True Concurrency can be Traced*

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Abstract

In this paper sets of labelled partial orders are employed as fundamental mathematical entities for modelling nondeterministic and concurrent processes thereby obtaining so-called noninterleaving semantics. Based on closures of sets of labelled partial orders, a simple recursive algebraic language with refinement is given denotational models fully abstract w.r.t. corresponding behaviourally motivated equivalences.

1 Introduction

During the last two decades a great deal of the research has been made in order to achieve a good understanding of the meaning of concurrent systems and how to reason about them, an understanding comparable to that of sequential programs. Whereas it is standard to take the meaning of a sequential program as a function from input to output there is no prevailing agreement on what the meaning of concurrent programs should be. As De Nicola and Hennessy reason in [DNH84] it is necessary to search for counterparts to functions when forming semantic theories for concurrency.

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In this research the algebraic framework has showed off valuable and for CCS, TCSP and other process algebras a whole spectrum of behavioural equivalences ranging from trace equivalence (in the classical language theoretic sense) [Hoa85, OH86] over failure and testing [BHR84, DNH84, OH86] to bisimulation equivalence [Mil80, Par81, Mil84] have been studied. Operationally these equivalences differ mainly in their view of the branching structure of the labelled transition system associated with processes. Through the study of degrees of branching some of the equivalences have been given fully abstract denotational models where the counterparts to input-output functions can be viewed as abstractions of computation trees (also called synchronization trees) which in turn are slightly modified unfoldings of the corresponding labelled transitions systems. However these equivalences typically have the property that they identifies concurrent and purely nondeterministic sequential processes like

$$a \parallel b \quad \text{and} \quad a \,;\, b \oplus b \,;\, a \tag{1}$$

and the semantics is often described as being interleaving.

Partly because of this intuitive unpleasant property of interleaving semantics other approaches treat concurrency as independent of nondeterminism and the processes of (1) are distinguished. Among these approaches are the so-called partial order semantics where causality, respectively concurrency, is represented by means of partial orderings of actions. I.e., alternatively to computation trees, constructions containing labelled partial orders (lpos for short) [Pra86] are proposed as counterparts to functions. These constructions are often sets of some kind of lpos and so nondeterminism cannot be discriminated in the semantics using them. But, it is possible in the denotational semantics based on a generalization of lpos, labelled event structures [Win87], where nondeterminism is dealt with by means of a conflict relation. Alternatively it could be based on a generalization of computation tress, called causal trees [DD89]. See [BC87] for a good survey on the rôle of partial orders in semantics for concurrency. Apart from step semantics, different proposals for generalizations of existing behavioural equivalences (for nondeterminism) have been made with time-based equivalence [Hen88b] and distributed bisimulation [CH88, Kie89] among the most discriminating. See also the final remarks of these papers.

Whereas the work on interleaving semantics has led to a number of e.g., axiomatisation and full abstractness results, such results are more unusual when it comes to noninterleaving semantics, [Hen88b] and [CH88, Kie89] being among the few exceptions. Motivated by this we shall in this paper explore the possibility of defining "natural" operational semantics for a algebraic process language which at the same time open up opportunities for fully abstract denotational models with lpos as main ingredient of the entities modelling processes. That is to say we are seeking behavioural equivalences where lpos come "naturally" in to the corresponding models, thereby capturing nonsequentiality.

But rather than introducing some new elaborate labelled transition system or cunning equivalence we shall stick to one of the simplest and most established equivalences, trace equivalence, and follow [Pnu85, BIM88] where increasing discriminating equivalences are obtained from the trace equivalence by considering the congruence when different combinators are added. Finding a combinator uncovering an aspect of concurrency, the congruence will be forced to take the aspect into account.

The combinator we shall study makes it possible to prescribe through a map how atomic actions within the scope of the combinator should be refined or implemented in terms of basic processes (change of atomicity). Because the refinement combinator enables "overlapping" of refined actions, the equivalences are not preserved under the new combinator and their finer associated congruences are considered. The paper is largely a continuation/ extension of [Lar88] and [NEL89] to cope with auto-parallelism and recursion.

The paper is organized as follows. At first lpos, or rather equivalence classes of lpos, operations and relations, are defined and a few properties stated. Then the process language, $BL\mathcal{R}_\Omega^{rec}$, is introduced in section 3 and in the following three sections operational and denotational semantics are given and the denotational models are proved to be fully abstract w.r.t. the corresponding operational equivalences. These three sections follow the same general line—at first the topic of the section is treated for the full language, $BL\mathcal{R}_\Omega^{rec}$, whereupon it is carried over to the finite sublanguage (without recursion constructs), $BL\mathcal{R}_\Omega$.

The operational capabilities are in section 4 given via an extended labelled transition system in the style of [Nic87, Hen88a] where an internal step is used to resolve (internal) nondeterministic choice. It turns out that a simple operational "lazy substitution" of refinements can be given by means of the internal step relation and this operational "substitution" is shown to coincide with the textual substitutions of refinements (on the finite sublanguage).

