# On Action Algebras*

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**Abstract**

*Action algebras* have been proposed by Pratt [22] as an alternative to Kleene algebras [8, 9]. Their chief advantage over Kleene algebras is that they form a finitely-based equational variety, so the essential properties of * (iteration) are captured purely equationally. However, unlike Kleene algebras, they are not closed under the formation of matrices, which renders them inapplicable in certain constructions in automata theory and the design and analysis of algorithms.

In this paper we consider a class of action algebras called *action lattices*. An action lattice is simply an action algebra that forms a lattice under its natural order. Action lattices combine the best features of Kleene algebras and action algebras: like action algebras, they form a finitely-based equational variety; like Kleene algebras, they are closed under the formation of matrices. Moreover, they form the largest subvariety of action algebras for which this is true. All common examples of Kleene algebras appearing in automata theory, logics of programs,

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relational algebra, and the design and analysis of algorithms are action lattices.

# 1 Introduction

*Iteration* is an inescapable aspect of computer programs. One finds a bewildering array of formal structures in the literature that handle iteration in various ways. Many of these are based on the algebraic operator $^*$, a construct that originated with Kleene [12] and has since evolved in various directions. Among these one finds Kleene algebras [3, 8, 9], $^*$-continuous Kleene algebras [3, 7, 10], action algebras [22], dynamic algebras [7, 21], and closed semirings (see [1, 17, 10, 15, 6]), all of which axiomatize the essential properties of $^*$ in different ways.

The standard relational and language-theoretic models found in automata theory [4, 5, 16, 15, 14], program logic and semantics (see [11] and references therein), and relational algebra [19, 20] are all examples of such algebras. In addition one finds a number of nonstandard examples in the design and analysis of algorithms, among them the so-called min,+ algebras (see [1, 17, 10]) an certain algebra of polygons [6].

Of the three classes of algebra mentioned, the lest restrictive is the class of *Kleene algebras*. Kleene algebras have been studied under various definitions by various authors, most notably Conway [3]. We adopt the definition of [8, 9], in which Kleene algebras are axiomatized by a certain finite set of universally quantified equational implications over the regular operators $+$, $;$, $^*$, 0, 1. Thus the class of Kleene algebras forms a finitely-based equational quasivariety. The equational consequences of the Kleene algebra axioms are exactly the regular identities [3, 8, 13]. Thus the family of regular languages over an alphabet $\Sigma$ forms the free Kleene algebra on free generators $\Sigma$.

A central step in the completeness proof of [8] is the demonstration that the family of $n \times n$ matrices over a Kleene algebra again forms a Kleene algebra. This construction is also useful in several other applications: matrices over the two-element Kleene algebra are used to derive fast algorithms for reflexive transitive closure in directed graphs; matrices over min,+ algebras are used to compute shortest paths in weighted directed graphs; and matrices over the free monoid $\Sigma^*$ are used to construct regular expressions equivalent to given finite automata (see [1, 17, 10]). Using matrices over an arbitrary

Kleene algebra, one can give a single uniform solution from which each of these applications can be derived as a special case.

Besides equations, the axiomatization of Kleene algebra contains the two equational implications

$$ax \leq x \quad\Rightarrow\quad a^*x \leq x \tag{1}$$

$$xa \leq x \quad\Rightarrow\quad xa^* \leq x. \tag{2}$$

It is known that no finite equational axiomatization exists over this signature [23] (although well-behaved infinite equational axiomatizations have been given [13, 21]). Pratt [22] argues that this is due to an inherent nonmonotonicity associated with the $^*$ operator. This nonmonitonicity is handled in Kleene algebras with the equational implications (1) and (2).

In light of the negative result of [23], it is quite surprising that the essential properties of $^*$ should be captured purely equationally. Pratt [22] shows that this is possible over an expanded signature. He augments the regular operators with two *residuation operators* $\rightarrow$ and $\leftarrow$, which give a end of weak left and right inverse to the composition operator ;, and identifies a finite set of equations that entail all the Kleene algebra axioms, including (1) and (2). The models of these equations are called *action algebras*. The inherent nonmonotonicity associated with $^*$ is captured by the residuation operators, each of which is nonmonotonic in one of its arguments. Moreover, all the examples of Kleene algebra mentioned above have naturally defined residuation operations under which they form action algebras. Thus the action algebras form a finitely-based equational variety contained in the quasivariety of Kleene algebras and containing all the examples we are interested in. This is a desirable state of affairs, since one can now reason about $^*$ in a purely equational way.

