 3 for the given R. Define the set $S(a)$ as below

$$S(a) = \{b : R(a, V/\{a, b\}, b), b \in V\}.$$

By the definition of G_0, the set $S(a)$ is the set of vertices in G_0 which are not connected by an edge to a in G_0. It follows that $V/\{S(a) \cup a\}$ is the set of neighbours of a in G_0, the set of vertices in G_0 connected by an edge to a in G_0. Denote this set by $N(a)$, i.e. $N(a) = V/\{S(a) \cup a\}$. For the given R, V and $a \in V$ we define also the following sets

$$\begin{array}{ll} \varphi(a) & = \{X : R(a, X, V/\{X \cup a\})\} \\ Bl(a) & = \text{the set in } \varphi(a) \text{ whose size (= number of elements) is minimal} \end{array}$$

$Bl(a)$ stands for the "blanket of a" since it is a set of minimal size shielding a from the rest of the vertices.

Theorem 7 *Let R be a relation closed under the axioms of symmetry, decomposition, weak union and intersection. Then Bl(a) is uniquely de fined and Bl(a) = N(a).*

Proof: We will show that for any set $X \subseteq V$ such that $a \notin X$, $X \in \varphi(a)$ if and only if $X \supseteqq N(a)$. This implies the claim of the theorem since it shows that all the sets in $\varphi(a)$ are supersets of $N(a)$ and it shows that $N(a)$ itself is a set in $\varphi(a)$ so the set of minimal size in $\varphi(a)$ must equal to $N(a)$.

Proof of the "if" part. Assume $X \supseteq N(a)$. Then $X = V/\{Y \cup a\}$ where $Y \subseteq S(a)$. We prove, by induction on the size of Y that $X \in \varphi(a)$.

Basis. $\mid Y \mid= 1$ or $Y = y \in S(a)$ then, by the definition of $S(a)$, $R(a, V/\{a, y\}, y) = R(a, X, y)$ by our assumption on X. Thus $X \in \varphi(a)$.

Step. Assume that the claim is true for Y_1, $\mid Y_1 \mid\geq 1$ and let $Y = \{Y_1 \cup y\} \subseteq S(a)$ where y is a singleton.

We can assume, by induction, that $X_1 = V/\{Y_1 \cup a\}$ and $X_2 = V/\{y \cup a\}$ are in $\varphi(a)$ so that

$$R(a, V/\{Y_1 \cup a\}, Y_1) \wedge R(a, V/\{y \cup a\}, y).$$

By intersection we get from the above that

$$R(a, V/\{Y_1 \cup a \cup y\}, Y_1 \cup y\} = R(a, V/\{Y \cup a\}, Y)$$

or $Y \in \varphi(a)$ as required.

Proof of the “only if” part. Assume that $X \in \varphi(a)$. Then $R(a, X, V/\{X \cup a\})$. If X is not a superset of $N(a)$ then we can set $N(a) = X_1 \cup X_2$, $X = X_1 \cup X_3$ and $X_2 \notin \emptyset$; X_1, X_2, X_3 mutually disjoint.

Now $R(a, X, V/\{X \cup a\}) = R(a, X_3 \cup X_1, V/\{Xcupa\})$ is given. Also $R(a, N(a), V/\{N(a) \cup a\}) = R(a, X_1 \cup X_2, V/\{N(a) \cup a\})$ since, by the first part of the proof $N(a)$ is in $\varphi(a)$.

All the elements of V are included in the above triplets with X_2 in the righthand side of the first triplet and X_3 in the righthand side of the second. We can therefore move, by weak union, all variables in the righthand side not in X_2 to the middle part in the first triplet and all the variables in the righthand side not in X_3 to the middle part of the second triplet resulting in

$$R(a, V/\{X_2 \cup a\}, X_2) \wedge R(a, V/\{X_3 \cup a\}, X_3).$$

By intersection we get

$$R(a, V/\{X_2 \cup X_3 \cup a\}, X_2 \cup X_3).$$

But we assumed that $X_2 \notin \emptyset$. Let b be a variable in X_2. Using weak union again, we can move all the elements except b from the righthand side of the above triplet to the middle, resulting in

$$R(a, V/\{a, b\}, b)$$

On the other hand $b \in X_2 \subseteq N(a)$ and $N(a)$ is disjoint from $S(a)$, implying that $b \notin S(a)$ which, by the definition of $S(a)$ implies that $\neg R(a, V/\{a, b\}, b)$, a contradiction. We must therefore conclude that $X \supseteq N(a)$. $\square$

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References

[Garrey, GJ-79] Garrey, M.R. and Johnson, D.S. :
Computers and Intractability: A Guide to the Theory of NP-completeness,
Freeman, San Francisco, 1979.

[Isham, I-81] Isham, V.
An Introduction to Spatial Proint Processes and Markov Random Fields,
International Statist Review, 49, 21-43,1981.

[Lauritzen, L-82] Lauritzen, S.L.
Lectures on Contingency Tables, 2nd Edition,
Aalborg, Denmark, University of Aalborg Press, 1982.