In the following section 5 motivating examples are used to get an idea of how a model for the finite sublanguage should look like—a model which build on closures of sets of lpos with the property that the lpos can reflect the "overlapping" capabilities of the refinement combinator. Based on this models for the full language are given—acquaintance with standard denotational techniques for dealing with recursion as presented in [Hen88a] is assumed.

In section 6 the full abstractness results of $BL\mathcal{R}_\Omega^{rec}$ are lifted from $BL\mathcal{R}_\Omega$ via the notion of algebraicity. In this course a new criterion for algebraicity of precongruences—a language being expressive w.r.t. a preorder—turn out to be very useful.

Finally we in section 7 conclude the paper and give a brief discussion of possible extensions.

2 Pomsets

As it appears from the introduction, the concept of labelled partial orders will be central for the models we are going to present. The basic idea is that lpos will represent individual behaviours of processes. In particular we will look at pomsets. We shall use the interpretation and graphical representation of pomsets from [Gra81]. That is

(2)
$$\begin{tikzcd}[row sep=tiny] & b \arrow[r] \arrow[ddr] & d \\ a \arrow[ur] \arrow[dr] & & \\ & c \arrow[r] \arrow[uur] & d \end{tikzcd}$$

is used to represent a behaviour of a process with five action occurrences, where the d occurrences are causally mutual independent, but dependent on the others, the b occurrence is causally dependent on a, but not on c, a.s.o.

Pomsets are usually defined as proper classes of isomorphic lpos ([Gis88, Pra86]). However by introducing lpos on basis of an appropriate ground set, we shall in this section see how pomsets, as well as their operations, partial orders and related notions, smoothly can be defined and reasoned about entirely within the set-theory. In addition with alternative characterizations of the partial orders isomorphy considerations are rarely necessary.

Basic Definitions

We will look at lpos, over an action alphabet Δ—a countably infinite alphabet (fixed through out the rest of the paper). We assume Δ to be disjoint from $I\!N$—the nonnegative integers. Furthermore we assume a fixed ground set closed under pairing and containing $I\!N$ and Δ.

Definition 2.1 *Lpos and Pomsets*

An lpo, p, is a tuple $\langle X_p, \leq_p, \ell_p \rangle$, where X_p is a subset of the ground set together with a partial order (reflexive, transitive and antisymmetric), $\leq_p$, and a labelling function $\ell_p : X_p \to \Delta$.

A *morphism* $f : p \to q$ *of lpos* is a function $f : X_p \to X_q$ such that

$$\begin{array}{ll} x \leq_p y \Rightarrow f(x) \leq_q f(y) & \text{for all } x, y \in X_p \\ \ell_p(x) = \ell_q(f(x)) & \text{for all } x \in X_p \end{array}$$

f is furthermore an *isomorphism of lpos* if $f : X_p \to X_q$ is a bijection and f^{-1} also is a morphisms of lpos. When such an isomorphism exists we write $p \cong q$.

A *pomset* is an equivalence class of an lpo p under $\cong$ and is denoted $[p]$; p is called a *representative* of the equivalence class. Whenever an lpo is denoted by a single symbol, p, we define for convenience $\mathbf{p}$ to be $[p]$. The set of pomsets is denoted $\mathbf{P}$.

A pomset $\mathbf{p}$ is contained in pomset $\mathbf{q}$ if a representative of $\mathbf{p}$ can be embedded in a representative of $\mathbf{q}$. Formally: $\mathbf{p}$ is a *subpomset* of $\mathbf{q}$, written $\mathbf{p} \hookrightarrow \mathbf{q}$, *iff* $\exists Y.\, \mathbf{p} = [q|_Y]$, where for a set, Y, the *restriction of* p *to* Y, $p|_Y$, is the lpo $\langle X_p|_Y, \leq_p|_{Y^2}, \ell_p|_Y \rangle$.
For $x \in X_p$ we sometimes (ambiguously) abbreviate $p|_{\{x\}}$ by x. □

Observe that $\mathbf{p}$ as well as $\mathbf{P}$ indeed are sets (follows from the ground set being one). The notion of subpomset is defined by means of a single representative so the reader should check that the definition is independent of the representative used in the definition.

For a pomset $\mathbf{p}$ and a set of pomsets Q we denote by $Q(\mathbf{p})$ those pomsets of Q which are contained in $\mathbf{p}$, i.e., $Q(\mathbf{p}) = \{\mathbf{q} \in Q \mid \mathbf{q} \hookrightarrow \mathbf{p}\}$.

Example: If $\mathbf{p}$ is the pomset represented in (2) then e.g.,

$$\mathbf{p} \hookrightarrow \mathbf{p}, \quad a \rightarrow c \rightarrow d \hookrightarrow \mathbf{p}, \quad a \rightarrow d \hookrightarrow \mathbf{p}$$

and

$$\left\{ c, a \rightarrow d, \begin{smallmatrix} b \searrow \\ c \nearrow \end{smallmatrix} d \right\} \subseteq \mathbf{P}(\mathbf{p})$$

Notice that we use $x, y, \ldots$ to range over elements of X_p. x and y are said to be concurrent/causally independent in an lpo p,

$$x \; co_p \; y \; \textit{iff} \; x \not\leq_p y \; \textit{and} \; y \not\leq_p x$$

With this definition co_p is *not* reflexive! We say that $Y \subseteq X_p$ is a co_p-*set* if all the elements of Y are mutual concurrent in p, i.e., if $co_p|_{Y^2} = (Y \times Y) \setminus \{\langle y, y \rangle \mid y \in Y\}$.