However, one disadvantage of action algebras is that they are not closed under the formation of matrices. In Example 3.3 below we construct an action algebra $\mathcal{U}$ for which the $2 \times 2$ matrices over $\mathcal{U}$ do not form an action algebra. Thus one cannot carry out the program of [8] or use action algebra to give a genera treatment of the applications mentioned above that require matrices.

In this paper we show that the situation can be rectified by father augmenting the signature with a meet operator $\cdot$ and imposing lattice axioms, and that this step is unavoidable if closure under the formation of matrices is desired. Specifically, we show that for $n \geq 2$, the family of $n \times n$ matrices over an action algebra $A$ is again an aition algebra if and only if $A$ has finite meets under its natural order. An action algebra with this property is called an *action lattice*. Action lattices have a finite equational axiomatization and are closed under the formation of matrices; moreover, they form the largest subvariety of action algebras for which this is true.

In specializing from action algebras to action lattices, we do not lose any of the various models of interest mentioned above. We have thus identified a class that combines the best features of Kleene algebras and action algebras:

- like action algebras, action lattices form a finitely-bred equations variety;
- like Kleene algebras, the $n \times n$ matrices over an action lattice again form an action lattice;
- all the Kleene algebras that normally arise in applications in logics of programs, automata theory, relations algebra, and the design and analysis of algorithms are examples of action lattices.

# 2 Definitions

With so many operators and axioms, it is not hard to become confused. Not the lest problem is conflict of notation in the literature. For the purposes of this paper, we follow [22] and use $+$ and $\cdot$ for join and meet, respectively, and ; for composition ([8, 9, 10] use $\cdot$ for composition).

For ease of reference, we collect all operators, signatures, axioms, and closes of structures together in four tables. All closes of algebraic structures we consider will have signatures consisting of some subset of the operators in Table 1 and axioms consisting of some subset of the formulars of Table 3. The signatures and closes themselves are defined in Tables 2 and 4, respectively.

The binary operators are written in infix. We normally omit the operator ; from expressions, writing $ab$ for $a;b$. We avoid parentheses by assigning $^*$ highest priority, then ;, then all the other operators. Thus $a + bc^*$ should be parsed $a + (b(c^*))$.

The expression $a \leq b$ is considered an abbreviation for the equation $a + b = b$.

| *symbol* | *name* | *arity* |
|---|---|---|
| $+$ | sum, join, plus | 2 |
| ; | product, (sequential) composition | 2 |
| $\cdot$ | meet | 2 |
| $\leftarrow$ | left residuation | 2 |
| $\rightarrow$ | right residuation | 2 |
| $^*$ | star, iteration | 1 |
| 0 | zero, additive identity | 0 |
| 1 | one, multiplicative identity | 0 |

Table 1: Operators

As shown in [22], the two definitions of **RES** given in Table 4 are equivalent. The first gives a useful characterization of $\rightarrow$ and $\leftarrow$ in succinct terms, and the second gives a purely equations characterization. With the second definition, **RES** and **ACT** are defined by pure equations.

| *short name* | *name* | *operators* |
|---|---|---|
| **is** | idempotent semirings | $+$, ;, 0, 1 |
| **ka** | Kleene algebras | **is**, $^*$ |
| **res** | residuation algebras | **is**, $\leftarrow$, $\rightarrow$ |
| **act** | action algebras | **ka**, **res** |
| **al** | action lattices | **act**, $\cdot$ |

