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A Logical Characterization of Asynchronously Communicating Agents

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Abstract

In this paper we study the behaviour of distributed systems. We consider systems composed of a *fixed* number of sequential processes communicating by asynchronous message passing. The behaviour is represented by a subclass of partial orders called *asynchronously communicating agent structures*, abbreviated ACA structures.

We present a logical characterization of ACA structures in the framework of temporal logic. The modalities of the logic capture the concepts of *communication*, *concurrency* and *locality*. We define an axiomatic basis for the logic and show both soundness and completeness.

Contents

1 Introduction

Models of distributed systems are traditionally defined using interleaving semantics: the behaviour is either represented as linear sequences or as tree structures. One consequence of this choice is that events which can occur concurrently will be constrained by causality, e.g. two events e and e' which can occur concurrently will be modelled either as e occurring before e' or as e' occurring before e.

In this paper we model the behaviour of distributed systems by structures of partial orders. Concurrent events will be modelled as events which are unordered and thus interpreted as events occurring independently.

We will model the behaviour of distributed systems composed of a *fixed* number of sequential processes communicating by asynchronous message passing. The behaviour of such systems defines a class of structures called asynchronously communicating agent structures, abbreviated to ACA structures.

An ACA structure consists of set of *event occurrences* together with a *causality relation.* The causality relation expresses the *comes before* relation between events, i.e. it reflects the case that one event has to occur before another event can occur. An ACA structure is partitioned into a fixed number of *disjoint substructures* each of which is totally ordered by the causality relation. Thus two different events e and e' from the same substructure will be ordered: either e comes before e' or e' comes before e. The substructures are called *agents.*

We can think of agents as modelling the behaviours or *runs* of sequential processes. A distributed system composed of several sequential processes will be modelled by several agents–one agent for each process–organized in an ACA structure. Asynchronous communication between the processes will be reflected by causal dependencies between events from different agents and expressed through the causality relation.

The main purpose of this paper is to give a logical characterization of ACA structures in the framework of temporal logic. We adapt the idea from [LT87] and develop *indexed* temporal operators which are used to describe the agents. The logical language consists basically of three different kinds of operators. One operator which is taken from [LT87] reflects communication in the past, and one operator reflects communication in the future. Finally we define an operator reflecting concurrency. We also define other temporal operators but they are all derived from these three operators. In particular we define local past and local future operators which are identical to the operators P, F, H and G in the framework of temporal logic [Bur84]. They are called local operators because they can only be used in connection with single agents.

The logical language we develop is closely related to the concepts of communication, concurrency and locality. We think it is reasonable to focus on these concepts in the specification and verification of concurrent systems. The inclusion of locality is very important because it provides the possibilities of describing concurrent systems at the level of individual components. In specifying large or complex concurrent systems it is desirable to focus on the individual components of the system at first, because then the number of properties to specify and verify will be minimised and a better understanding of the overall system can be achieved. Having specified and verified the individual components it becomes necessary to broaden attention to the entire system, where we wish to reason about concurrency and communication between two components.

Related work: The foundation for the work presented in this paper is the work of K. Lodaya and P.S. Thiagarajan [LT87]. In [LT87] a logical characterization of a class of structures called *n-agent event structures* is presented. An n-agent event structure is a structure consisting of n disjoint subsets each of which can exhibit causality and non-determinism but no concurrency. Thus each subset can be organized as a tree structure. The class of ACA structures which we consider in this paper may be considered as the subclass of n-agent event structures in which each substructure is a sequence and not a tree structure. The logical language developed in [LT87] mainly consists of temporal operators reflecting communication that have occurred in the past.

Our work is also related to the work of M. Mukund and P.S. Thiagarajan [MT89]. In [MT89] a logical characterization of a class of structures called prime event structures is presented. The logic designed consists of operators reflecting the concepts of causality, non-determinism and concurrency. Our work is related to [MT89] especially concerning the modalities reflecting concurrency. Because of the concurrency operators a rather remarkable inference rule (which is adapted from [Bur80]) is included both in our work and in [MT89] in order to obtain the proof of completeness. However, the style of the two completeness proofs differ; we use a so-called Henkin proof style [Hen49] whereas in [MT89] a novel proof style is considered.

Finally, but in a less direct way, our work is related to the work of W. Penczek [Pen88]. In [Pen88] a logical characterization of prime event structures capturing the concepts of causality and non-determinism is presented.

The outline of the paper: In section 2 we define the ACA structures which are used in a Kripke-style interpretation of the logical language defined in section 3. In section 4 we present an axiomatic basis which is shown to be sound. In section 5 we show completeness of the logic w.r.t. the axiomatic basis and in section 6 we finish the paper with a conclusion. Finally we have included four appendices containing rather comprehensive proofs of some of the theorems presented in the paper.

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2 ACA structures

Throughout the paper we shall let ω denote the non-negative integers and let $n \in \omega$ be a fixed number. The variables i, j and k will range over the set $\{1, \dots, n\}$ if nothing else is stated.

We begin by defining the class of ACA structures.

Definition 2.1 An *ACA structure* is a pair $(E, <)$ where

(i) E is a set of *event occurrences*,

(ii) $< \subset E \times E$ is the *causality relation* which is irreflexive and transitive,

(iii) $E = E_1 \uplus E_2 \uplus \dots \uplus E_n$ where $\uplus$ denotes disjoint union, and

(iv) for each i the structure $(E_i, <_i)$ where $<_i$ is the relation $<$ restricted to $E_i \times E_i$ is a totally ordered set, i.e.

$$\forall e, e' \in E : e \neq e' \text{ implies } e < e' \text{ or } e' < e.$$

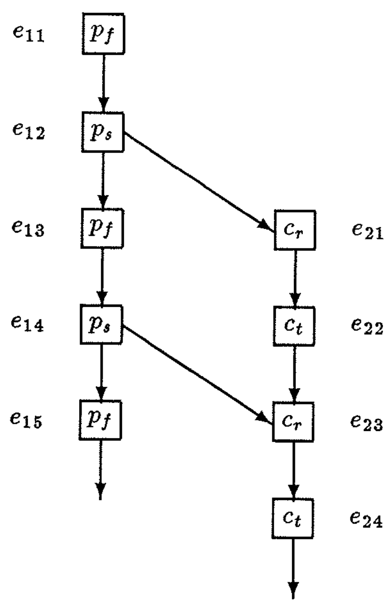
The structure $(E_i, <_i)$ will be called the i^{th} *agent.* ∎

We let $A, A', A'', \dots$ denote ACA structures and $A_i, A_i', A_i'', \dots$ the i^{th} agents of the structures $A, A', A'', \dots$ respectively. Finally we let $e, e_1, e_2, \dots, e', e'', \dots$ denote events.

We present an example of an ACA structure in which the value of n is two.

Example 2.2 Let $A = (E_1 \uplus E_2, <)$ with $E_1 = \{e_{1j} \mid j = 1, \dots\}$, $E_2 = \{e_{2j} \mid j = 1, \dots\}$ and the causality relation given by figure 2.1 below.

Figure 2.1



The first agent is supposed to model a producer and the second a consumer. The events have been labelled in order to express that the producer is sending products via an unbound buffer to the consumer. The p_f and p_s actions of the producer indicate fabrication and sending of the product respectively. The c_r and c_t actions of the consumer indicate reception and treatment of the product respectively. For convenience we have just indicated the "minimal" elements of the causality relation as directed arcs. For instance, e_{13} is causal dependent on e_{11} but will not be drawn. Throughout the paper we will draw the agents upwards down as the choice of the shape of the temporal operators are better justified by this picture of agents. ■

Intuitively ACA structures can be thought of as composed of a fixed number of sequences denoting the agents. Between the sequences there can exist causal dependencies and this will reflect communication between agents. We would like to emphasize that ACA structures are capable of expressing causality and concurrency. Concurrency is expressed through the lack of ordering between events. For instance, the events e_{13} and e_{21} of the ACA structure defined in example 2.2 are out of order and thus interpreted as concurrent. However, ACA structures are not capable of expressing non-determinism or choice available to the individual processes as the agents are sequences and *not* tree structures.

We think there are good reasons to study ACA structures. Firstly, they have been considered as models of distributed systems in the literature, e.g. [Lam78, CL85]. Secondly, ACA structures are attractive in the semantic description of process languages based on the denotational approach, e.g. [Chr89]. Thirdly, ACA structures can be viewed as a special class of elementary event structures defined in [NPW80] and later developed in many different ways by G. Winskel [Win80]. Event structures are by now appreciated as attractive models for distributed systems.

For use later on we introduce some derived relations. Assume that $A = (E, <)$ is an ACA structure. Then

(i) $id \stackrel{\text{def}}{=} \{(e, e) \mid e \in E\}$,

(ii) $> \stackrel{\text{def}}{=} \{(e, e') \mid (e', e) \in <\}$,

(iii) $\leq \stackrel{\text{def}}{=} < \cup\, id$,

(iv) $\geq \stackrel{\text{def}}{=} > \cup\, id$,

(v) $co \stackrel{\text{def}}{=} E \times E - (\leq \cup \geq)$.

Observe that the relation co is symmetric and irreflexive. It reflects the lack of ordering between events, hence it models concurrency.

An important concept in connection with ACA structures is the concept of *configurations.* A configuration is a set of events which have happened in the process of observing events.

Definition 2.3 Let $A = (E, <)$ be an ACA structure and E' a subset of E. The set E' is a *configuration* iff:

$$\forall e \in E', \forall e' \in E : e' < e \text{ implies } e' \in E'.$$

Let C_A denote the class of all configurations for the ACA structure A. ■

Thus the notion of configurations captures the intuition that an event can only occur if all events which lie in its past have occurred. In general configurations will represent global states of affairs. For instance, a configuration could capture the state of affairs for two totally independent agents. Such configurations will be called *global* configurations. Informally, a global configuration is a configuration which cannot be defined as the past of a single event. In contrast to a global configuration, a *local* configuration represents a state of affairs which can be defined as the past of a single event. Local configurations are captured through the left closure of events.

Definition 2.4 Let $A = (E, <)$ be an ACA structure and let $e \in E$. The *left closure* of e, denoted $\lceil e \rceil$, is defined as:

$$\lceil e \rceil = \{e' \in E \mid e' \leq e\}.$$

■

Proposition 2.5 Let $A = (E, <)$ be an ACA structure and let $e \in E$. Then $\lceil e \rceil$ is a configuration.

Proof Follows easily from the definition of configurations. ■

We let $LC_A = \{\lceil e \rceil \mid e \in E\}$ denote the subclass of C_A of local configurations for the ACA structure $A = (E, <)$.

As in [LT87] we would like to justify viewing ACA structures as *frames* for our logic to be developed in the next section. Standard frames for temporal logic or in general modal logic consist of tuples (W, R) where W is a set of *possible worlds* and $R \subseteq W \times W$ is the *accessibility relation* ordering the set of worlds.

Considering an ACA structure $A = (E, <)$ we will view the set of local configurations LC_A as the set of worlds and the strict inclusion relation $\subset$ as the accessibility relation. But why only consider LC_A as the set of possible worlds and not C_A? As explained, members of C_A will in general represent global configurations. We agree with K. Lodaya and P.S. Thiagarajan in [LT87] finding it very difficult to justify asserting the truth or falsity of a formula at global configurations without an omnipotent observer capable of recording global configurations. Moreover, as the concept of locality is vital in our approach, we necessarily have to represent the behaviour of concurrent systems by local configurations and *not* by global configurations as it is seen in many other approaches, e.g. [EH82, Pnu85, Maz86]. Thus we will only consider local configurations as the set of possible worlds.

Given an ACA structure $A = (E, <)$ it is easily seen that $A = (E, <)$ and $(LC_A, \subset)$ are isomorphic structures, hence we will perceive ACA structures themselves as frames. Thus speaking of a formula α being true at the event e what we really mean is that the formula holds at the local configuration $\lceil e \rceil$.

3 The logic and its semantics

The logical language used to characterize ACA structures is based on Propositional Calculus. We define a countable infinite set $P = \{p_0, p_1, \ldots\}$ of *atomic* propositions. Furthermore, we let $T = \{\tau_1, \tau_2, \ldots, \tau_n\}$ be a set consisting of n atomic *type* propositions. We assume that $P \cap T = \emptyset$.

