A Note on the Jacobian Conjecture

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Abstract

We extend a corollary in [2], yielding a sufficient and necessary condition for a polynomial map to have an inverse of the simplest form, and give a surprisingly simple proof for the Jacobian Conjecture in two variables of the case $f_i = x_i - h_i$, where h_i is homogeneous of degree ≥ 2, $i = 1, 2$.

1 The Jacobian conjecture

Let k be a fiels of characteristic 0, and let $f = (f_1, \ldots, f_n)$ be a polynomial map from k^n to k^n, $f_i \in k[x_1, \ldots, x_n], 1 \leq i \leq n$.
The Jacobian matrix for f is:

$$J(f) = \left[\frac{\partial f_i}{\partial x_j} \right], \quad j(f) = det J(f)$$

The *Jacobian Conjecture* states that if $J(f)$ is invertible, *i.e.* $j(f)$ is a nonzero constant in k, then f has a polynomial inverse.

Although it is trivially true when $n = 1$, the Jacobian Conjecture has not been generally resolved even when $n = 2$. Only in some special cases has it been proved true([1]):

1. if the degrees of f_1 and f_2 do not exceed 100 (Moh).

2. if one of the degrees is of the form pq where p (resp. q) is 1 or a prime (Abhyankar and Moh, Nakai and Baba).

3. if one of the degrees is 4 (Nakai and Babai).

4. if the larger of the two degrees is $2p$ for some odd prime p (Nakai and Baba).

In section 3, we give a surprisingly simple proof for the case $f_i = x_i - h_i$, where h_i is homogeneous of degree $\geq 2, i = 1, 2$ by using a corollary in [2]. Unfortunately, this simple proof only works for $n = 2$.

In the general n-variable case, Wang ([1]) proved the Jacobian Conjecture is true if all f_i's have degree 2. Wright, et al ([2]) reduced the problem to the case where the degree of each f_i is at most 3 at the cost of introducing extra variables.

In section 2, we give a weaker condition for the aforementioned corollary in [2] and prove under that condition the converse holds, too. This yields a sufficient and necessary condition for a polynomial map to have an inverse of the simplest form.

2 The Simplest Inverse

Without loss of generality ([2]), we assume f_i has the canonical form $f_i(\underline{x}) = x_i - h_i(\underline{x})$, where h_i has no constant or linear parts. Then, the Jacobian matrix for f is:

$$J(f) = \left[\frac{\partial f_i}{\partial x_j}\right] = I - \left[\frac{\partial h_i}{\partial x_j}\right],\ j(f) = det J(f)$$

Furthermore, if $J(f)$ is invertible, we assume $j(f) = 1$. Observe that if $h(\underline{x})$ is homogeneous, then $J(h)$ is a nilpotent matrix.

The following lemma describes the sufficient and necessary condition for the inverse of $f(x)$ to have the *simplest* form $g(\underline{x}) = \underline{x} + h(\underline{x})$, when $h(\underline{x})$ is homogeneous. From *Abhyankar Inversion Formula* in [2], we see for $f(\underline{x})$ of the canonical form with $h(\underline{x}) \neq 0$ and homogeneous, the inverse of $f(\underline{x})$ contains $\underline{x} + h(\underline{x})$ as the first two lower degree parts, this justifies our usage of the word *simplest*.

Lemma 1 *Let* $f(\underline{x}) = \underline{x} - h(\underline{x})$, $h(\underline{x})$ *homogeneous of degree* $d \geq 2$, *and* *assume* $j(f) = 1$, *then* f *is invertible with inverse* $g(\underline{x}) = \underline{x} + h(\underline{x})$ *iff* $J(h(\underline{x})) \cdot h(\underline{x}) = 0$ i.e. $J(h(\underline{x}))^2 \underline{x} = 0$.