Table 2: Signatures

$$a + (b + c) = (a + b) + c \quad (3)$$
$$a + b = b + a \quad (4)$$
$$a + a = a \quad (5)$$
$$a + 0 = 0 + a = a \quad (6)$$
$$a(bc) = (ab)c \quad (7)$$
$$a1 = 1a = a \quad (8)$$
$$a(b + c) = ab + ac \quad (9)$$
$$(a + b)c = ac + bc \quad (10)$$
$$a0 = 0a = 0 \quad (11)$$
$$1 + a + a^*a^* \leq a^* \quad (12)$$
$$ax \leq x \Rightarrow a^*x \leq x \quad (13)$$
$$xa \leq x \Rightarrow xa^* \leq x \quad (14)$$
$$ax \leq b \iff x \leq a \rightarrow b \quad (15)$$
$$xa \leq b \iff x \leq b \leftarrow a \quad (16)$$
$$a(a \rightarrow b) \leq b \quad (17)$$
$$(b \leftarrow a)a \leq b \quad (18)$$
$$a \rightarrow b \leq a \rightarrow (b + c) \quad (19)$$
$$b \leftarrow a \leq (b + c) \leftarrow a \quad (20)$$
$$x \leq a \rightarrow ax \quad (21)$$
$$x \leq xa \leftarrow a \quad (22)$$
$$(x \rightarrow x)^* = x \rightarrow x \quad (23)$$
$$(x \leftarrow x)^* = x \leftarrow x \quad (24)$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad (25)$$
$$a \cdot b = b \cdot a \quad (26)$$
$$a \cdot a = a \quad (27)$$
$$a + (a \cdot b) = a \quad (28)$$
$$a \cdot (a + b) = a \quad (29)$$

Table 3: Axioms

| *class* | *name* | *signature* | *defining axioms* |
|---|---|---|---|
| **US** | upper semilattices | $+$ | (3)–(5) |
| **IS** | idempotent semirings | **is** | (3)–(11) |
| **KA** | Kleene algebras | **ka** | **IS**, (12)–(14) |
| **RES** | residuation algebras | **res** | **IS**, (15)–(16) |
| **RES** | residuation algebras | **res** | **IS**, (17)–(22) |
| **RKA** | residuated Kleene algebras | **act** | **KA**, **RES** |
| **ACT** | action algebras | **act** | **RES**, (12), (23), (24) |
| **LS** | lower semilattices | $\cdot$ | (25)–(27) |
| **L** | lattices | $+, \cdot$ | **US**, **LS**, (28), (29) |
| **AL** | action lattices | **it** | **ACT**, **L** |

Table 4: Algebraic structures

Let **C** be a class of algebraic structures with signature $\sigma$ and let $\mathcal{A}$ be an algebraic structure with signature $\tau$. We say that $\mathcal{A}$ *expands to* an algebra in **C** if the operators in $\sigma - \tau$ can be defined on $\mathcal{A}$ in such a way that the resulting algebra, restricted to signature $\sigma$, is in **C**.

# 3 Main Results

## 3.1 Action algebras are residuated Kleene algebras

We first give an alternative characterization of action algebras that we will later find useful: action algebras are exactly the residuated Kleene algebra.

**Lemma 3.1 ACT = RKA**.

*Proof.* Every action algebra is a residuation algebra by definition. As shown in [22], every action algebra is a Kleene algebra. This establishes the forward inclusion.

Conversely, we show that the properties (23) and (24) hold in all residuated Kleene algebra. By symmetry, it will suffice to show (23). The inequality $x \to x \leq (x \to x)^*$ follows from (12) and the **IS** axioms. For the reverse inequality, we have

$$\begin{aligned} x(x \to x) &\leq x && \text{by (17)} \\ x(x \to x)^* &\leq x && \text{by (14), and} \\ (x \to x)^* &\leq x \to x && \text{by (15).} \end{aligned}$$

### 3.2 Matrices

Let $\mathcal{R}$ be an idempotent semiring and let $\mathbf{Mat}(n, \mathcal{R})$ denote the family of $n \times n$ matrices over $\mathcal{R}$, with $+$ interpreted as the usual matrix addition, ; the usual matrix multiplication, 0 the zero matrix, and 1 the identity matrix. Under these definitions, $\mathbf{Mat}(n, \mathcal{R})$ forms an idempotent semiring. Moreover, if $\mathcal{R}$ is also a Kleene algebra, we define $^*$ on $\mathbf{Mat}(n, \mathcal{R})$ in the usual way (see [3, 8, 10]); then $\mathbf{Mat}(n, \mathcal{R})$ forms a Kleene algebra [8].