The intuition behind the set T of type propositions is that we would like to identify particular agents in our logical language. Thus τ_i is an identification for the i^{th} agent. For convenience we define $\hat{P} = P \cup T$. We let p range over $\hat{P}$.

Definition 3.1 Let W be the set of well-formed formulas. It is defined as the least set satisfying:

(i) $\hat{P} \subseteq W$, and

(ii) if $\alpha, \beta \in W$ then $\sim\alpha, \alpha \vee \beta, \downarrow_i \alpha, \uparrow_i \alpha, \triangle_i \alpha \in W$.

We let α, β and γ range over W. ∎

The interpretation of the logic is given by the so-called possible worlds semantics or Kripke semantics [BS84]. The logic is interpreted in *models* where a model consists of a *frame* and a *valuation function.*

Definition 3.2 A *model* is a pair $M = (A, V)$ where

(i) $A = (E, <)$ is an ACA structure called the *frame*, and

(ii) $V : E \rightarrow 2^{\hat{P}}$ is the *valuation function* satisfying:

$$\forall e \in E : \tau_i \in V(e) \text{ iff } e \in E_i.$$

We assume that $M, M', \ldots$ denote models. ∎

The next definition captures the semantics of the logic.

Definition 3.3 Let $M = (A, V)$ with $A = (E, <)$ be a model. Let $e \in E_i$ and $\alpha, \beta \in W$ be formulas. The notion of a formula α being true at e in the model M is denoted $e, M \models \alpha$. The relation $\models$ is defined by structural induction and given according to the following rules:

(i) $e, M \models p$ iff $p \in V(e)$,

(ii) $e, M \models \sim\alpha$ iff $e, M \not\models \alpha$,

(iii) $e, M \models \alpha \vee \beta$ iff $e, M \models \alpha$ or $e, M \models \beta$,

(iv) $e, M \models \downarrow_j \alpha$ iff $\exists e' \in E_j : e < e'$ and $e', M \models \alpha$,

(v) $e, M \models \uparrow_j \alpha$ iff $\exists e' \in E_j : e' < e$ and $e', M \models \alpha$,

(vi) $e, M \models \triangle_j \alpha$ iff $\exists e' \in E_j : e \; co \; e'$ and $e', M \models \alpha$.

∎

If a formula α is true at an event belonging to a model we will call α satisfiable. If α is true at every event for all models we will call α valid. More formally we have:

Definition 3.4 Let $\mathcal{K}$ denote the class of all models. Suppose that $\alpha \in W$. Then

(i) α is *satisfiable* iff there exists a model $M = (A, V) \in \mathcal{K}$ with $A = (E, <)$ and an event $e \in E$ such that $e, M \models \alpha$,

(ii) α is *M–valid* iff $e, M \models \alpha$ for every event e at the model M, and

(iii) $\alpha \in W$ is *valid* (denoted $\models \alpha$) iff α is M–valid for every model $M \in \mathcal{K}$.

■

The modalities $\downarrow_j$ and $\uparrow_j$ are used to reflect communication. The relationship $e, M \models \downarrow_j \alpha$ expresses intuitively that at the event e it is known that in the future related to e there will be an event e' belonging to the j^{th} agent satisfying α. On the other hand $e, M \models \uparrow_j \alpha$ expresses intuitively that in the past related to e there exists an event e' in the j^{th} agent satisfying α. Finally the modality $\triangle_j$ is supposed to reflect concurrency. The relationship $e, M \models \triangle_j \alpha$ expresses intuitively that concurrently with the event e there exists an event e' in the j^{th} agent satisfying α.

We will also need derived logical connectives and modalities. The well-known connectives of Propositional Calculus such as $\wedge$, $\supset$ and $\equiv$ are defined in terms of $\sim$ and $\vee$ in the usual way. In addition we define:

Definition 3.5

(i)	$\top$	$\stackrel{\text{def}}{=}$	$p \vee \sim p$
(ii)	$\bot$	$\stackrel{\text{def}}{=}$	$\sim \top$
(iii)	$\Downarrow_i \alpha$	$\stackrel{\text{def}}{=}$	$\sim \downarrow_i \sim \alpha$
(iv)	$\Uparrow_i \alpha$	$\stackrel{\text{def}}{=}$	$\sim \uparrow_i \sim \alpha$
(v)	$\nabla_i \alpha$	$\stackrel{\text{def}}{=}$	$\sim \triangle_i \sim \alpha$
(vi)	$\ominus_i \alpha$	$\stackrel{\text{def}}{=}$	$\tau_i \wedge \uparrow_i \alpha$
(vii)	$\Diamond_i \alpha$	$\stackrel{\text{def}}{=}$	$\tau_i \wedge \downarrow_i \alpha$
(viii)	$\boxminus_i \alpha$	$\stackrel{\text{def}}{=}$	$\tau_i \wedge \Uparrow_i \alpha$
(ix)	$\Box_i \alpha$	$\stackrel{\text{def}}{=}$	$\tau_i \wedge \Downarrow_i \alpha$

In the above clause (i) the atomic proposition $p \in P$ is arbitrary chosen. We note that this is possible as P is assumed to be non-empty. The formulas $\top$ and $\bot$ represent truth and false respectively. ■

Proposition 3.6 Let $M = (A, V)$ with $A = (E, <)$ be a model. Let $e \in E_i$ and $\alpha \in W$ be a formula. Then we have the following:

(i) $e, M \models \ominus_j \alpha$ iff $i = j$ and $\exists e' \in E_j : e' <_j e$ and $e', M \models \alpha$,

(ii) $e, M \models \Diamond_j \alpha$ iff $i = j$ and $\exists e' \in E_j : e <_j e'$ and $e', M \models \alpha$,

(iii) $e, M \models \boxminus_j \alpha$ iff $i = j$ and $\forall e' \in E_j$: if $e' <_j e$ then $e', M \models \alpha$,

(iv) $e, M \models \Box_j\alpha$ iff $i = j$ and $\forall e' \in E_j$: if $e <_j e'$ then $e', M \models \alpha$,

(v) $e, M \models \Downarrow_j \alpha$ iff $\forall e' \in E_j$: if $e < e'$ then $e', M \models \alpha$,

(vi) $e, M \models \Uparrow_j \alpha$ iff $\forall e' \in E_j$: if $e' < e$ then $e', M \models \alpha$,

(vii) $e, M \models \nabla_j\alpha$ iff $\forall e' \in E_j$: if e *co* e' then $e', M \models \alpha$.

Proof Is easily seen to be a consequence of definition 3.3. ■

We note that the modalities $\ominus_j$, $\Diamond_j$, $\boxminus_j$ and $\Box_j$ are purely local as they are restricted to the j^{th} agent. Intuitively they are equivalent to the operators P, F, H and G in the framework of temporal logic.

The modalities $\Downarrow_j$ and $\Uparrow_j$ are the *dual* operators of $\downarrow_j$ and $\uparrow_j$ respectively and used to reflect communication. The modality $\Downarrow_j$ is related to future communication whereas $\Uparrow_j$ is related to communication in the past. The relationship $e, M \models \Downarrow_j \alpha$ expresses intuitively that when events e' in the j^{th} agent can happen *only* if e has occurred (i.e. $e < e'$) then at e' the formula α will be satisfied. The relationship $e, M \models \Uparrow_j \alpha$ expresses intuitively that if events e' in the j^{th} agent are demanded to occur *before* e can occur (i.e. $e' < e$) then α is satisfied at e'. The last derived modality ∇_j is the dual of $\triangle_j$ and it is supposed to reflect concurrency. The relationship $e, M \models \nabla_j\alpha$ expresses intuitively that for all events e' belonging to the j^{th} agent, which are in no causal order with e (i.e. e *co* e'), the formula α has to be satisfied.

We conclude this section by arguing for the choice of the shape of the temporal operators. We have followed tradition and used box and diamond notation for the local operators whereas the shape of the global operators are new, though the triangle shape of the strong concurrency operator, i.e. $\triangle$, is also used in [MT89]. We will justify the choice of the shape of the communication operators by referring to figure 2.1. As agents are pictured upwards down communication in the future will occur following arrows in their directions, i.e. down, and communication in the past have occurred following arrows opposite their directions, i.e. up. For instance, if the formula $\downarrow_2 c_t$ is considered at the event e_{11} in figure 2.1 then it could express that moving down the arrows there will be a treatment event belonging to the second agent which is modelling the behaviour of the consumer. If on the other hand the formula $\uparrow_1 p_f$ is viewed at the event e_{22} then it could reflect that moving up, i.e. in the opposite direction of the arrows, there has been a fabrication event in the first agent which is modelling the producer.

4 The axiomatic basis

In this section we present an axiomatic basis for the logic defined in the previous section. In search of an axiomatic basis we are guided both by tradition and by the proof of completeness of the logic. Thus we have indexed versions of standard axioms and inference rules taken from [Bur84] but also special axioms and inference rules needed in the proof of completeness. We first present the axiomatic basis in full and then go into details about some of the axioms and inference rules afterwards.

Axioms

(A0) All the substitutional instances of the tautologies of propositional logic.

Deductive Closure:

(A1.a) $\Downarrow_i (\alpha \supset \beta) \supset (\Downarrow_i \alpha \supset \Downarrow_i \beta)$
(A1.b) $\Uparrow_i (\alpha \supset \beta) \supset (\Uparrow_i \alpha \supset \Uparrow_i \beta)$
(A1.c) $\bigtriangledown_i(\alpha \supset \beta) \supset (\bigtriangledown_i\alpha \supset \bigtriangledown_i\beta)$

Transitivity:

(A2.a) $\downarrow_j \downarrow_i \alpha \supset \downarrow_i \alpha$
(A2.b) $\uparrow_j \uparrow_i \alpha \supset \uparrow_i \alpha$

Relating past and future:

(A3.a) $\tau_i \wedge \alpha \supset \Downarrow_j \uparrow_i \alpha$
(A3.b) $\tau_i \wedge \alpha \supset \Uparrow_j \downarrow_i \alpha$

Type axioms:

(A4.a) $\downarrow_i \alpha \supset \downarrow_i (\tau_i \wedge \alpha)$
(A4.b) $\uparrow_i \alpha \supset \uparrow_i (\tau_i \wedge \alpha)$
(A4.c) $\triangle_i\alpha \supset \triangle_i(\tau_i \wedge \alpha)$
(A4.d) $\bigvee_{i=1}^{n}(\tau_i \wedge \bigwedge_{j=1}^{i-1} \sim\tau_j \wedge \bigwedge_{j=i+1}^{n} \sim\tau_j)$

Concurrent axioms:

(A5.a) $\tau_i \supset \bigtriangledown_i\bot$
(A5.b) $\tau_i \wedge \alpha \supset \bigtriangledown_j\triangle_i\alpha$

Relating communication and concurrency:

(A6.a) $\tau_i \wedge \downarrow_j \triangle_i \alpha \supset \downarrow_i \alpha$
(A6.b) $\tau_i \wedge \uparrow_j \triangle_i \alpha \supset \uparrow_i \alpha$
(A6.c) $\leftrightsquigarrow_{ji} \alpha \supset \downarrow_i \alpha \vee \uparrow_i \alpha \vee \triangle_i \alpha \vee \alpha$
(A6.d) $p^i \supset \Downarrow_j (\sim\downarrow_i p \wedge \sim\triangle_i p) \wedge \Uparrow_j (\sim\uparrow_i p \wedge \sim\triangle_i p) \wedge \nabla_j(\sim\downarrow_i p \wedge \sim\uparrow_i p)$

In the above axiom (A6.c) the "formula" $\leftrightsquigarrow_{ji} \alpha$ is syntactic sugar for the formula $\downarrow_j \uparrow_i \alpha \vee \uparrow_j \downarrow_i \alpha \vee \downarrow_j \triangle_i \alpha \vee \uparrow_j \triangle_i \alpha \vee \triangle_j \downarrow_i \alpha \vee \triangle_j \uparrow_i \alpha \vee \triangle_j \triangle_i \alpha$. In axiom (A6.d) p^i denotes the formula $p \wedge \boxminus_i(\sim p) \wedge \Box_i(\sim p)$ for a fixed $p \in P$.