ε is used to denote the empty lpo, $\langle \emptyset, \emptyset, \emptyset \rangle$. We overload notation and use ε $[\langle \emptyset, \emptyset, \emptyset \rangle]$ and the singleton pomset $[\langle \{a\}, \{\langle a, a \rangle\}, a \mapsto a \rangle]$ respectively.

It will not be necessary to deal with infinite pomsets in the following so we will throughout the rest of this paper assume *pomsets* to be *finite*, i.e., we shall only consider pomsets $\mathbf{p}$ where X_p is finite.

Having restricted ourselves to finite pomsets we can now for a pomset associate a unique multiplicity function over Δ which for each action tells how many elements in the pomsets that are labelled with this action. The (finite) multiplicity function, $m_{\mathbf{p}}$, of a pomset $\mathbf{p}$ is simply $m_p : \Delta \longrightarrow I\!N$, where $\forall a \in \Delta. \, m_p(a) = |\{x \in X_p \mid \ell_p(x) = a\}|$. Multiplicity functions are partially ordered pointwise and clearly every finite set of multiplicity functions has a lub (least upper bound) which is finite.

Definition 2.2 *Pomset Property*

An lpo property, P_*, is $\cong$-invariant if it is preserved under lpo isomorphism, i.e., $p \cong q$ and $P_*(p)$ implies $P_*(q)$.
P_* is a *pomset property* if it is induced from a $\cong$-invariant lpo property, Q_*, as follows: $P_*(\mathbf{p})$ *iff* $Q_*(p)$ $\square$

In the sequel we shall make no distinction between a pomset property and the lpo property it is induced from. An example of a pomset property, P_*, is where $P_*(\mathbf{p})$ demands $\leq_p$ to satisfy the trichotomy law: $\forall x, y \in X_p.\ x \leq_p y$ *or* $y \leq_p x$, i.e., that $\leq_p$ shall be total. The set of pomsets having this property is denoted W (words) and we write the property as P_{w}. Pomsets of W are by Gischer [Gis88] alternatively called tomsets. We shall often write w for $\mathbf{w} \in W$, because of the one to one correspondence between Δ^* and W (see [Sta81]).

Operations on Pomsets

Pomsets have been equipped with a variety of operations ([Gra81, Sta81, Pra86, Gis88]). In this paper we need only a few of these. The following two are both natural generalizations of concatenation of words: sequential and parallel composition.

Definition 2.3 *Sequential and Parallel Composition of Pomsets*

For two pomsets, $\mathbf{p}_0$ and $\mathbf{p}_1$, their sequential/ parallel composition, $\mathbf{p}_0 \cdot \mathbf{p}_1$/ $\mathbf{p}_0 \times \mathbf{p}_1$, is obtained (informally) by taking their disjoint union (component wise), and in the case of sequential composition making all elements of $\mathbf{p}_1$ causally dependent on all elements of $\mathbf{p}_0$. Formally the operations are defined via the corresponding operations on the representatives:
$p_0 \cdot p_1 = \langle X, \leq, \ell \rangle$ and $p_0 \times p_1 = \langle X, \leq', \ell \rangle$, where

$$
\begin{array}{rl}
X & \text{is the set } \{0\} \times X_{p_0} \cup \{1\} \times X_{p_1} \\
\leq & \text{is the partial order defined by} \\
 & \langle i, x \rangle \leq \langle j, y \rangle \ \textit{iff} \quad i = j \ \textit{and} \ x \leq_{p_i} y \\
 & \qquad\qquad\qquad\qquad \textit{or} \ i = 0, j = 1 \\
\leq' & \text{is the partial order defined by} \\
 & \langle i, x \rangle \leq \langle j, y \rangle \ \textit{iff} \quad i = j \ \textit{and} \ x \leq_{p_i} y \\
\ell & \text{is the function } \langle i, x \rangle \mapsto \ell_{p_i}(x)
\end{array}
$$

(So $\mathbf{p}_0 \cdot \mathbf{p}_1 = [p_0 \cdot p_1]$ and $\mathbf{p}_0 \times \mathbf{p}_1 = [p_0 \times p_1]$). $\square$

Example:
$$
\begin{tikzcd}[row sep=tiny, column sep=small]
 & b \\
a \arrow[ur] \arrow[dr] & \\
 & a
\end{tikzcd}
\cdot c \rightarrow d =
\begin{tikzcd}[row sep=tiny, column sep=small]
 & b \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & c \arrow[r] & d \\
 & a \arrow[ur] &
\end{tikzcd}
\text{ and }
\begin{tikzcd}[row sep=tiny, column sep=small]
 & b \\
a \arrow[ur] \arrow[dr] & \\
 & a
\end{tikzcd}
\times c \rightarrow d =
\begin{tikzcd}[row sep=tiny, column sep=small]
 & b \\
a \arrow[ur] \arrow[dr] & \\
 & a \\
c \arrow[r] & d
\end{tikzcd}
$$

Sets of pomsets and operators on them are used extensively in the models we shall present, so we briefly treat them here. The two operations on pomsets $\cdot$ and $\times$ generalize to sets in the natural way, e.g., $P \cdot Q = \{\mathbf{p} \cdot \mathbf{q} \mid \mathbf{p} \in P, \mathbf{q} \in Q\}$. We shall also use $\cup$, the normal set union, as operator on sets of pomsets. In the following $\mathcal{P}(\_)$ will denote the powerset operator.