We say that an ordered structure $\mathcal{R}$ *has finite meets* if every finite set of elements has a meet or greatest lower bond. An upper semilattice $(R, +)$ has (nonempty) finite meets if and only if it expands to a lattice $(R, +, \cdot)$; the operation $\cdot$ gives the meet of its arguments.

**Lemma 3.2** *Let $\mathcal{R} = (R, +, ;, 0, 1, \leftarrow, \rightarrow)$ be a residuation algebra. For $n \geq 2$, the indempotent semiring $\mathbf{Mat}(n, \mathcal{R})$ expands to a residuation algebra if and only if $\mathcal{R}$ has finite meets.*

*Proof.* Suppose first that $\mathcal{R}$ has finite meets, and expand $\mathcal{R}$ to a lattice $(\mathcal{R}, +, \cdot)$ accordingly. Using the notation $\sum$ for iterated $+$ and $\prod$ for iterated $\cdot$, we define the operations $\rightarrow$ and $\leftarrow$ on $\mathbf{Mat}(n, \mathcal{R})$ as follows:

$$(A \rightarrow B)_{ij} = \prod_{k=1}^{n} (A_{ki} \rightarrow B_{kj}) \tag{30}$$

$$(B \leftarrow A)_{ij} = \prod_{k=1}^{n} (B_{ik} \leftarrow A_{jk}) \tag{31}$$

Then for all $n \times n$ matrices $X$,

$$\begin{aligned}
AX \leq B &\iff \bigwedge_{ij} (AX)_{ij} \leq B_{ij} \\
&\iff \bigwedge_{ij} (\sum_{k} A_{ik} X_{kj}) \leq B_{ij} \\
&\iff \bigwedge_{ij} \bigwedge_{k} A_{ik} X_{kj} \leq B_{ij} \\
&\iff \bigwedge_{jk} \bigwedge_{i} X_{kj} \leq A_{ik} \rightarrow B_{ij} \\
&\iff \bigwedge_{jk} X_{ik} \leq \prod_{i} A_{ik} \leq B_{ij} \\
&\iff \bigwedge_{jk} X_{kj} \leq (A \rightarrow B)_{kj}
\end{aligned}$$

$$\iff \quad X \leq A \to B$$

The property

$$XA \leq B \iff X \leq B \leftarrow A$$

follows from a symmetric argument. Thus the residuation axioms (15) and (16) are satisfied in $\mathbf{Mat}(n, \mathcal{R})$ with these definitions.

Conversely, suppose $\mathbf{Mat}(2, \mathcal{R})$ expands to a residuation algebra (the argument is similar for any $n > 2$). Then with respect to the natural order $\leq$ in $\mathcal{R}$ defined in terms of $+$, there exist maximum $x, y, z, w$ such that

$$\left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right] \left[\begin{array}{cc} x & y \\ z & w \end{array}\right] \leq \left[\begin{array}{cc} a & a \\ b & b \end{array}\right]$$

componentwise; i. e., $x, y, z, w$ are maximum such that $x \leq a$, $x \leq b$, $y \leq a$, and $y \leq b$. Then $x$ and $y$ are the greatest lower bound of $a$ and $b$ with respect to $\leq$. Since $a$ and $b$ were arbitrary, $R$ contains all binary meets, hence all nonempty finite meets. The empty meet is given by the top element $0 \to 0$.

□

Not every residuation algebra has finite meets; we construct a counterexample below. Thus the family of $n \times n$ matrices over a residuation algebra does not in general form a residuation algebra. The same is true for action algebras. Hence, in order to obtain a subvariety of action algebras closed under the formation of matrices, we will be forced to account for $\cdot$ explicitly.