Inference Rules

Let $\boxdot_i \in \{\Downarrow_i, \Uparrow_i, \nabla_i\}$. We have the following inference rules:

(MP)
$$\frac{\alpha, \alpha \supset \beta}{\beta}$$

(R1)
$$\frac{\alpha}{\boxdot_i \alpha}$$

(R2)
$$\frac{p^i \supset \alpha}{\tau_i \supset \alpha}$$

In (R2) we demand that α is free of p.

■

Axiom (A0) expresses that any substitution instance of valid formulas of propositional logic are axioms of the logic. For instance, $p \vee \sim p$, where $p \in P$, is a valid formula of propositional logic, hence $\alpha \vee \sim\alpha$ is an axiom of the logic. Axiom (A1) and inference rules (MP) and (R1) are standard and require no explanation. Axiom (A2) captures the transitivity of the causality relation. Axiom (A3) are standard and adapted from [Bur84]. Axiom (A4) captures the identification of agents through the set $\{\tau_1, \ldots, \tau_n\}$ of formulas. In particular, axiom (A4.d) is supposed to reflect the fact that each local stage belongs to exactly one agent. Axiom (A5) are related to the concurrency operator: (A5.a) reflects that the individual agents contain no concurrent events while (A5.b) expresses the symmetry of the *co* relation. The remaining axioms may at first look a little remarkable. They will be used in the proof of completeness to ensure that models for particular formulas can be build. Axiom (A6.a) and (A6.b) reflect the way communication and concurrency must be organized between agents. Axiom (A6.c) is supposed to ensure that so-called maximal consistent sets have the proper orderings between each other, but more about this in the next section. In order to explain the intuition behind the axiom assume that the formula $\leftrightsquigarrow_{ji} \alpha$ is satisfied at an event $e \in E_k$ in the model $M = (A, V)$ with $A = (E, <)$. The formula expresses that via an event $e' \in E_j$ which we could call the connecting link there exists an event $e'' \in E_i$ such that α is satisfied at e''. The axiom expresses intuitively that the connecting link e' can be ruled out. The axiom makes essential use of the fact that $\{<, >, co, id\}$ is a partitioning of $E \times E$; a fact which implies that the events e and e'' satisfy $(e, e'') \in < \cup > \cup\, co \cup id$. Axiom (A6.d) will be used in connection with (A6.c) to ensure not only the proper relations between maximal consistent sets but also *uniqueness* w.r.t. the possible relations. The formula p^i expresses that the atomic proposition p is true exactly at one event in the i^{th} agent. Finally we give a remark on inference rule (R2). It is adapted from [Bur80] and will be used to label events uniquely in relation to the individual agents containing the events.

A formula α will be called a *thesis* iff it can be derived using the axioms and inference rules of the axiomatic basis. We will let $\vdash \alpha$ denote the fact that α is a thesis.

Theorem 4.1 (Soundness) We have: If $\vdash \alpha$ then $\models \alpha$.

Proof It is rather easy to verify all the axioms and all the inference rules except (R2). The proof of soundness of (R2) is presented in appendix A. ∎

Before going into the proof of completeness we present some derived inference rules and theses.

Derived inference rules

(PR)

$$\frac{\alpha_1, \ldots, \alpha_m, \alpha_1 \wedge \ldots \wedge \alpha_m \supset \beta}{\beta}$$

(DR.1)

$$\frac{\alpha \supset \beta}{\boxdot_i \alpha \supset \boxdot_i \beta}$$

(SU)

$$\frac{\alpha \equiv \beta, \Phi(\alpha/\gamma)}{\Phi(\beta/\gamma)}$$

In (DR.1) $\boxdot_i$ is supposed to range over $\{\Downarrow_i, \Uparrow_i, \bigtriangledown_i\}$. In (SU) $\Phi(\alpha/\gamma)$ denotes a formula Φ in which α is substituted for γ.

∎

Theses

(T1.a) $\triangle_i \alpha \equiv \sim\bigtriangledown_i\sim\alpha$
(T1.b) $\sim\triangle_i \alpha \equiv \bigtriangledown_i\sim\alpha$
(T1.c) $\triangle_i\sim\alpha \equiv \sim\bigtriangledown_i \alpha$
(T1.d) $\downarrow_i \alpha \equiv \sim\Downarrow_i \sim\alpha$
(T1.e) $\sim\downarrow_i \alpha \equiv \Downarrow_i \sim\alpha$
(T1.f) $\downarrow_i \sim\alpha \equiv \sim\Downarrow_i \alpha$
(T1.g) $\uparrow_i \alpha \equiv \sim\Uparrow_i \sim\alpha$
(T1.h) $\sim\uparrow_i \alpha \equiv \Uparrow_i \sim\alpha$
(T1.i) $\uparrow_i \sim\alpha \equiv \sim\Uparrow_i \alpha$

(T2.a) $\Downarrow_i \alpha \wedge \downarrow_i \beta \supset \downarrow_i (\alpha \wedge \beta)$
(T2.b) $\Uparrow_i \alpha \wedge \uparrow_i \beta \supset \uparrow_i (\alpha \wedge \beta)$
(T2.c) $\nabla_i \alpha \wedge \triangle_i \beta \supset \triangle_i (\alpha \wedge \beta)$

(T3.a) $\downarrow_i (\alpha \wedge \beta) \supset \downarrow_i \alpha \wedge \downarrow_i \beta$
(T3.b) $\uparrow_i (\alpha \wedge \beta) \supset \uparrow_i \alpha \wedge \uparrow_i \beta$
(T3.c) $\triangle_i (\alpha \wedge \beta) \supset \triangle_i \alpha \wedge \triangle_i \beta$

(T4.a) $\tau_i \wedge \beta \wedge \downarrow_j \alpha \supset \downarrow_j (\alpha \wedge \uparrow_i \beta)$
(T4.b) $\tau_i \wedge \beta \wedge \uparrow_j \alpha \supset \uparrow_j (\alpha \wedge \downarrow_i \beta)$
(T4.c) $\tau_i \wedge \beta \wedge \triangle_j \alpha \supset \triangle_j (\alpha \wedge \triangle_i \beta)$

(T5.a) $\tau_i \wedge \downarrow_j \Uparrow_i \alpha \supset \alpha$
(T5.b) $\tau_i \wedge \uparrow_j \Downarrow_i \alpha \supset \alpha$

(T6) $\tau_i \wedge \triangle_j \nabla_i \alpha \supset \alpha$

(T7) $\tau_i \wedge \leftrightsquigarrow_{ji} \alpha \supset \downarrow_i \alpha \vee \uparrow_i \alpha \vee \alpha$

$\blacksquare$

Proof of derived inference rules and theses are presented in appendix B.

5 Completeness

The completeness proof will be a so-called Henkin proof [Hen49], i.e. we prove that every consistent formula can be satisfied. The proof method is strongly guided by [Bur84].

As usual, by a *consistent* formula we mean a formula whose negation is not a thesis of our axiom system. The finite set of formulas $\{\alpha_1, \ldots, \alpha_m\}$ is consistent iff $\alpha_1 \wedge \ldots \wedge \alpha_m$ is consistent. A set of formulas is consistent if every finite subset is consistent. We let MCS denote the class of *maximal consistent* sets of formulas, i.e. consistent sets which are not properly included in any other consistent sets. We assume that Q, R, S range over the set MCS. Finally we shall assume Lindenbaum's lemma that any consistent set of formulas can be extended to a maximal consistent set.

The next result concerning maximal consistent sets will be used often and sometimes tacitly in what follows.

Proposition 5.1 Let $Q \in MCS$. Then

(i) ${\sim}\alpha \in Q$ iff $\alpha \notin Q$,

(ii) $\alpha \vee \beta \in Q$ iff $\alpha \in Q$ or $\beta \in Q$,

(iii) $\alpha \wedge \beta \in Q$ iff $\alpha \in Q$ and $\beta \in Q$,

(iv) if α is a thesis then $\alpha \in Q$, and

(v) if $\alpha_1, \alpha_2, \ldots, \alpha_m \in Q$ and $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m \supset \beta$ is a thesis then $\beta \in Q$.

Proof The proposition is shown by standard arguments. Consult for instance [HC68]. ■

Definition 5.2 The function *type* : $MCS \rightarrow \{1, \ldots, n\}$ is given by:

$$\forall Q \in MCS: \ \mathit{type}(Q) = i \text{ iff } \tau_i \in Q.$$

■

By proposition 5.1 and axiom (A4.d) it follows that the function *type* specified above is well-defined.

Lemma 5.3 Let $\alpha \in W$ and let $\odot_i \in \{\downarrow_i, \uparrow_i, \triangle_i\}$. Then

if $\odot_i\alpha$ is consistent then α is consistent.

Proof Assume that $\odot_i\alpha$ is consistent but that α is inconsistent. Then $\vdash {\sim}\alpha$. By inference rule (R1) and theses (T1.b), (T1.e) and (T1.h) we conclude that $\vdash {\sim}\odot_i\,\alpha$ which cannot be the case. ■

The next result follows from standard arguments. Consult for instance [Bur84].

Lemma 5.4 Let $Q, R \in MCS$ such that $\mathit{type}(Q) = i$ and $\mathit{type}(R) = j$. Then the following statements are equivalent:

(i) if $\alpha \in Q$ then $\uparrow_i \alpha \in R$,

(ii) if $\alpha \in R$ then $\downarrow_j \alpha \in Q$,

(iii) if $\Downarrow_j \alpha \in Q$ then $\alpha \in R$,

(iv) if $\Uparrow_i \alpha \in R$ then $\alpha \in Q$.

■

We define a relation between arbitrary elements from MCS. The relation is based on the modalities reflecting communication.

Definition 5.5 The relation $\prec \subseteq MCS \times MCS$ is defined as follows:

$$\forall Q, R \in MCS : Q \prec R \overset{\text{def}}{\Longleftrightarrow} \forall \alpha \in W : \text{if } \alpha \in Q \text{ then } \uparrow_{type(Q)} \alpha \in R.$$

■

Observe that the definition of $\prec$ is build upon clause (i) of lemma 5.4. By the equivalences stated in the lemma we could have build the relation upon any of the clauses given in the lemma.

Intuitively the relation $\prec$ and the causality relation $<$ are connected. This will become clear once we have introduced the notions of chronicles and chronicle structures which are concepts that combine frames and members of MCS. But more about this subject later on.

Lemma 5.6 Let $Q \in MCS$ such that $type(Q) = i$. Then

(i) if $\downarrow_j \alpha \in Q$ then $\exists R \in MCS : type(R) = j$ and $Q \prec R$ and $\alpha \in R$

(ii) if $\uparrow_j \alpha \in Q$ then $\exists R \in MCS : type(R) = j$ and $R \prec Q$ and $\alpha \in R$.

Proof We first prove (i). Let $R^- = \{\uparrow_i \gamma \mid \gamma \in Q\} \cup \{\tau_j, \alpha\}$. As every consistent set of formulas can be extended to a maximal consistent set it suffices to show that R^- is consistent. By proposition 5.1 and thesis (T3.b) it is enough to prove that if $\gamma \in Q$ then $\tau_j \wedge \alpha \wedge \uparrow_i \gamma$ is consistent in order to prove that R^- is consistent. Suppose $\gamma \in Q$. Then $\gamma \wedge \downarrow_j \alpha \in Q$ by proposition 5.1. As $type(Q) = i$ it follows by thesis (T4.a) that $\downarrow_j (\alpha \wedge \uparrow_i \gamma) \in Q$ which by axiom (A4.a) implies $\downarrow_j (\tau_j \wedge \alpha \wedge \uparrow_i \gamma) \in Q$. We therefore conclude that the formula is consistent. By lemma 5.3 it follows that $\tau_j \wedge \alpha \wedge \uparrow_i \gamma$ is consistent. Proof of (ii) can be given by similar arguments, hence we omit it. ■

The next lemma is in nature similar to lemma 5.4. The proof follows from standard arguments. Again we refer to [Bur84]

Lemma 5.7 Let $Q, R \in MCS$ such that $type(Q) = i$ and $type(R) = j$. Then the following statements are equivalent:

(i) if $\alpha \in Q$ then $\triangle_i \alpha \in R$,

(ii) if $\alpha \in R$ then $\triangle_j \alpha \in Q$,

(iii) if $\nabla_j \alpha \in Q$ then $\alpha \in R$,

(iv) if $\nabla_i \alpha \in R$ then $\alpha \in Q$.