The next operator refines the different elements of a pomset into different pomsets (a formalization of the concept of "change of atomicity").

Example: Consider the pomset $a\!\begin{array}{l}\nearrow b\\ \searrow b\end{array}$. Suppose we would like to refine the upper occurrence of b to $\begin{array}{r} d \searrow \\ e \nearrow \end{array} d$, the lower to $c \rightarrow a$ and the a occurrence to $\begin{array}{r} b \searrow \\ a \nearrow \end{array} a$. Call this refinement π and the associated operator $<\pi>$—then we would expect:

$$a\!\begin{array}{l}\nearrow b\\ \searrow b\end{array}\!<\pi> \;=\; \begin{tikzcd}[row sep=small, column sep=small] & & d \arrow[r] & d \\ b \arrow[r] & a \arrow[ur] \arrow[r] \arrow[dr] & e \arrow[ur] & \\ a \arrow[ur] & & c \arrow[r] & a \end{tikzcd}$$

Actually it does not make sense to talk about the upper, lower, etc. occurrence of b in a pomset, but for a particular representative each individual element can be replaced by "its own" pomset (representative) thus obtaining the representative of, a new pomset.

The construction is not as simple as the others and we need to introduce some additional notions.

Definition 2.4 *Particular Refinement*
Let p be an lpo. A *particular refinement* (abbreviated p. ref.) for p is a mapping, π_p, which for each element of X_p associates an lpo. For such a mapping we can construct a new lpo $p<\pi_p> = \langle X, \leq, \ell \rangle$, where

$$\begin{array}{rl} X & \text{is the set } \{\langle x, x' \rangle \mid x \in X_p, x' \in X_{\pi_p(x)}\} \\ \leq & \text{is the partial order defined by} \\ & \langle x, x' \rangle \leq \langle y, y' \rangle \ \textit{iff} \ \ x \leq_p y \ \textit{and} \\ & \qquad\qquad\qquad\qquad\quad x = y \Rightarrow x' \leq_{\pi_p(x)} y' \\ \ell & \text{is the function } \langle x, x' \rangle \mapsto \ell_{\pi_p(x)}(x') \end{array}$$

Notice that $p<\pi_p>$ is a finite lpo. It is not hard to see that sequential (parallel) composition can be derived from particular refinements of an ordered (unordered) two element pomset; see [Gis84, Eng89] for the details. That is to say with the words of Gischer [Gis88] $\cdot$ and $\times$ are pomset definable operations on pomsets. Gischer actually make a kind of particular refinement into an operation on pomsets (called substitution) but it would not allow the type of refinements we shall need.

The refinement operator for pomsets can defined using the particular refinement construction for lpos.

> **Definition 2.5** *Refinement of Pomsets*
> A $\mathcal{P}(\mathbf{P})$-*refinement* is a mapping $\varrho : \Delta \to \mathcal{P}(\mathbf{P})$. We say that a $\mathcal{P}(\mathbf{P})$-refinement, ϱ, is ε-*free* *iff* $\forall a \in \Delta.\, \varepsilon \notin \varrho(a)$ and ϱ is *image finite* if $\varrho(a)$ is finite for every $a \in \Delta$.
> A p. ref. π_p for an lpo p is *consistent* with a $\mathcal{P}(\mathbf{P})$-refinement ϱ *iff*
>
> $$\forall x \in X_p.\, [\pi_p(x)] \in \varrho(\ell_p(x))$$
>
> The mapping associated with ϱ is now defined as $<\!\varrho\!> : \mathbf{P} \to \mathcal{P}(\mathbf{P})$ with $\mathbf{p}<\!\varrho\!> = \{[p<\!\pi_p\!>] \mid \pi_p \text{ is a } \varrho\text{-consistent p. ref. for } p\}$ and generalized to sets of pomsets by $P<\!\varrho\!> = \bigcup_{\mathbf{p} \in P} \mathbf{p}<\!\varrho\!>$.

In general $\mathbf{p}<\!\varrho\!>$ is a finite set of pomsets when ϱ is image finite because we only work with finite pomsets.