Proof: *if part*: Recall Taylor Expansion Formula on vector space of functions:

$$f(\underline{x} + \triangle \underline{x}) = f(\underline{x}) + \nabla f(\underline{x}) \cdot \triangle \underline{x} + \ldots + \nabla^t f(\underline{x})(\triangle \underline{x})^t + \cdots$$

where ∇ is the differential operator, $\nabla^t f(\underline{x})$ is a t-dimensional matrix.

$$\nabla^t f(\underline{x})(\triangle \underline{x}))^t = (\ldots(\nabla^t f(\underline{x}) \underbrace{\triangle \underline{x}) \ldots) \triangle \underline{x}}_{t}$$

Apply the above formula to $h(\underline{x} - h(\underline{x}))$: $\quad (\triangle \underline{x} = -h(\underline{x}))$

$$h(\underline{x} - h(\underline{x}) \ ) = h(\underline{x}) - \nabla h(\underline{x}) \cdot h(\underline{x}) + \cdots + (-1)^t \nabla^t h(\underline{x}) \cdot h^t(\underline{x}) + \cdots$$

By inducing on t, we prove that $0 = J(h(\underline{x})h(\underline{x}) \ ) = \nabla h(\underline{x}) \cdot h(\underline{x})$ implies

$$\nabla^t h(\underline{x}) \cdot h^t(\underline{x}) = 0, \text{ for all } t \geq 1, \text{ thus } h(\underline{x} - h(\underline{x})) = h(\underline{x}).$$

This yields $h(\underline{x}) = h(g)$, but $g(\underline{x}) = \underline{x} + h(g)$,
therefore $g(\underline{x}) = \underline{x} + h(\underline{x})$.

Now assume $\nabla^{t-1} h(\underline{x}) \cdot h^{t-1}(\underline{x}) = 0$.
Apply ∇ once more, by chain rule, we have

$$0 = \nabla(\nabla^{t-1} h(\underline{x}) \ \cdot \ h^{t-1}(\underline{x})) = \nabla^t h(\underline{x}) \ \cdot \ h^{t-1}(\underline{x}) +$$
$$+ \nabla^{t-1} h(\underline{x}) (\sum_{\substack{i+j=t-2 \\ 0 \leq i,j \leq t-2}} h^i \nabla h(\underline{x}) \ \cdot \ h^j(\underline{x}))$$

Multiply $h(\underline{x})$ to the right, and notice that $\nabla h(\underline{x}) \cdot h(\underline{x}) = 0$.

We conclude $\nabla^t h(\underline{x}) \cdot h^t(\underline{x}) = 0$, this completes the induction.

only if part: As $g(\underline{x}) = \underline{x} + h(g)$, we have $h(\underline{x}) = h(g) = h(x + h(\underline{x}))$.
Apply Taylor Expansion to $h(x + h(\underline{x}) \ )$ with $\triangle \underline{x} = h(\underline{x})$.
As $h(\underline{x})$ is homogeneous of degree $d \geq 2$, and $\nabla^t h(\underline{x}) \cdot h^t(\underline{x})$ has degree

$(d-t)+td$, for $1 \leq t \leq d$, whereas
$\nabla^t h(\underline{x}) = 0$ for $t > d$, it follows that

$$\nabla^t h(\underline{x}) \cdot h^t(\underline{x}) = 0, \text{for all } t \geq 1$$

In particular, $\nabla h(\underline{x}) \cdot h(\underline{x}) = J(h)h = 0$.

□

Remark If h is homogeneous of degree $d \geq 2$, by Euler's Theorem for homogeneous functions, $h_i = \frac{1}{d}(\sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} x_j)$, hence $h(\underline{x}) = \frac{1}{d} J(h(\underline{x}))\underline{x}$. It is clear that $J(h)^2 = 0$ implies that $J(h)h = \frac{1}{d} J(h)^2 \underline{x} = 0$. On the other hand, for a general matrix M over $k[\underline{x}]$, $M^2 \underline{x} = 0$ for all $\underline{x} \in k^n$ does not necessarily imply $M^2 = 0$.
For example, let

$$M = \begin{bmatrix} x_2 & -x_1 \\ x_2 & -x_1 \end{bmatrix}$$

we have $M\underline{x} = 0$, hence $M^2\underline{x} = 0$, but $M^2 \neq 0$.