■

We define a relation, based upon the modalities reflecting concurrency, between arbitrary elements of MCS.

Definition 5.8 The relation $\bowtie \subseteq MCS \times MCS$ is defined as follows:

$$\forall Q, R \in MCS : Q \bowtie R \overset{\text{def}}{\Longleftrightarrow} \forall \alpha \in W : \text{if } \alpha \in Q \text{ then } \triangle_{type(Q)}\alpha \in R.$$

■

Observe that by lemma 5.7 the relation $\bowtie$ is symmetric; a fact that justifies the choice of the shape of the symbol expressing the relation. Intuitively $\bowtie$ is connected to the relation *co* reflecting events that are concurrent. This will become clear once we have introduced the notions of chronicles and chronicle structures which, as mentioned, will be done later on.

Lemma 5.9 Let $Q \in MCS$ such that $type(Q) = i$. Then

$$\text{if } \triangle_j \alpha \in Q \text{ then } \exists R \in MCS : type(R) = j \text{ and } Q \bowtie R \text{ and } \alpha \in R.$$

Proof Let $R^- = \{\triangle_i \gamma \mid \gamma \in Q\} \cup \{\tau_j, \alpha\}$. As every consistent set of formulas can be extended to a maximal consistent set it suffices to show that R^- is consistent. By proposition 5.1 and thesis (T3.c) it is enough to prove that if $\gamma \in Q$ then $\tau_j \wedge \alpha \wedge \triangle_i \gamma$ is consistent in order to prove that R^- is consistent. Suppose $\gamma \in Q$. Then $\gamma \wedge \triangle_j \alpha \in Q$. By thesis (T4.c) and the case that $type(Q) = i$ we have $\triangle_j(\alpha \wedge \triangle_i \gamma) \in Q$. Then by axiom (A4.c) it follows that $\triangle_j(\tau_j \wedge \alpha \wedge \triangle_i \gamma) \in Q$, hence the formula is consistent. Finally by lemma 5.3 we conclude that $\tau_j \wedge \alpha \wedge \triangle_i \gamma$ is consistent. ■

In the following we present some results concerning the relations $\prec$ and $\bowtie$. The properties we are going to show about the relations will be used later on when we have to build models for particular formulas.

Lemma 5.10 Let $Q, R \in MCS$ such that $type(Q) = i$. Then

$$\text{if } \alpha \in Q \text{ implies } \downarrow_i \alpha \vee \uparrow_i \alpha \vee \triangle_i \alpha \vee \alpha \in R \text{ then } (Q, R) \in \prec \cup \succ \cup \bowtie \cup \subseteq,$$

where $\succ = \{(R, Q) \mid Q \prec R\}$ and $\subseteq$ is the normal inclusion relation on sets.

Proof Suppose first that $(Q, R) \notin \prec \cup \succ \cup \bowtie$. We want to show that $Q \subseteq R$ must be the case. Now $(Q, R) \notin \prec \cup \succ \cup \bowtie$ implies the existence of $\alpha_0, \alpha_1, \alpha_2 \in Q$ such that $\uparrow_i \alpha_0 \notin R$, $\downarrow_i \alpha_1 \notin R$ and $\triangle_i \alpha_2 \notin R$. Suppose $\alpha \in Q$. Then we have $\alpha \wedge \hat{\alpha} \in Q$, where $\hat{\alpha} = \alpha_0 \wedge \alpha_1 \wedge \alpha_2$. By the hypothesis of the lemma we have $\downarrow_i (\alpha \wedge \hat{\alpha}) \vee \uparrow_i (\alpha \wedge \hat{\alpha}) \vee \triangle_i(\alpha \wedge \hat{\alpha}) \vee (\alpha \wedge \hat{\alpha}) \in R$. It follows readily that $\alpha \wedge \hat{\alpha} \in R$ is the only possibility and therefore that $\alpha \in R$ is the case.

In order to complete the lemma we have to consider the cases $(Q, R) \notin \prec \cup \succ \cup \subseteq$, $(Q, R) \notin \prec \cup \bowtie \cup \subseteq$ and finally $(Q, R) \notin \succ \cup \bowtie \cup \subseteq$. By a proof method similar to the above it follows easily that in the first case $Q \bowtie R$ is satisfied, in the second case $R \prec Q$ is satisfied and in the last case $Q \prec R$ is satisfied. ■

Corollary 5.11 Let $Q, R \in MCS$ such that $type(Q) = type(R) = i$. Then

$$\text{if } \alpha \in Q \text{ implies } \downarrow_i \alpha \vee \uparrow_i \alpha \vee \alpha \in R \text{ then } (Q, R) \in \prec \cup \succ \cup \subseteq.$$

Proof Follows easily from lemma 5.10 and axiom (A5.a). ∎

Lemma 5.12 The relation $\prec$ is transitive.

Proof Let $Q, R, S \in MCS$ such that $type(Q) = i$, $type(R) = j$ and $type(S) = k$. Assume $Q \prec R$ and $R \prec S$. We have to show that $Q \prec S$. By definition of $\prec$ and lemma 5.4 it is enough to show that if $\Downarrow_k \alpha \in Q$ then $\alpha \in S$. Assume therefore that $\Downarrow_k \alpha \in Q$. Then, as $Q \prec R$ and $R \prec S$, it follows that $\uparrow_j \uparrow_i \Downarrow_k \alpha \in S$ which by axiom (A2.b) implies $\uparrow_i \Downarrow_k \alpha \in S$. As $type(S) = k$ it follows that $\tau_k \wedge \uparrow_i \Downarrow_k \alpha \in S$, hence by thesis (T5.b) we conclude that $\alpha \in S$. ∎

As mentioned earlier, the relation $\prec$ is intuitively related to the causality relation $<$. And, as observed by the above lemma, $\prec$ is transitive like the causality relation. However, $\prec$ will in general not be irreflexive as it is the case for the causality relation. We will not go into further details as there will be no need to require irreflexivity of $\prec$ in order to obtain the proof of completeness.

We proceed by showing two more properties concerning the relations $\prec$ and $\bowtie$ before we define the notions of chronicles and chronicle structures.

Lemma 5.13 Let $Q, R, S \in MCS$ such that $type(Q) = i$ and $type(R) = type(S) = j$. Then we have the following:

(i) if $Q \bowtie R$ and $Q \prec S$ then $R \prec S$, and

(ii) if $Q \bowtie R$ and $S \prec Q$ then $S \prec R$.

Proof We only prove (i) as proof of (ii) can be given by similar arguments. Assume that $Q \bowtie R$ and $Q \prec S$. Let $\alpha \in R$. Now $Q \bowtie R$ implies $\triangle_j \alpha \in Q$ and $Q \prec S$ implies $\uparrow_i \triangle_j \alpha \in S$. As $type(S) = j$ we conclude that $\tau_j \wedge \uparrow_i \triangle_j \alpha \in S$ which by axiom (A6.b) implies $\uparrow_j \alpha \in S$, hence $R \prec S$. ∎

Lemma 5.14 Let $Q, R, S \in MCS$ such that $type(Q) = i$ and $type(R) = type(S) = j$. The relations $\prec$ and $\bowtie$ will satisfy the following:

(i) if $Q \prec R$ and $Q \prec S$ then $(R, S) \in \prec \cup \succ \cup =$,

(ii) if $R \prec Q$ and $S \prec Q$ then $(R, S) \in \prec \cup \succ \cup =$, and

(iii) if $Q \bowtie R$ and $Q \bowtie S$ then $(R, S) \in \prec \cup \succ \cup =$,

where $=$ is the normal equality relation on sets.

Proof We first prove (i). Assume $Q \prec R$ and $Q \prec S$. Let $\alpha \in R$. Now $Q \prec R$ implies $\downarrow_j \alpha \in Q$ and $Q \prec S$ implies $\uparrow_i \downarrow_j \alpha \in S$. As $type(S) = j$ it follows that $\tau_j \wedge \uparrow_i \downarrow_j \alpha \in S$. By Propositional Calculus it follows readily that $\tau_j \wedge \uparrow_i \downarrow_j \alpha$ implies $\tau_j \wedge \leftrightsquigarrow_{ij} \alpha$ because $\uparrow_i \downarrow_j \alpha$ is equal to one of the disjunctions constituting the formula $\leftrightsquigarrow_{ij} \alpha$ (the formula is defined on page 12). Hence $\tau_j \wedge \leftrightsquigarrow_{ij} \alpha \in S$ and by thesis (T7) we conclude that $\downarrow_j \alpha \vee \uparrow_j \alpha \vee \alpha \in S$ which by corollary 5.11 implies $(R, S) \in \prec \cup \succ \cup \subseteq$. Letting $\alpha \in S$ we get by the same arguments that $(S, R) \in \prec \cup \succ \cup \subseteq$. We now conclude that $(R, S) \in \prec \cup \succ \cup =$. Proof of (ii) and (iii) follows by similar arguments. ∎

We need the notions of chronicles and chronicle structures before going into the proof of completeness.

Definition 5.15 Let $A = (E, <)$ be a frame. A *chronicle* on A is a function $X : E \longrightarrow MCS$. We characterize a chronicle X as follows:

(i) X is *strict* iff for all $e, e' \in E$ it is the case that $X(e)$ and $X(e')$ are related only by one of the four relations $\prec$, $\succ$, $\bowtie$ and $=$,

(ii) X is *coherent* iff

(a) $\forall e, e' \in E$: if $e < e'$ then $X(e) \prec X(e')$,

(b) $\forall e, e' \in E$: if e *co* e' then $X(e) \bowtie X(e')$, and

(c) $\forall e \in E : \tau_i \in X(e)$ iff $e \in E_i$,

(iii) X is *prophetic* iff

$$\forall e \in E : \text{if } e \in E_i \text{ and } \downarrow_j \alpha \in X(e) \text{ then } \exists e' \in E_j : e < e' \text{ and } \alpha \in X(e'),$$

(iv) X is *historic* iff

$$\forall e \in E : \text{if } e \in E_i \text{ and } \uparrow_j \alpha \in X(e) \text{ then } \exists e' \in E_j : e' < e \text{ and } \alpha \in X(e'),$$

(v) X is *concurrent* iff

$$\forall e \in E : \text{if } e \in E_i \text{ and } \triangle_j \alpha \in X(e) \text{ then } \exists e' \in E_j : e' \text{ co } e \text{ and } \alpha \in X(e'), \text{ and}$$

(vi) X is *perfect* iff X is coherent, prophetic, historic and concurrent.

■

Definition 5.16 Let $A = (E, <)$ be a frame and X a coherent chronicle on A. The structure $CS = (A, X)$ is called a *chronicle structure*. CS is called a *strict chronicle structure* provided X is strict. Finally CS is called a *(strict) finite chronicle structure* if the set E is finite (and X is strict). ■

Next we present some results relating models and chronicles. The results provide a method of showing that consistent formulas can be satisfied.

Definition 5.17 Let $M = (A, V)$ with $A = (E, <)$ be a model. X_V is called the *chronicle induced by* V and is defined as follows:

$$\forall e \in E : X_V(e) = \{\alpha \mid e, M \models \alpha\}.$$

■

Definition 5.18 Let $A = (E, <)$ be a frame and X a perfect chronicle on A. V_X is called the *valuation induced by* X and is defined as follows:

$$\forall e \in E : V_X(e) = \{p \in \hat{P} \mid p \in X(e)\}.$$

■

Lemma 5.19 Let X be a perfect chronicle on the frame $A = (E, <)$. Then $X_{V_X} = X$.