Example: Consider the pomset of the last example and suppose ϱ is a $\mathcal{P}(\mathbf{P})$-refinement such that $a \mapsto \left\{\begin{tikzcd}[row sep=tiny, column sep=small] b \arrow[dr] & \\ & a \\ a \arrow[ur] & \end{tikzcd}\right\}$ and $b \mapsto \left\{ c \to a,\ \begin{tikzcd}[row sep=tiny, column sep=small] d \arrow[dr] & \\ & d \\ e \arrow[ur] & \end{tikzcd}\right\}$. Then

$$\begin{tikzcd}[row sep=tiny, column sep=small] & b \\ a \arrow[ur] \arrow[dr] & \\ & b \end{tikzcd}<\!\varrho\!> \;=\; \left\{ \begin{tikzcd}[row sep=tiny, column sep=small] & & c \arrow[r] & a \\ b \arrow[r] & a \arrow[ur] \arrow[dr] & & \\ a \arrow[ur] & & c \arrow[r] & a \end{tikzcd},\ \begin{tikzcd}[row sep=tiny, column sep=small] & & d \arrow[r] & d \\ b \arrow[r] & a \arrow[ur] \arrow[r] \arrow[dr] & e \arrow[ur] & \\ a \arrow[ur] & & c \arrow[r] & a \end{tikzcd},\ \begin{tikzcd}[row sep=tiny, column sep=small] & & d \arrow[dr] & \\ & & & d \\ b \arrow[r] & a \arrow[uur] \arrow[r] \arrow[dr] \arrow[ddr] & e \arrow[ur] & \\ a \arrow[ur] & & d \arrow[dr] & \\ & & & d \\ & & e \arrow[ur] & \end{tikzcd} \right\}$$

The difference between our refinement operation and Gischers substitution can be illustrated by this example. The result of Gischers substitution would be without the pomset in the "middle".

The operations enjoy a number of properties of which some are:

Proposition 2.6

- $\cdot$, $\times$ and $\cup$ are associative with neutral elements $\{\varepsilon\}$, $\{\varepsilon\}$ and $\emptyset$ respectively
- $\times$ and $\cup$ are commutative
- $\{\varepsilon\}{<}\varrho{>} = \{\varepsilon\}$, $\{a\}{<}\varrho{>} = \varrho(a)$ and ${<}\varrho{>}$ distributes over $\cdot$, $\times$ and $\cup$
- $\cdot$, $\cup$, $\times$ and ${<}\varrho{>}$ are $\subseteq$-monotone in all their arguments

Two Partial Orders on Pomsets

The first relation on pomsets we are going to present is used to compare the "concurrency" of two pomsets.

Definition 2.7 $\preceq$*-ordering on Pomsets*

A pomset $\mathbf{p}$ is smoother than [Gra81]/ subsumed by [Gis88]/less nonsequential than the pomset $\mathbf{q}$, $\mathbf{p} \preceq \mathbf{q}$, if $\mathbf{p}$ except perhaps for some additional order on elements equals $\mathbf{q}$. Formally this partial order on pomsets is induced from the corresponding lpo preorder by: $\mathbf{p} \preceq \mathbf{q}$ *iff* $p \preceq q$, where

> $p \preceq q$ *iff* there exists bijective function $X_q \to X_p$which is a morphism of lpos.

The $\preceq$-downwards closure of a pomset $\mathbf{p}$, $\{\mathbf{p}' \in \mathbf{P} \mid \mathbf{p}' \preceq \mathbf{p}\}$, is denoted $\delta(\mathbf{p})$. □

Notice that for lpos p and q, $p \preceq q$ does *not* imply $p \cong q$.

Example: $a \to b \to c \;\preceq\; \begin{array}{l} a \searrow \\ \quad\;\; c \\ b \nearrow \end{array}$ and $\begin{array}{ccc} a & \to & b \\ & \times & \\ c & \to & d \end{array} \;\preceq\; \begin{array}{ccc} a & \to & b \\ & \searrow & \\ c & \to & d \end{array} \;\preceq\; \begin{array}{ccc} a & \to & b \\ c & \to & d \end{array}$

If P_* is a property of pomsets then $\delta_*(\mathbf{p})$ will be a shorthand for the semi $\preceq$-downwards closure $\{\mathbf{p}' \in \mathbf{P} \mid \mathbf{p}' \preceq \mathbf{p} \text{ and } P_*(\mathbf{p}')\}$. E.g., $\delta_{\mathsf{w}}(\mathbf{p}) = \{\mathbf{p}' \in \mathbf{P} \mid \mathbf{p}' \preceq \mathbf{p} \text{ and } P_{\mathsf{w}}(\mathbf{p}')\} = \{\mathbf{p}' \in W \mid \mathbf{p}' \preceq \mathbf{p}\}$. Though we might have $\mathbf{p} \notin \delta_*(\mathbf{p})$ for some pomset property P_*, we call it the δ_*-closure. δ_* also generalize to sets: $\delta_*(Q) = \bigcup_{\mathbf{q} \in Q} \delta_*(\mathbf{q})$ and is $\subseteq$-monotone.

The following alternative characterization of $\preceq$ is often more convenient to use.