**Example 3.3** We construct an action algebra that does not have finite meets. Let $\mathcal{U}$ be an arbitrary upper semilattice containing three elements $0, 1, T$ such that $0 < 1 \leq u \leq T$ for all $u \neq 0$. Let $+$ be the join operation of $\mathcal{U}$, let the distinguished elements $0, 1$ be as given, and define the remaining action algebra operations as follows:

$$ab \;=\; ba \;=\; \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0 \\ b & \text{if } a = 1 \\ T & \text{if both } a, b > 1 \end{cases}$$

$$a^* \;=\; \begin{cases} 1 & \text{if } a = 0 \text{ or } a = 1 \\ T & \text{if } a > 1 \end{cases}$$

$$a \rightarrow b \;=\; b \leftarrow a \;=\; \begin{cases} 0 & \text{if } a \not\leq b \\ 1 & \text{if } 1 < a \leq b < T \\ b & \text{if } a = 1 \\ T & \text{if } a = 0 \text{ or } b = T \end{cases}$$

It is straightforward to check that the resulting structure is an action algebra. Moreover, $\mathcal{U}$ can certainly be chosen without finite meets; for example, let $\mathcal{U}$ consist of the natural numbers, two incomparable elements above the nature numbers, and a top element. Then the two incomparable elements have no meet. □

We show now that if $\cdot$ is added to the signature of action algebras along with the lattice equations, we obtain a finitely-based subvariety **AL** of **ACT** closed under the formation of matrices. Moreover, it is the largest subvariety of **ACT** with this property, by the direction ($\leftarrow$) of Lemma 3.2.

**Theorem 3.4** *The Kleene algebra* $\mathbf{Mat}(n, \mathcal{A})$ *of* $n \times n$ *matrices over an action lattice* $\mathcal{A}$ *expands to an action lattice.*

*Proof.* As remarked previously, $\mathbf{Mat}(n, \mathcal{A})$ forms a Kleene algebra under the usual definitions of the Kleene algebra operations $+$, ;, $^*$, , 0, 1 [8]. Let the residuation operations be defined as in (30) and (31); by Lemma 3.2, $\mathbf{Mat}(n, \mathcal{A})$ is a residuation algebra. Then by Lemma 3.1, $\mathbf{Mat}(n, \mathcal{A})$ is an action algebra. Finally, let $\cdot$ be defined on matrices componentwise. Since $\mathcal{A}$ is a lattice and since $+$ and $\cdot$ are defined componentwise, $\mathbf{Mat}(n, \mathcal{A})$ is also a lattice (it is isomorphic to the direct product of $n^2$ copies of $\mathcal{A}$). Thus $\mathbf{Mat}(n, \mathcal{A})$ is an action lattice. □

All the examples given in §1, under the nature definitions of the residuation and meet operators, are easily seen to be examples of action lattices. Thus we have given a finitely-based variety **AL** that contains all these natural examples and is closed under the formation of $n \times n$ matrices.

# 4 Conclusions and Open Questions

The Kleene algebras have a natural free model on free generators $\Sigma$, namely the regular sets $\mathbf{Reg}_\Sigma$ [8]. This structure expands to an action algebra under the natural definition of the residuation operators

$$\begin{aligned} A \to B &= \{x \in \Sigma^* \mid \forall y \in A\ yx \in B\} \\ B \leftarrow A &= \{x \in \Sigma^* \mid \forall y \in A\ xy \in B\} \end{aligned}$$

and to an action lattice under the definition

$$A \cdot B = A \cap B \ .$$

Thus the axioms of action algebras and action lattices do not entail any more identities over the signature **ka** than do the Kleene algebra axioms.

One might suspect from this that $\mathbf{Reg}_\Sigma$ with residuation is the free action algebra on $\Sigma$ and $\mathbf{Reg}_\Sigma$ with residuation and meet is the free action lattice on $\Sigma$, but this is not the case: the identity

$$a \to (a + ba) = 1$$

holds in $\mathbf{Reg}_{\{a,b\}}$ but is not a consequence of the axioms of action algebras or action lattices, as can be seen by reinterpreting $a \mapsto a$ and $b \mapsto a$.

We conclude with some open questions.

1. What is the complexity of the equations theory of action algebras and action lattices? (The equational theory of Kleene algebras is *PSPACE*-complete [18].)

2. Every $^*$-continuous Kleene algebra extends universally to a closed semiring in the sense that the forgetful functor from closed semirings to $^*$-continuous Kleene algebras has a left adjoint [9]. In a sense, this says that it does not matter which of the two classes one chooses to work with. Is there such a relationship between Kleene algebras and action algebras, or between action algebras and action lattices?

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