Proof Let $M = (A, V_X)$. By definition of $X_{V_X}(e)$ we have to show that

$$\forall \alpha \in W, \forall e \in E : \alpha \in X(e) \text{ iff } e, M \models \alpha \qquad (*)$$

in order to prove that $X_{V_X} = X$. Hence suppose $\alpha \in W$ and $e \in E_i$. The proof of $(*)$ is by induction on the structure of α. We omit the details. ■

The strategy in showing completeness is to show that every consistent formula is satisfiable. Now lemma 5.19 suggests an obvious way of proving that a consistent formula α is satisfiable: construct a perfect chronicle X on a frame $A = (E, <)$ such that there exists an $e \in E$ satisfying $\alpha \in X(e)$. From now on we will concentrate on showing that the perfect chronicle X on the frame A exists. First we show that a frame with a coherent but not perfect chronicle can be extended to a frame with an improved chronicle. In this connection it will be convenient to work with live requirements.

Definition 5.20 Let $CS = (A, X)$ with $A = (E, <)$ be a chronicle structure. Let $e \in E_i$. Then

(i) $(e, \downarrow_j \alpha)$ is called a *live prophetic requirement* in CS iff

$$\downarrow_j \alpha \in X(e) \text{ and } \nexists e' \in E_j : e < e' \text{ and } \alpha \in X(e'),$$

(ii) $(e, \uparrow_j \alpha)$ is called a *live historic requirement* in CS iff

$$\uparrow_j \alpha \in X(e) \text{ and } \nexists e' \in E_j : e' < e \text{ and } \alpha \in X(e'),$$

(iii) $(e, \triangle_j \alpha)$ is called a *live concurrent requirement* in CS iff

$$\triangle_j \alpha \in X(e) \text{ and } \nexists e' \in E_j : e \ co\ e' \text{ and } \alpha \in X(e'), \text{ and}$$

(iv) (e, β) is called a *live requirement* in CS iff one of the above mentioned (i), (ii) or (iii) is satisfied.

■

Obviously a chronicle structure $CS = (A, X)$ containing a live requirement is a chronicle structure in which the chronicle is not perfect. We want to show that if $CS = (A, X)$ contains a live requirement then CS can be extended to a chronicle structure $CS' = (A', X')$ such that the live requirement in CS is no longer a live requirement in CS'. But first we are going to work with strict finite chronicle structures.

Lemma 5.21 Let $CS = (A, X)$ with $A = (E, <)$ be a strict finite chronicle structure. Let (e_1, β) be a live requirement in CS. Then there exists a finite chronicle structure $CS' = (A', X')$ with $A' = (E', <')$ such that:

(i) $E' = E \cup \{e_2\}$ for some $e_2 \notin E$,

(ii) $<'$ restricted to $E \times E$ is $<$,

(iii) X' restricted to E is X, and

(iv) (e_1, β) is no longer a live requirement in CS'.

Proof The proof is rather involved and must be divided into three parts according to the three types of live requirements. See appendix C for a proof. ∎

If we are capable of removing live requirements iteratively from a chronicle structure then the limit of this process will be a chronicle structure with a perfect chronicle.

The next lemma enables live requirements to be removed iteratively.

Lemma 5.22 Let $CS = (A, X)$ with $A = (E, <)$ be a finite chronicle structure. Let q denote the number of elements in E. Let $f(\alpha)$, where $\alpha \in W$, denote the result of having substituted p_{i+1} for p_i in α. Let $\tilde{f}$ denote the extension of f to sets of formulas defined in the obvious way. Finally let $SCS = (A, Y)$ be a strict finite chronicle structure satisfying:

$$\forall e \in E : \tilde{f}^q(X(e)) \subseteq Y(e),$$

where $\tilde{f}^q(X(e))$ denotes the result of having performed $\tilde{f}$ on $X(e)$ q times.

Suppose (e_1, β) is a live requirement in CS. Then there exists a finite chronicle structure $CS' = (A', X')$ with $A' = (E', <')$ such that:

(i) $E' = E \cup \{e_2\}$ for some $e_2 \notin E$,

(ii) $<'$ restricted to $E \times E$ is $<$,

(iii) X' restricted to E is X, and

(iv) (e_1, β) is no longer a live requirement in CS'.

Furthermore, a strict finite chronicle structure $SCS' = (A', Y')$ can be constructed such that:

$$\forall e \in E' : \tilde{f}^{q+1}(X'(e)) \subseteq Y'(e).$$

Proof It follows readily that if (e_1, β) is a live requirement in CS then $(e_1, f^q(\beta))$[1] is a live requirement in SCS. By help of lemma 5.21 we can remove the live requirement $(e_1, f^q(\beta))$ in SCS, i.e. there exists a chronicle structure $CS'' = (A'', X'')$ with $A'' = (E'', <'')$ satisfying:

(i) $E'' = E \cup \{e_2\}$ for some $e_2 \notin E$,

(ii) $<''$ restricted to $E \times E$ is $<$,

(iii) X'' restricted to E is Y, and

(iv) $(e_1, f^q(\beta))$ is no longer a live requirement in CS''.

We now define the chronicle structure $CS' = (A', X')$ as follows:

(i) $A' = A''$,

[1] $f^q(\beta)$ denotes the result of having performed the substitution f on β q times.

(ii) $X'(e_2) = \tilde{f}^{-q}(X''(e_2))$, and

(iii) $\forall e \in E'$: if $e \neq e_2$ then $X'(e) = X(e)$,

where $\tilde{f}^{-1}(X''(e_2)) = \{\alpha \mid f(\alpha) \in X''(e_2)\}$ and $\tilde{f}^{-q}(X''(e_2))$ denotes the result of having performed this transformation q times. It is readily verified that CS' is the required chronicle structure.

We finally define the chronicle structure $SCS' = (A', Y')$, i.e. we have to define Y'. We proceed as follows (assuming $e_2 \in E'_j$):

(i) $Y'(e_2) \approx \tilde{f}(Y'(e_2)) \cup \{p_0^j\}$, and

(ii) $\forall e \in E' - \{e_2\}$:

 (a) if $e <' e_2$ then $Y'(e) \approx \tilde{f}(Y(e)) \cup \{\downarrow_j \alpha \mid \alpha \in Y'(e_2)\}$,

 (b) if $e_2 <' e$ then $Y'(e) \approx \tilde{f}(Y(e)) \cup \{\uparrow_j \alpha \mid \alpha \in Y'(e_2)\}$, and

 (c) if $e \; co' \; e_2$ then $Y'(e) \approx \tilde{f}(Y(e)) \cup \{\triangle_j \alpha \mid \alpha \in Y'(e_2)\}$.

In the above specifications p_0^j is equal to $p_0 \wedge \boxminus_j {\sim} p_0 \wedge \Box_j {\sim} p_0$ and $\approx$ denotes maximal consistent extension. These maximal extensions are of course only successful if the sets which are being extended are consistent. But this will be the case. Consult appendix D for proofs.

Before completing the lemma we have to show that SCS' is a strict finite chronicle structure satisfying:

$$\forall e \in E' : \tilde{f}^{q+1}(X'(e)) \subseteq Y'(e). \qquad (*)$$

By definition $(*)$ is satisfied, hence we only have to prove that SCS' is a strict finite chronicle structure. As $A' = (E', <')$ is a frame and E' is finite we just have to prove that Y' is strict and coherent. By definition of Y' and axiom (A6.d) it is easily seen that for all $e \in E' - \{e_2\}$ we have the following:

(i) if $e <' e_2$ then $Y'(e) \prec Y'(e_2)$ and none of the other three relations $\succ$, $\bowtie$ and $=$ are satisfied between $Y'(e)$ and $Y'(e_2)$,

(ii) if $e_2 <' e$ then $Y'(e_2) \prec Y'(e)$ and none of the other three relations $\succ$, $\bowtie$ and $=$ are satisfied between $Y'(e_2)$ and $Y'(e)$, and

(iii) if $e \; co' \; e_2$ then $Y'(e) \bowtie Y'(e_2)$ and none of the other three relations $\prec$, $\succ$ and $=$ are satisfied between $Y'(e)$ and $Y'(e_2)$.

Furthermore, by definition of Y' and axiom (A6.d) we have $Y'(e_2) = Y'(e_2)$ and none of the other three relations $\prec$, $\succ$ and $\bowtie$ are satisfied between $Y'(e_2)$ and $Y'(e_2)$.

Finally we have to treat events $e, e' \in E' - \{e_2\}$. Assume $e \in E_i$ and $e' \in E_k$. By considering the possible relations between e, e' and e_2 we get according to the definition of S', axiom (A6.c), lemma 5.10 and the transitivity of $\prec$ that at least one of the relations $\prec$, $\succ$, $\bowtie$ and $=$ must be satisfied between $Y'(e)$ and $Y'(e')$. For instance, if $e \; co' \; e_2$ and $e' \; co' \; e_2$ then by definition of Y' is follows that if $\alpha \in Y'(e)$ then $\triangle_j \triangle_i \alpha \in Y'(e')$ and if $\beta \in Y'(e')$ then $\triangle_j \triangle_k \beta \in Y'(e)$. By Propositional Calculus it follows that $\leftrightsquigarrow_{ji} \alpha \in Y'(e)$ and $\leftrightsquigarrow_{jk} \beta \in Y'(e')$ respectively and by using axiom (A6.c) and lemma 5.10 the required result follows. The other possibilities are treated in a similar way except for the cases where it is enough to use the transitivity of $\prec$, i.e. in the cases $e < e_2 < e'$ and $e' < e_2 < e$.

By analysing the possible relations between e and e' we can finish the lemma.

(i) Suppose $e <' e'$. Then $e < e'$ will be satisfied which implies that $Y(e) \prec Y(e')$. As SCS is a strict chronicle structure we conclude that $(Y(e), Y(e')) \notin \succ \cup \bowtie \cup =$. As for all $e'' \in E$ we have that $\tilde{f}(Y(e'')) \subseteq Y'(e'')$ it is readily observed that $(Y'(e), Y'(e')) \notin \succ \cup \bowtie \cup =$ is satisfied. The only possibility left is therefore $Y'(e) \prec Y'(e')$.

(ii) The possibilities $e' <' e$, $e\ co'\ e'$ and $e = e'$ are treated by arguments similar to those given in the previous case.

■

Theorem 5.23 (Completeness) Let $\alpha \in W$. If $\models \alpha$ then $\vdash \alpha$

Proof We will show that every consistent formula is satisfiable. Let E be a countable set of events. Fix an enumeration $e_1, e_2, \ldots$ of E and fix an enumeration $\alpha_1, \alpha_2, \ldots$ of W, the set of formulas. Fix an *injective* function $g : E \times W \to \omega$. Since $E \times W$ is a countable set, there will be no trouble in finding such an injective function. In what follows, for $(e, \alpha) \in E \times W$, we will refer to $g((e, \alpha))$ as the *code number* of (e, α).

Now, assume that α is a consistent formula. Pick an $Q \in MCS$ containing α. Assume $type(Q) = k$. Let $CS^1 = (A^1, X^1)$ where $A^1 = (\{e_1\}, \emptyset)$ and $X^1(e_1) = Q$. Clearly CS^1 is a chronicle structure. Next define $SCS^1 = (A^1, Y^1)$, where $Y^1(e_1) \approx \tilde{f}(X^1(e_1)) \cup \{p_0^k\}$ and $\tilde{f}$ is the function defined in lemma 5.22. Clearly SCS^1 is a strict chronicle structure and it is easily seen that CS^1 and SCS^1 satisfy the hypothesis of lemma 5.22.