Proposition 2.8 For pomsets $\mathbf{p}$ and $\mathbf{q}$ we have:

$$\begin{aligned}\mathbf{p} \preceq \mathbf{q} \ &\textit{iff} \ \ p = \langle X_{q'}, \leq_p, \ell_{q'} \rangle \text{ and } \leq_p \supseteq \leq_{q'} \text{ for some } q' \in \mathbf{q} \\ &\textit{iff} \ \ \langle X_{p'}, \leq_q, \ell_{p'} \rangle = q \text{ and } \leq_{p'} \supseteq \leq_q \text{ for some } p' \in \mathbf{p}\end{aligned}$$

Extend $\preceq$ to sets by: $P \preceq Q$ *iff* $\forall \mathbf{p} \in P, \mathbf{q} \in Q.\, \mathbf{p} \preceq \mathbf{q}$; and to refinements by: $\varrho \preceq \varrho'$ *iff* $\forall a \in \Delta.\, \varrho(a) \preceq \varrho'(a)$. From the last proposition and the refinement construction it is not hard to see that $P \preceq Q$ and $\varrho \preceq \varrho'$ implies $P{<}\varrho{>} \preceq Q{<}\varrho'{>}$. Since $\cdot$ and $\times$ can be obtained as appropriate refinements of two element pomsets we then have

(3) $\qquad \cdot$ and $\times$ are $\preceq$-monotone in their left and right arguments

and by refinements of the pomsets in the last example also

(4) $$(\mathbf{p} \times \mathbf{q}) \cdot (\mathbf{p}' \times \mathbf{q}') \preceq (\mathbf{p} \cdot \mathbf{p}') \times (\mathbf{q} \cdot \mathbf{q}').$$

We now turn to the second partial order on pomsets.

Definition 2.9 $\sqsubseteq$*-ordering on Pomsets*

$\mathbf{p}$ is a *prefix* of $\mathbf{q}$, $\mathbf{p} \sqsubseteq \mathbf{q}$, if $\mathbf{p}$ is a subpomset of $\mathbf{q}$ and the elements of $\mathbf{p}$ only dominates the elements of $\mathbf{p}$ in $\mathbf{q}$. Formally the corresponding lpo preorder, $\sqsubseteq$, is defined $p \sqsubseteq q$ *iff* there exists a $\leq_q$-downwards closed set Y such that p is isomorphic to the restriction of q to Y:

$$p \sqsubseteq q \ \textit{iff} \ \ \exists Y.\, p \cong q|_Y \ \textit{and} \ \{x \in X_q \mid \exists y \in Y.\, x \leq_q y\} \subseteq Y$$

The $\sqsubseteq$-downwards closure of a pomset $\mathbf{p}$ is: $\pi(\mathbf{p}) = \{\mathbf{p}' \in \mathbf{P} \mid \mathbf{p}' \sqsubseteq \mathbf{p}\}$.
$\square$

That $\mathbf{p} \sqsubseteq \mathbf{q}$ implies $\mathbf{p} \hookrightarrow \mathbf{q}$ follows from $p \cong q|_Y$. Observe that $\{x \in X_q \mid \exists y \in Y.\, x \leq_q y\} \subseteq Y$ just is a formalization of: Y is $\leq_q$-downwards closed.

Example: $\begin{tikzcd}[row sep=tiny] & b \\ a \arrow[ur] \arrow[dr] & \\ & c \end{tikzcd} \ \sqsubseteq \ \begin{tikzcd}[row sep=tiny] & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}$, but $a \to b \to d \ \not\sqsubseteq \ \begin{tikzcd}[row sep=tiny] & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}$

As for the partial order $\preceq$ there is an alternative characterization of $\sqsubseteq$:

Proposition 2.10 For pomsets $\mathbf{p}$ and $\mathbf{q}$ we have:

$\mathbf{p} \sqsubseteq \mathbf{q}$
iff $p' = q|_{X_{p'}}$ for some $p' \in \mathbf{p}$ with $\{x \in X_q \mid \exists y \in X_{p'}.\, x \leq_q y\} \subseteq X_{p'}$
iff $p = q'|_{X_p}$ for some $q' \in \mathbf{q}$ with $\{x \in X_{q'} \mid \exists y \in X_p.\, x \leq_{q'} y\} \subseteq X_p$

The following proposition shade some light over over $\sqsubseteq$ and its relation to $\preceq$.

Proposition 2.11 Given pomsets $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. Then

a) $\mathbf{p} \sqsubseteq \mathbf{q} \times \mathbf{r}$ *iff* $\exists \mathbf{q}' \sqsubseteq \mathbf{q}, \mathbf{r}' \sqsubseteq \mathbf{r}.\, \mathbf{p} = \mathbf{q}' \times \mathbf{r}'$

b) $\mathbf{p} \sqsubseteq \mathbf{q} \cdot \mathbf{r}$ *implies* $\mathbf{p} \sqsubseteq \mathbf{q}$ *or* $\exists \mathbf{r}' \sqsubseteq \mathbf{r}.\, \mathbf{p} = \mathbf{q} \cdot \mathbf{r}'$

c) $\mathbf{p} \sqsubseteq \mathbf{q}$ *implies* $\exists \mathbf{r}.\, \mathbf{p} \cdot \mathbf{r} \preceq \mathbf{q}$

d) $\exists \mathbf{r}.\, \mathbf{p} \preceq \mathbf{r} \sqsubseteq \mathbf{q}$ *iff* $\exists \mathbf{s}.\, \mathbf{p} \sqsubseteq \mathbf{s} \preceq \mathbf{q}$

Proof a) – c) are proven using the alternative characterizations of the two preorders.