Therefore, in genreal, the condition of Lemma 1 is slightly weaker than the condition of *Corollary 5.4* in [2], an under this weaker condition the converse holds, too.

However, as the matrix in question is the Jacobian matrix $J(h(\underline{x}))$ for homogeneous functions $h(\underline{x})$, it could happen that $J(h(\underline{x}))^2 = 0$ is equivalent to $J(h(\underline{x}))^2 = 0$ in this specific setting. This is the case when $n = 2$, as $j(h) = 1$ implies $J(h)^2 = 0$ (see the proof of Theorem 2). When $n = 3$, $J(h)^2 = 0$ impies that the rank of $J(h)$ is 1, or the compound matrix of $J(h)$ is zero, whereas $J(h)^2\underline{x} = J(h)h = 0$ gives no hint of the rank of $J(h)$. For $n > 3$, no simple things can be said. We believe the two conditions are not equivalent when $n \geq 3$.

As for homogeneous $h(\underline{x})$, we know $J(h(\underline{x}))$ is nilpotent. Lemma 1 points out a simple relation between the nilpotency of $J(h)$ (or rather, a modified condition on the nilpotency of $J(h)$) and the form the inverse of $f(\underline{x}) = \underline{x} - h(\underline{x})$ may take. One might like to further investigate this relationship and ask:

> Does $J(h(\underline{x}))^k = 0$ or $J(h(\underline{x}))^k\underline{x} = 0$ or other similar expressions give a sufficient and/or necessary condition for the inverse

of $f(\underline{x}) = \underline{x} - h(\underline{x})$ to take some simple form, *e.g.* as might be suggested by the Abhyankar Inversion Formula?

The answer seems to be negative.

3 The Jacobian Conjecture In Two Variables

In this section, we prove that when $n = 2$ and $h(\underline{x})$ homogeneous, $f(\underline{x}) = \underline{x} - h(\underline{x})$ is invertible, with the *simplest* inverse $g(\underline{x}) = \underline{x} + h(\underline{x})$ by showing $J(h)^2 = 0$. Homogeneity of $h(\underline{x})$ plays the key role in the proof.

Theorem 2 *For $f = (f_1, f_2)$, $f_i = x_i - h_i$, where h_i is homogeneous of degree ≥ 2, $i = 1, 2$. Assume $j(f) = 1$, then f is invertible.*

Proof: As h_1, h_2 are homogeneous,

$$1 = j(f) = 1 - \frac{\partial h_1}{\partial x_1} - \frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2}$$

implies

$$\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} = 0, \quad \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} = \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}$$

Therefore

$$\left(\frac{\partial h_1}{\partial x_1}\right)^2 + \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} = \left(\frac{\partial h_1}{\partial x_1}\right)^2 + \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} = 0$$

Similarly,

$$\frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2}\frac{\partial h_2}{\partial x_1} = 0, \quad \frac{\partial h_1}{\partial x_1}\frac{\partial h_1}{\partial x_2} + \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_2} = 0, \quad \frac{\partial h_2}{\partial x_1}\frac{\partial h_1}{\partial x_2} + \left(\frac{\partial h_2}{\partial x_2}\right)^2 = 0$$

Thus, we have shown $J(h)^2 = 0$. By *Lemma 1*, f is invertible. □

References

[1] S. Wang *A Jacobian criterion for separability* J.Algebra **65** (1980), 453—494.

[2] H. Bass, E. Connell, and D. Wright *The Jacobian Conjucture: Reduction of Degree and Formal Expansion of the Inverse* Bull. of the Amer. Math. Soc. *Vol* 7, *No* 2 (1982), 287—330.