We now proceed by iteratively removing live requirements from the chronicle structures CS^1 and SCS^1 according to lemma 5.22. For $m \geq 1$, suppose the chronicle structure $CS^m = (A^m, X^m)$ is defined with $A^m = (E^m, <^m)$ where $E^m = \{e_1, e_2, \ldots, e_m\}$. Also suppose that the strict chronicle structure $SCS^m = (A^m, Y^m)$ is defined with the chronicle Y^m satisfying:

$$\forall e \in E^m : \tilde{f}^m(X^m(e)) \subseteq Y^m(e).$$

Suppose CS^m does not have any live requirements. Then set $CS^{m+1} = CS^m$. Otherwise consider a live requirement (e, β) in CS^m which has – among all the live requirements in CS^m – the least code number. By lemma 5.22 the structure CS^m can be extended to the chronicle $CS^{m+1} = (A^{m+1}, X^{m+1})$ with $A^{m+1} = (E^{m+1}, <^{m+1})$ and $E^{m+1} = E^m \cup \{e^{m+1}\}$ so that (e, β) is no longer a live requirement in CS^{m+1}. Furthermore, we know by lemma 5.22 that there exists a strict chronicle structure $SCS^{m+1} = (A^{m+1}, Y^{m+1})$ satisfying:

$$\forall e \in E^{m+1} : \tilde{f}^{m+1}(X^{m+1}(e)) \subseteq Y^{m+1}(e).$$

Finally set $CS = (A, X)$ where $A = (E, <)$, $E = \bigcup_{m=1}^{\infty} E^m$ and $< = \bigcup_{m=1}^{\infty} <^m$. X is given by:

$$\forall e \in E : X(e) = X^m(e) \text{ where } e \in E^m.$$

It is routine to verify that X is a perfect chronicle on ES. Hence by lemma 5.19, $M = (A, V_X)$ is a model in which $e_1, M \models \alpha$. ■

6 Conclusion

In this paper we have succeeded in giving a logical characterization of ACA structures. The logic designed is strongly related to the concepts of concurrency, communication and locality. We have presented an axiomatic basis for the logic and have obtained proofs of both soundness and completeness. The proof of completeness is considered the main result of the paper.

A number of interesting extensions to the presented work can be considered. First of all it would be preferable to allow the number of agents in the ACA structures to be unbound. At present we consider only a fixed number of agents. A solution to this extension would require a modification of the axiomatic basis. In particularly the type axiom (A4.d) will have to be reconsidered and perhaps formulated as an inference rule. Secondly it would be worth investigating whether the logic presented is decidable. At present we do not know how to solve the question of decidability as the logic does not have the finite model property.

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Appendix A

The purpose of this appendix is to prove that inference rule (R2):

$$\frac{p^i \supset \alpha}{\tau_i \supset \alpha},$$

where α is free of p, is sound. First we present a lemma.

Lemma Let $g : P \to P$. Let $\tilde{g} : W \to W$ denote the extension of g to arbitrary formulas defined in the obvious way. Then

$$\forall \beta \in W : \text{ if } \models \beta \text{ then } \models \tilde{g}(\beta).$$

Proof Suppose $\beta \in W$ such that $\models \beta$. Let $M = (A, V)$ where $A = (E, <)$ be an arbitrary model. Let $e \in E$. We want to show that $e, M \models \tilde{g}(\beta)$ is the case. To this end we construct a model $M' = (A', V')$ from M in the following way:

(i) $A' = A$, and

(ii) $\forall e \in E : V'(e) = \{p \mid g(p) \in V(e)\}$.

The models M and M' satisfy:

$$\forall \gamma \in W, \forall e \in E : e, M' \models \gamma \text{ iff } e, M \models \tilde{g}(\gamma).$$

The proof of this statement is by induction on the structure of γ.

(i) Suppose $\gamma = p_k$. We have $e, M' \models p_k$ iff $p_k \in V'(e)$. By definition of V' it follows that $p_k \in V'(e)$ iff $g(p_k) \in V(e)$, hence iff $e, M \models \tilde{g}(p_k)$.

(ii) Suppose $\gamma = {\sim}\gamma'$. By definition we have $e, M' \models {\sim}\gamma'$ iff $e, M' \not\models \gamma'$. By the induction hypothesis this is the case iff $e, M \not\models \tilde{g}(\gamma')$, hence iff $e, M \models \tilde{g}({\sim}\gamma')$.

(iii) Suppose $\gamma = \gamma_1 \vee \gamma_2$. By definition we have $e, M' \models \gamma_1 \vee \gamma_2$ iff $e, M' \models \gamma_1$ or $e, M' \models \gamma_2$. Once again the required result follows from the induction hypothesis.

(iv) Suppose $\gamma = \downarrow_j \gamma'$. By definition we have $e, M' \models \downarrow_j \gamma'$ iff there exists $e' \in E_j$ such that $e < e'$ and $e', M' \models \gamma'$. By the induction hypothesis this is the case iff there exists $e' \in E_j$ such that $e < e'$ and $e', M \models \tilde{g}(\gamma')$, hence iff $e, M \models \tilde{g}(\downarrow_j \gamma')$.

The last two possibilities (i.e. $\gamma = \uparrow_j \gamma'$ and $\gamma = \triangle_j \gamma'$) follow the same line as the case $\gamma = \downarrow_j \gamma'$.

We now return to the proof of the lemma. As $\models \beta$ is assumed it follows that $e, M' \models \beta$. By the above result we immediately have $e, M \models \tilde{g}(\beta)$. ■

Now we can return to the proof of inference rule (R2). We have to show that if $\models p^i \supset \alpha$ then $\models \tau_i \supset \alpha$ where $p \in P$, $p^i = p \wedge \boxminus_i({\sim}p) \wedge \Box_i({\sim}p)$ and α is free of p. Assume therefore that $\models p^i \supset \alpha$ is the case. Suppose that $P = \{p_0, p_1, \ldots\}$. Assume without loss of generality that $p = p_0$. Let $M = (A, V)$ with $A = (E, <)$ be an arbitrary model. Assume that $e_0 \in E$. We want to show that $e_0, M \models \tau_i \supset \alpha$. Obviously we only have to consider the case $e_0 \in E_i$, and to show that $e_0, M \models \alpha$ is fulfilled in this case.

We define the transformation $f : W \to W$ such that $f(\beta)$, where $\beta \in W$, denotes the result of having substituted p_{i+1} for p_i in β. Let $\tilde{f}$ denote the extension of f to sets of formulas defined in the obvious way.

For technical reasons we define a model $M' = (A', V')$ from M as follows:

(i) $A' = A$,

(ii) $V'(e_0) = \{p_0\} \cup \tilde{f}(V(e_0))$, and

(iii) $\forall e \in E$: if $e \neq e_0$ then $V'(e) = \tilde{f}(V(e))$.

At first we present three claims.

Claim 1 We have $e_0, M' \models p_0^i$.

Proof Obviously by construction of V'. ∎

Claim 2 Let $\beta \in W$. Then

$$\forall e \in E : e, M \models \beta \text{ iff } e, M' \models \tilde{f}(\beta),$$

Proof The proof is by induction on the structure of β. We leave out the details as the proof is very similar to the induction proof presented in the lemma stated at the beginning of the appendix. ∎

Let $h : P \to P$ be defined as follows:

(i) $h(p_0) = p_0$, and

(ii) $\forall p \in P - \{p_0\} : h(p) = f(p)$,

Let $\tilde{h}$ denote the extension of h to W defined in the obvious way.

Claim 3 We have $\tilde{h}(p_0^i \supset \alpha) = p_0^i \supset \tilde{f}(\alpha)$.

Proof Follows immediately from the fact that α is free of p_0. ∎

We can now return to the proof of inference rule (R2). As hypothesis we have $\models p_0^i \supset \alpha$. According to the lemma stated at the beginning of the appendix it follows that $\models \tilde{h}(p_0^i \supset \alpha)$ which, by claim 3, implies $\models p_0^i \supset \tilde{f}(\alpha)$. As a special case we have $e_0, M' \models p_0^i \supset \tilde{f}(\alpha)$. By claim 1 and Modus Ponens we conclude that $e_0, M' \models \tilde{f}(\alpha)$ which, according to claim 2, implies $e_0, M \models \alpha$.

∎

Appendix B

Proofs of derived inference rules and theses. PC denotes Propositional Calculus.

(PR)

(1)	$\vdash \alpha_1 \wedge \ldots \wedge \alpha_m \supset \beta$	(Given)
(2)	$\vdash (\alpha_1 \supset (\alpha_2 \supset (\ldots(\alpha_m \supset \beta)\ldots)))$	(PC)
(3)	$\vdash \alpha_1, \ldots, \alpha_m$	(Given)
(4)	$\vdash \beta$	(2, 3, MP m times)

(DR.1)

(1)	$\vdash \alpha \supset \beta$	(Given)
(2)	$\vdash \odot_i(\alpha \supset \beta)$	(R1)
(3)	$\vdash \odot_i\alpha \supset \odot_i\beta$	(A1, MP)

(SU)

It suffices to prove that if $\alpha \supset \beta$ and $\beta \supset \alpha$ are theses then so are $\Phi(\alpha/\gamma) \supset \Phi(\beta/\gamma)$ and $\Phi(\beta/\gamma) \supset \Phi(\alpha/\gamma)$. This is proved by induction on the structure of Φ and is straightforward.

(T1)

All equalities follow easily from definition of dual modalities, i.e. definition 3.5.

(T2.a)

(1)	$\vdash (\alpha \supset \beta) \supset (\sim\beta \supset \sim\alpha)$	(PC)
(2)	$\vdash \Downarrow_i (\alpha \supset \beta) \supset \Downarrow_i (\sim\beta \supset \sim\alpha)$	(DR.1)
(3)	$\vdash \Downarrow_i (\alpha \supset \beta) \supset (\Downarrow_i \sim\beta \supset \Downarrow_i \sim\alpha)$	(A1.a, PR)
(4)	$\vdash \Downarrow_i (\alpha \supset \beta) \supset (\downarrow_i \alpha \supset \downarrow_i \beta)$	(T1.e, SU, PC)
(5)	$\vdash \Downarrow_i (\beta \supset \alpha \wedge \beta) \supset (\downarrow_i \beta \supset \downarrow_i (\alpha \wedge \beta))$	(Subst. in 4)
(6)	$\vdash \alpha \supset (\beta \supset \alpha \wedge \beta)$	(PC)
(7)	$\vdash \Downarrow_i \alpha \supset \Downarrow_i (\beta \supset \alpha \wedge \beta)$	(DR.1)
(8)	$\vdash \Downarrow_i \alpha \supset (\downarrow_i \beta \supset \downarrow_i (\alpha \wedge \beta))$	(PC, PR, 5)
(9)	$\vdash (\Downarrow_i \alpha \wedge \downarrow_i \beta) \supset \downarrow_i (\alpha \wedge \beta)$	(PC, PR)

(T2.b) and (T2.c)

Shown by arguments similar to those in the proof of (T2.a).

(T3.a)

(1)	$\vdash \alpha \supset \alpha \vee \beta$	(PC)
(2)	$\vdash \Downarrow_i \alpha \supset \Downarrow_i (\alpha \vee \beta)$	(DR.1)
(3)	$\vdash \Downarrow_i \beta \supset \Downarrow_i (\alpha \vee \beta)$	(Subst. in (2)
(4)	$\vdash \Downarrow_i \alpha \vee \Downarrow_i \beta \supset \Downarrow_i (\alpha \vee \beta)$	(PC, PR, 2, 3)
(5)	$\vdash \Downarrow_i \sim\alpha \vee \Downarrow_i \sim\beta \supset \Downarrow_i (\sim\alpha \vee \sim\beta)$	(Subst. in 4)
(6)	$\vdash \sim\Downarrow_i (\sim\alpha \vee \sim\beta) \supset \sim(\Downarrow_i \sim\alpha \vee \Downarrow_i \sim\beta)$	(PC, PR)
(7)	$\vdash \downarrow_i \sim(\sim\alpha \vee \sim\beta) \supset \downarrow_i \alpha \wedge \downarrow_i \beta$	(T1.e, PC, SU)
(8)	$\vdash \downarrow_i (\alpha \wedge \beta) \supset \downarrow_i \alpha \wedge \downarrow_i \beta$	(PC, SU)

(T3.b) and (T3.c)

Shown by arguments similar to those in the proof of (T3.a).

(T4.a)

(1)	$\vdash \Downarrow_j \alpha \wedge \downarrow_j \beta \supset \downarrow_j (\alpha \wedge \beta)$	(T2.a)
(2)	$\vdash \Downarrow_j \uparrow_i \alpha \wedge \downarrow_j \beta \supset \downarrow_j (\uparrow_i \alpha \wedge \beta)$	(Subst. in 1)
(3)	$\vdash \tau_i \wedge \alpha \supset \Downarrow_j \uparrow_i \alpha$	(A3.a)
(4)	$\vdash \tau_i \wedge \alpha \wedge \downarrow_j \beta \supset \downarrow_j (\uparrow_i \alpha \wedge \beta)$	(PC, PR, 2, 3)

(T4.b)

Shown by arguments similar to those in the proof of (T4.a).