In [Pra86, page 49] Pratt outlines a proof of d). He defines prefix in another, but equivalent way: $\mathbf{p}$ is a prefix of $\mathbf{q}$ if $\exists Y.\, p \cong q|_Y$ and $X_q \setminus Y$ is $\leq_q$-upwards closed. A more formalized proof is:

only if: Assume $\mathbf{p} \preceq \mathbf{r} \sqsubseteq \mathbf{q}$. By c) we know there is a pomset $\mathbf{r}'$ such that $\mathbf{r} \cdot \mathbf{r}' \preceq \mathbf{q}$. From $\mathbf{p} \preceq \mathbf{r}$ and $\preceq$-monotonicity of $\cdot$ then $\mathbf{p} \cdot \mathbf{r}' \preceq \mathbf{q}$. But $\mathbf{p} \sqsubseteq \mathbf{p} \cdot \mathbf{r}'$ so we can just choose $\mathbf{s} = \mathbf{p} \cdot \mathbf{r}'$.

if: Suppose $\mathbf{p} \sqsubseteq \mathbf{s} \preceq \mathbf{q}$. Then by the alternative characterizations of $\preceq$ and $\sqsubseteq$ there are representatives p' and q' of $\mathbf{p}$ and $\mathbf{q}$ respectively such that $p' = s|_{X_{p'}}$, $X_{p'}$ is $\leq_s$-downwards closed and $q' = \langle X_s, \leq_{q'}, \ell_s \rangle$ with $\leq_s \supseteq \leq_{q'}$. Define r to be $q'|_{X_{p'}}$. Then r is an lpo and to see that $X_{p'}$ is $\leq_r$-downwards closed assume $x \leq_r y \in X_{p'}$. Then $x \leq_{q'} y$ and from $\leq_{q'} \subseteq \leq_s$ also $x \leq_s y$. $x \in X_{p'}$ follows now from the $\leq_s$-downwards closure of $X_{p'}$. Hence $r \sqsubseteq q'$. We also have $r = \langle X_{p'}, \leq_{q'}|_{X_{p'}{}^2}, \ell_{p'} \rangle$, so from $\leq_{q'} \subseteq \leq_s$ then $\leq_r = \leq_q|_{X_{p'}{}^2} \subseteq \leq_s|_{X_{p'}{}^2} = \leq_{p'}$. Thus $p' \preceq r \sqsubseteq q'$ and $\mathbf{p} = \mathbf{p}' \preceq \mathbf{r} \sqsubseteq \mathbf{q}' = \mathbf{q}$. $\square$

Two Types of Pomset Properties

The first type of pomset properties we shall consider is those where the property of a pomset is inherited to all subpomsets. Following [BC87] we call such a property hereditary and define it:

Definition 2.12 *Hereditary Pomset Properties*

A pomset property, P_*, is *hereditary*, *iff*

$$\forall \mathbf{p} \in \mathbf{P}.\ P_*(\mathbf{p}), \mathbf{q} \hookrightarrow \mathbf{p} \Rightarrow P_*(\mathbf{q}) \qquad \Box$$

The P_{w}-property is an example of a hereditary pomset property. To give an example of the consequences of property being hereditary we state:

Proposition 2.13 Let P_* be a hereditary pomset property. Then

$$\mathbf{q} \preceq \mathbf{p}_0 \cdot \mathbf{p}_1, P_*(\mathbf{q}) \Rightarrow \exists \mathbf{q}_0, \mathbf{q}_1.\ \mathbf{q} = \mathbf{q}_0 \cdot \mathbf{q}_1 \ \textit{and}\ \mathbf{q}_i \preceq \mathbf{p}_i, P_*(\mathbf{q}_i) \text{ for } i = 0, 1$$

Of course there is a similar proposition for parallel composition.

We shall now deal with a certain type of pomset properties where one can deduce/ synthesize the property for the sequential composition of two pomsets if they both have the property.

Definition 2.14 *Dot Synthesizable Pomset Properties*

A pomset property, P_*, is *dot synthesizable*, *iff*

$$\forall \mathbf{p}, \mathbf{q} \in \mathbf{P}.\ P_*(\mathbf{p}) \ \textit{and}\ P_*(\mathbf{q}) \ \textit{implies}\ P_*(\mathbf{p} \cdot \mathbf{q}) \qquad (5) \qquad \Box$$

The P_{w}-property is also an example of a dot synthesizable pomset property.

Of course we cannot be sure that $\delta_*(\mathbf{p})$ is nonempty no matter whether we have to do with hereditary or dot synthesizable pomset properties. Take for instance the pomset property which is not fulfilled by any pomset. However it can be shown that if P_* is a dot synthesizable pomset property holding for the empty and singleton pomsets then $\delta_*(\mathbf{p})$ is nonempty for every pomset $\mathbf{p}$. For example this is the case for P_{w} and we conclude $\delta_{\mathsf{w}}(\mathbf{p}) \neq \emptyset$ for every pomset $\mathbf{p}$.