(T4.c)

(1)	$\vdash \nabla_j \alpha \wedge \Delta_j \beta \supset \Delta_j(\alpha \wedge \beta)$	(T2.c)
(2)	$\vdash \nabla_j(\Delta_i \alpha) \wedge \Delta_j \beta \supset \Delta_j(\Delta_i \alpha \wedge \beta)$	(Subst. in 1)
(3)	$\vdash \tau_i \wedge \alpha \supset \nabla_j \Delta_i \alpha$	(A5.b)
(4)	$\vdash \tau_i \wedge \alpha \wedge \Delta_j \beta \supset \Delta_j(\Delta_i \alpha \wedge \beta)$	(PC, PR, 2, 3)

(T5.a)

(1)	$\vdash \tau_i \wedge \alpha \supset \Downarrow_j \uparrow_i \alpha$	(A3.a)
(2)	$\vdash \sim\Downarrow_j \uparrow_i \alpha \supset \sim(\tau_i \wedge \alpha)$	(PC, PR)
(3)	$\vdash \downarrow_j \Uparrow_i \sim\alpha \supset \sim(\tau_i \wedge \alpha)$	(T1.f, T1.h, PR, SU)
(4)	$\vdash \tau_i \wedge \downarrow_j \Uparrow_i \sim\alpha \supset \sim\alpha$	(PC, SU)
(5)	$\vdash \tau_i \wedge \downarrow_j \Uparrow_i \alpha \supset \alpha$	(Subst. in 4)

(T5.b)

Shown by arguments similar to those in the proof of (T5.a).

(T6)

(1)	$\vdash \tau_i \wedge \alpha \supset \nabla_j \triangle_i \alpha$	(A5.b)
(2)	$\vdash \sim\nabla_j \triangle_i \alpha \supset \sim(\tau_i \wedge \alpha)$	(PC, PR)
(3)	$\vdash \nabla_j \triangle_i \sim\alpha \supset \sim(\tau_i \wedge \alpha)$	(T1.c, T1.b, SU)
(4)	$\vdash \tau_i \wedge \nabla_j \triangle_i \alpha \supset \alpha$	(PC, SU)

(T7)

(1)	$\vdash \leftrightsquigarrow_{ji} \alpha \supset \downarrow_i \alpha \vee \uparrow_i \alpha \vee \triangle_i \alpha \vee \alpha$	(A6.c)
(2)	$\vdash \tau_i \wedge \leftrightsquigarrow_{ji} \alpha \supset \tau_i \wedge (\downarrow_i \alpha \vee \uparrow_i \alpha \vee \triangle_i \alpha \vee \alpha)$	(PC, PR)
(3)	$\vdash \tau_i \wedge \leftrightsquigarrow_{ji} \alpha \supset \nabla_i \bot \wedge (\downarrow_i \alpha \vee \uparrow_i \alpha \vee \triangle_i \alpha \vee \alpha)$	(A5.a, PR)
(4)	$\vdash \tau_i \wedge \leftrightsquigarrow_{ji} \alpha \supset \downarrow_i \alpha \vee \uparrow_i \alpha \vee \alpha$	(PC, PR)

Appendix C

The purpose of this appendix is to prove lemma 5.21 stating that live requirements can be eliminated. The proof will fall into three parts according to the three types of live requirements that exists. We will only show that live concurrent requirements can be removed as the arguments needed in the other two cases more or less are the same.

Lemma Let $CS = (A, X)$ with $A = (E, <)$ be a strict finite chronicle structure. Let (e_1, β) be a live concurrent requirement in CS. Then there exists a finite chronicle structure $CS' = (A', X')$ with $A' = (E', <')$ such that:

(i) $E' = E \cup \{e_2\}$ for some $e_2 \notin E$,

(ii) $<'$ restricted to $E \times E$ is $<$,

(iii) X' restricted to E is X, and

(iv) (e_1, β) is no longer a live requirement in CS'.

Proof As (e_1, β) is a live concurrent requirement it follows that $\beta = \triangle_j \alpha$. Assume that $e_1 \in E_i$. By axiom (A5.a) we have $i \neq j$. By lemma 5.9 there exists an $Q \in MCS$ such that $type(Q) = j$, $X(e_1) \bowtie Q$ and $\alpha \in Q$. Fix e_2 such that $e_2 \notin E$ and define for all $k \in \{1, \ldots, n\}$,

$$E'_k = \begin{cases} E_k \cup \{e_2\} & \text{if } k = j \\ E_k & \text{otherwise} \end{cases}$$

Set $E' = \bigcup_{k=1}^{n} E'_k$ and define

(i) $lpre(e_2) = \{e \in E_j \mid X(e) \prec Q\}$,

(ii) $lpost(e_2) = \{e \in E_j \mid Q \prec X(e)\}$,

(iii) $pre(e_2) = \{e \in E \mid \exists e' \in lpre(e_2) : e \leq e'\}$, and

(iv) $post(e_2) = \{e \in E \mid \exists e' \in lpost(e_2) : e' \leq e\}$.

Let now

$$<' = (< \cup R_1 \cup R_2)^+,$$

where $R_1 = pre(e_2) \times \{e_2\}$ and $R_2 = \{e_2\} \times post(e_2)$. Let finally

$$\forall e \in E' : X'(e) = \begin{cases} Q, & \text{if } e = e_2 \\ X(e), & \text{otherwise} \end{cases}$$

If we are capable of proving that $CS' = (A', X')$ is a chronicle structure, in which $e_1\ co'\ e_2$, then obviously it is a chronicle structure in which (e_1, β) is no longer a live requirement. As it turns out we cannot always prove that CS' is a chronicle structure. If CS' is not a chronicle structure we prove that CS' can be changed by imposing further causality between e_2 and events from E such that the resulting structure is a chronicle structure in which (e_1, β) is no longer a live requirement. The proof is divided into 11 claims, each stated and proved below.

Claim 1: $pre(e_2) \cap post(e_2) = \emptyset$.

Proof Suppose $pre(e_2) \cap post(e_2) \neq \emptyset$. Let $e \in pre(e_2) \cap post(e_2)$. By definition of $pre(e_2)$ we know that there exists $e' \in lpre(e_2)$ such that $e \leq e'$. As X is coherent we have $X(e) \prec Q$. By definition of $post(e_2)$ we know that there exists $e'' \in lpost(e_2)$ such that $e'' \leq e$. Once again, by the fact that X is coherent, it follows that $Q \prec X(e)$. By the transitivity of $\prec$ we now conclude that $X(e) \prec X(e)$ which cannot be the case as X is strict. ∎

Claim 2: Assume $e, e' \in E'$ such that $(e, e') \in (<' - <)$. Then $e \leq' e_2$ and $e_2 \leq' e'$.

Proof As $e <' e'$ we have by definition of $<'$ that there exists $\{x_0, \ldots, x_m\}$ such that $x_0 = e$, $x_m = e'$ and $(x_h, x_{h+1}) \in < \cup R_1 \cup R_2$ for each $h \in \{0, \ldots, m-1\}$. By the fact that $(e, e') \in (<' - <)$ it follows that there has to exist $h \in \{0, \ldots, m-1\}$ such that $(x_h, x_{h+1}) \in R_1 \cup R_2$ and the result follows at once. ∎

Claim 3: Assume $e \in E'$ such that $e \neq e_2$ and $e <' e_2$. Then $e \in pre(e_2)$.

Proof As $e <' e_2$ we have by definition of $<'$ that there exists $\{x_0, \ldots, x_m\}$ such that $x_0 = e$, $x_m = e_2$ and $(x_h, x_{h+1}) \in < \cup R_1 \cup R_2$ for each $h \in \{0, \ldots, m-1\}$. By definition we also know that $m \geq 1$. The rest follows by induction on m. We leave out the details. ∎

Claim 4: Assume $e \in E'$ such that $e \neq e_2$ and $e_2 <' e$. Then $e \in post(e_2)$.

Proof More or less the same arguments as in the previous claim. We leave out the details. ∎

Claim 5: The relation $<'$ is transitive and irreflexive.

Proof By definition $<'$ is transitive, hence we only have to show that $<'$ is irreflexive. Suppose $e \in E'$ such that $e <' e$. We want to show that this assumption leads to an inconsistency. We cannot have $e < e$ because $<$ is known to be irreflexive. Hence $(e, e) \in (<' - <)$ must be the case which by claim 2 implies $e \leq' e_2$ and $e_2 \leq' e$. There are two possibilities:

(i) Suppose $e = e_2$. Then we have $e_2 <' e_2$. By definition of $<'$ it is easily seen that either $(e_2, e_2) \in R_1$ or $(e_2, e_2) \in R_2$. Assume $(e_2, e_2) \in R_1$. Then $e_2 \in pre(e_2)$ which implies that there exists $e' \in lpre(e_2)$ such that $e_2 \leq e'$. As $e' \neq e_2$ we conclude from claim 4 that $e' \in post(e_2)$. This cannot be the case by claim 1. If $(e_2, e_2) \in R_2$ the result follows by first using claim 3 and then claim 1.

(ii) Suppose $e \neq e_2$. Then $e <' e_2$ and $e_2 <' e$ which, by claim 3 and claim 4, implies that $e \in pre(e_2)$ and $e \in post(e_2)$. This cannot be the case by claim 1.

∎

Claim 6: $E' = E'_1 \uplus \ldots \uplus E'_n$.

Proof Follows immediately from the definition of E'. ∎

Claim 7: The two events e_1 and e_2 are concurrent, i.e. $e_1\ co'\ e_2$.

Proof

(i) Suppose $e_2 <' e_1$. Then, as $e_1 \neq e_2$, it follows by claim 4 that $e_1 \in post(e_2)$. As $e_1 \notin E_j$ (remember that $i \neq j$) we have the existence of $e \in lpost(e_2)$ such that $e < e_1$. As X is coherent we conclude that $X(e) \prec X(e_1)$. Since $X(e_1) \bowtie Q$ and $type(Q) = type(X(e)) = j$ it follows by lemma 5.13 that $X(e) \prec Q$. As $e \in lpost(e_2)$ we have $Q \prec X(e)$. By the transitivity of $\prec$ we conclude that $X(e) \prec X(e)$ which is absurd as X is strict.

(ii) Suppose $e_1 <' e_2$. Then, as $e_1 \neq e_2$, we have according to claim 3 that $e_1 \in pre(e_2)$. By a proof similar to the previous case it is easily seen that there exists $e \in lpre(e_2)$ satisfying $X(e) \prec X(e)$. Again, as X is strict, the result follows.

■

Claim 8: For each $k \in \{1, \ldots, n\} : (E'_k, <'_k)$ is totally ordered.

Proof Let $k \in \{1, \ldots, n\}$. We have to show that any two events from E'_k will be comparable, i.e. if $e, e' \in E'_k$ then $e \leq'_k e'$ or $e' \leq'_k e$. By definition of $<'$ we only have to treat the case $k = j$.

It is easily observed that if $E_j \subseteq lpre(e_2) \cup lpost(e_2)$ then any two events from E'_j will be comparable. Suppose therefore that $e \in E_j$. We analyse the possible relations between e_1 and e where e_1 is the event belonging to the live requirement.

(i) Suppose $e < e_1$. Then, as X is coherent, we have $X(e) \prec X(e_1)$. Since $X(e_1) \bowtie Q$ and $type(Q) = type(X(e)) = j$ it follows by lemma 5.13 that $X(e) \prec Q$, hence $e \in lpre(e_2)$.

(ii) Suppose $e_1 < e$. Then by arguments similar to those given in the previous case it follows that $Q \prec X(e)$, hence $e \in lpost(e_2)$.

(iii) Suppose $e\ co\ e_1$. As X is coherent we have $X(e) \bowtie X(e_1)$. Since $X(e_1) \bowtie Q$ and $type(Q) = type(X(e)) = j$ it follows by lemma 5.14 that $X(e) \prec Q$, $Q \prec X(e)$ or $Q = X(e)$ is the case. We cannot have $Q = X(e)$, or else (e_1, β) would be no live requirement. We therefore conclude that $Q \prec X(e)$ or $X(e) \prec Q$ is the case, thus $e \in lpre(e_2) \cup lpost(e_2)$.

We finally remark that $e_1 = e$ cannot be the case as $e_1 \in E_i$, $e \in E_j$ and $i \neq j$. ■

Before ending the lemma we will have to show that X' is coherent. This is provided by the last three claims.

Claim 9: $\forall e, e' \in E'$: if $e <' e'$ then $X'(e) \prec X'(e')$.

Proof Suppose $e, e' \in E'$ such that $e <' e'$. By the definition of $<'$ we have that there exists $\{x_0, \ldots, x_m\}$ such that $x_0 = e$, $x_m = e'$ and $(x_h, x_{h+1}) \in \, < \cup R_1 \cup R_2$ for each $h \in \{0, \ldots, m-1\}$. By definition we also know that $m \geq 1$. We proceed by induction on m.

$\underline{m = 1}$ If $(x_0, x_1) \in R_1 \cup R_2$ then it is easily seen by definition of R_1 and R_2 that $X'(x_0) \prec X'(x_1)$. If $(x_0, x_1) \in \, <$ then the result follows because X is coherent.

$\underline{m > 1}$ We have $x_1 <' x_m$, hence by induction hypothesis that $X'(x_1) \prec X'(x_m)$. As $\prec$ is transitive we only have to show that $X'(x_0) \prec X'(x_1)$ where $(x_0, x_1) \in \, < \cup R_1 \cup R_2$. But this is already proved in the case $m = 1$ above.

■

Claim 10: We would like to show that

$$\forall e, e' \in E' : \text{if } e \; co' \; e' \text{ then } X'(e) \bowtie X'(e') \qquad (*)$$

As it turns out we cannot always prove $(*)$. If $(*)$ is not satisfied we change the structure CS' by imposing further causality between e_2 and events from E such that the resulting structure will satisfy $(*)$.

Proof Clearly we only have to consider the case where either $e = e_2$ or $e' = e_2$. Assume therefore that there exists $e \in E'_k$ such that $e_2 \; co' \; e$ and such that $X'(e_2) \bowtie X'(e)$ is not necessarily satisfied. The strategy is now to show that $X'(e_2) \bowtie X'(e)$ is the case or, if that is not possible, to make sure that $e_2 \; co' \; e$ is no longer satisfied.

Now, if $\alpha \in X'(e)$ and $\beta \in X'(e_2)$ then $\leftrightsquigarrow_{ik} \alpha \in X'(e_2)$ and $\leftrightsquigarrow_{ij} \beta \in X'(e)$, where $\leftrightsquigarrow_{ik} \alpha$ ($\leftrightsquigarrow_{ij} \beta$) is the formula displayed on page 12. This postulate can easily be shown by considering the events e, e_1 and e_2, and the corresponding maximal consistent sets $X'(e)$, $X'(e_1)$ and $X'(e_2)$. By axiom (A6.c) and lemma 5.10 we conclude that $(X'(e_2), X'(e)) \in \, \prec \cup \succ \cup \bowtie \cup =$. We cannot have $X'(e) = X'(e_2)$ as $e_2 \; co \; e$ implies $k \neq j$. (Remember that $\tau_j \in X'(e_2)$ and $\tau_k \in X'(e)$). If $X'(e_2) \bowtie X'(e)$ is the case then there is nothing to prove, hence assume that $X'(e_2) \not\bowtie X'(e)$. Then $X'(e_2) \prec X'(e)$ or $X'(e) \prec X'(e_2)$. Assume without loss of generality that $X'(e_2) \prec X'(e)$ is the case. A first guess would be to make a new structure $CS'' = (A'', X'')$ with $A'' = (E'', <'')$ just by taking X'' as X' and A'' as $(E', (<' \cup \{(e_2, e)\})^+)$. This new structure would surely satisfy:

(i) A'' is a frame[2], and

(ii) $\forall e', e'' \in E''$: if $e' <'' e''$ then $X''(e') \prec X''(e'')$.

[2]In general, if $\hat{A} = (\hat{E}, \hat{<})$ is a frame and $a, b \in \hat{E}$ such that a and b are concurrent then it is easily seen that $\hat{A}' = (\hat{E}, (\hat{<} \cup \{(a, b)\})^+)$ and $\hat{A}'' = (\hat{E}, (\hat{<} \cup \{(b, a)\})^+)$ also are frames. We only have to make sure that the new causal order is anti-symmetric and this will be the case, because if the new causal order is not anti-symmetric then this can only be the case if a and b are in order in $\hat{A}$ which was assumed not to be the case.

Furthermore, we would no longer have trouble with e_2 and e as they are in order now.

But we cannot always add (e_2, e) to the causality order because we have to take care of events $a, b \in E'$ satisfying $a\ co'\ b$ and $X'(a) \bowtie X'(b)$. We are not allowed to add (e_2, e) to the causality order if this implies that a and b become in order. Assume therefore that there exists $a, b \in E'$ such that $a\ co'\ b$ and $X'(a) \bowtie X'(b)$ and assume that adding (e_2, e) to the causality order will coerce a and b to be in order. Assume without loss of generality that $a <' e_2$ and $e <' b$ is the case[3]. It is easily seen that if we instead added (e, e_2) to the causality order then a and b would still be out of order, or else $a\ co'\ b$ cannot be satisfied from the beginning. We therefore seek to conclude that $X'(e) \prec X'(e_2)$ must be the case.

As a and b are different from e_2 we have by definition that $X'(a) = X(a)$ and $X'(b) = X(b)$. Now, as X is coherent, we conclude that $X'(a) \bowtie X'(b)$, and as CS is a strict chronicle structure this is the only relation among the four possibilities $\prec$, $\succ$, $\bowtie$ and $=$ satisfied between $X'(a)$ and $X'(b)$. As $a <' e_2$ and $e <' b$ we conclude by claim 9 that $X'(a) \prec X'(e_2)$ and $X'(e) \prec X'(b)$. Due to the transitivity of $\prec$ we are therefore constrainted to conclude that $X'(e_2) \prec X'(e)$ cannot be the case. The only possibility left is therefore $X'(e) \prec X'(e_2)$.

We can now construct a new structure $CS'' = (A'', T'')$ with $A'' = (E'', <'')$ just by taking X'' as X' and A'' as $(E', (<' \cup \{(e, e_2)\})^+)$. This new structure satisfies:

(i) A'' is a frame, and

(ii) $\forall e', e'' \in E''$: if $e' <'' e''$ then $X''(e') \prec X''(e'')$.

Furthermore, we would no longer have trouble with e and e_2 as they are in order now and in such a way that a and b is not coerced to be in order.

But have we completed the claim now? No! Once again we have to take into account events $c, d \in E'$ satisfying $c\ co'\ d$ and $X'(c) \bowtie X'(d)$. In adding (e, e_2) to the causality relation we must be sure that $c\ co'\ d$ is still satisfied. This we cannot be sure of, and as we cannot turn e and e_2 around again (i.e. adding (e_2, e) to the causality relation) as this would coerce a and b to be in order, we have to leave e and e_2 out of order and must force $X'(e) \bowtie X'(e_2)$ to be the case. Assume therefore that there exists $c, d \in E'$ such that $c\ co'\ d$ and $X'(c) \bowtie X'(d)$. Assume also that adding (e, e_2) to the causality relation will make c and d in order. Suppose without loss of generality that $e_2 <' c$ and $d <' e$.[4] By an argument similar to the above where we considered the events a and b it is easily seen that $X(e_2) \prec X(e)$ neither can be the case. We now conclude that the assumption $X'(e_2) \not\bowtie X'(e)$ is false, as $X'(e_2) \not\prec X'(e)$ and $X'(e) \not\prec X'(e_2)$ imply that $X'(e_2) \bowtie X'(e)$ *must* be satisfied.

To sum up: If we cannot order e and e_2 at all, because of events a, b, c and d, then we can show that $X'(e_2) \bowtie X'(e)$ must be satisfied. We can therefore leave the structure CS' as it is.

The above mentioned strategy is supposed to be used for every $e \in E'$ such that e_2 and e

[3]It could be the case that $a = e_2$, but we are only interested in assuring that events *different* from e_2 stay concurrent if that was the case from the beginning. On the other hand it could very well be the case that $e = b$, but this will only simplify the arguments, so we assume that $e <' b$ is satisfied.

[4]We assume that $c \neq e_2$ and $e \neq d$.

are in no causal order. The strategy will end as CS' is finite, and the final result will be a structure obeying $(*)$. This completes claim 10. ■

We finally need to prove the following claim in order to establish that X' is coherent.

Claim 11: $\forall e \in E' : \tau_i \in X'(e)$ iff $e \in E_i$.

Proof As X is a coherent chronicle structure it is only necessary to consider the case $e = e_2$. If $e = e_2$ then $X'(e) = Q$ and the result follows at once from the fact that $type(Q) = j$ and $e_2 \in E_j$. ■

Appendix D

In this appendix we will prove some postulates which were stated in lemma 5.22 without proofs.

Let $f(\alpha)$, where $\alpha \in W$, denote the result of having substituted p_{i+1} for p_i in α. Let $\tilde{f}$ denote the extension of f to sets of formulas defined in the obvious way.

Lemma Let $Q \in MCS$ such that $type(Q) = i$ and $P = \{p_0, p_1, \ldots\}$. Let finally $p_0^i = p_0 \wedge \boxminus_i(\sim p_0) \wedge \Box_i(\sim p_0)$. Then $\tilde{f}(Q) \cup \{p_0^i\}$ is consistent.

Proof As Q is consistent it follows that $\tilde{f}(Q)$ is consistent. Hence if $\tilde{f}(Q) \cup \{p_0^i\}$ is inconsistent then this can only be the case if there exists $f(\alpha) \in \tilde{f}(Q)$ such that $f(\alpha) \wedge p_0^i$ is inconsistent. We have

$\vdash \sim(f(\alpha) \wedge p_0^i)$	(Assumed)
$\vdash p_0^i \supset \sim f(\alpha)$	(PC)
$\vdash \tau_i \supset \sim f(\alpha)$	(R2 as $\sim f(\alpha)$ is free of p_0)
$\vdash \sim(\tau_i \wedge f(\alpha))$	(PC)

In the above deduction (PC) is shorthand for Propositional Calculus. We conclude that $\tau_i \wedge f(\alpha)$ is inconsistent which is absurd as both $\tau_i, f(\alpha) \in \tilde{f}(Q)$ and $\tilde{f}(Q)$ is consistent. ■

Lemma Let $Q, R \in MCS$ such that $type(Q) = i$ and $type(R) = j$. Let $P = \{p_0, p_1, \ldots\}$. Finally let R^+ be a maximal consistent set of formulas satisfying $\tilde{f}(R) \subseteq R^+$. Then

(i) If $Q \prec R$ then $\tilde{f}(Q) \cup \{\downarrow_j \beta \mid \beta \in R^+\}$ is consistent.

(ii) If $R \prec Q$ then $\tilde{f}(Q) \cup \{\uparrow_j \beta \mid \beta \in R^+\}$ is consistent.

(iii) If $Q \bowtie R$ then $\tilde{f}(Q) \cup \{\triangle_j \beta \mid \beta \in R^+\}$ is consistent.

Proof We only prove (i) as the others follow by similar arguments. Suppose $f(\alpha) \in \tilde{f}(Q)$ and $\beta \in R^+$. We only have to show that $f(\alpha) \wedge \downarrow_j \beta$ is consistent in having proved that $\tilde{f}(Q) \cup \{\downarrow_j \beta \mid \beta \in R^+\}$ is consistent. As $f(\alpha) \in \tilde{f}(Q)$ we have $\alpha \in Q$ and by the fact that $Q \prec R$ we conclude that $\uparrow_j \alpha \in R$. It follows that $\uparrow_i f(\alpha) \in R^+$. As $\beta \in R^+$ we have $\beta \wedge \uparrow_i f(\alpha) \in R^+$. As $type(R) = j$ we conclude by thesis (T4.b) that $\uparrow_i (f(\alpha) \wedge \downarrow_j \beta) \in R^+$, hence the formula is consistent. By lemma 5.3 we conclude that $f(\alpha) \wedge \downarrow_j \beta$ is consistent. ■