Proposition 2.15 If P_* hereditary and dot synthesizable then

a) $\delta_*(\mathbf{p}_0 \cdot \mathbf{p}_1) = \delta_*(\mathbf{p}_0) \cdot \delta_*(\mathbf{p}_1)$

b) $\delta_*(\mathbf{p}_0 \times \mathbf{p}_1) = \delta_*(\delta_*(\mathbf{p}_0) \times \delta_*(\mathbf{p}_1))$

c) $\delta_*\pi(\mathbf{p}) = \pi\delta_*(\mathbf{p})$, provided P_* holds for ε and the singleton pomsets.

Notice that since $\delta = \delta_{true}$ c) clearly is an extension of [Pra86].

Proof We just prove a) and c) since b) follows similar as a).

a) $\subseteq$: Suppose $\mathbf{q} \in \delta_*(\mathbf{p}_0 \cdot \mathbf{p}_1)$—i.e., $\mathbf{q} \preceq \mathbf{p}_0 \cdot \mathbf{p}_1$ and $P_*(\mathbf{q})$. Then by proposition 2.13 there exists pomsets $\mathbf{q}_0$ and $\mathbf{q}_1$ such that $\mathbf{q} = \mathbf{q}_0 \cdot \mathbf{q}_1$ and $\mathbf{q}_i \preceq \mathbf{p}_i, P_*(\mathbf{q}_i)$ for $i = 0, 1$. This implies $\mathbf{q}_i \in \delta_*(\mathbf{p}_i)$ for $i = 0, 1$ and $\mathbf{q} = \mathbf{q}_0 \cdot \mathbf{q}_1 \in \delta_*(\mathbf{p}_0) \cdot \delta_*(\mathbf{p}_1)$.

$\supseteq$: Given $\mathbf{q} \in \delta_*(\mathbf{p}_0) \cdot \delta_*(\mathbf{p}_1)$. Then $\mathbf{q} = \mathbf{p}'_0 \cdot \mathbf{p}'_1$ for some $\mathbf{p}'_i \in \delta_*(\mathbf{p}_i)$ and $i = 0, 1$. This implies $P_*(\mathbf{p}'_i)$ and $\mathbf{p}'_i \preceq \mathbf{p}_i$ for $i = 0, 1$, so as a consequence of the $\preceq$-monotonicity of $\cdot$ then $\mathbf{p}'_0 \cdot \mathbf{p}'_1 \preceq \mathbf{p}_0 \cdot \mathbf{p}_1$, and $P_*(\mathbf{p}'_0 \cdot \mathbf{p}'_1)$ since P_* is dot synthesizable. Hence $\mathbf{q} \in \delta_*(\mathbf{p}_0 \cdot \mathbf{p}_1)$.

c) $\subseteq$: Suppose $\mathbf{q} \in \delta_*\pi(\mathbf{p})$. Then $P_*(\mathbf{q})$ and there is a pomset $\mathbf{r}$ with $\mathbf{q} \preceq \mathbf{r} \sqsubseteq \mathbf{p}$. By c) of proposition 2.11 there is a $\mathbf{r}'$ such that $\mathbf{q} \cdot \mathbf{r}' \preceq \mathbf{p}$. From the proviso it then follows that there is a $\mathbf{p}' \in \delta_*(\mathbf{r}')$. Hence $P_*(\mathbf{p}')$ and by the $\preceq$-monotonicity of $\cdot$ also $\mathbf{q} \cdot \mathbf{p}' \preceq \mathbf{q} \cdot \mathbf{r}' \preceq \mathbf{p}$. $P_*(\mathbf{q} \cdot \mathbf{p}')$ follows from $P_*(\mathbf{q})$ and $P_*(\mathbf{p}')$. Because $\mathbf{q} \sqsubseteq \mathbf{q} \cdot \mathbf{p}'$ we actually have $\mathbf{q} \in \pi\delta_*(\mathbf{p})$.

$\supseteq$: Let a $\mathbf{q} \in \pi\delta_*(\mathbf{p})$ be given. This means there is a $\mathbf{s}$ such that $P_*(\mathbf{s})$ and $\mathbf{q} \sqsubseteq \mathbf{s} \preceq \mathbf{p}$. $\mathbf{q} \sqsubseteq \mathbf{s}$ implies $\mathbf{q} \hookrightarrow \mathbf{s}$, so because P_* is hereditary we also have $P_*(\mathbf{q})$. By proposition 2.11.d) there is a pomset $\mathbf{r}$ with $\mathbf{q} \preceq \mathbf{r} \sqsubseteq \mathbf{p}$. Hence $\mathbf{q} \in \delta_*\pi(\mathbf{p})$. $\square$

3 A Concurrent Process Language with Action Refinement

The process language, $BL\mathcal{R}^{rec}_{\Omega}$, we shall use will be an extension of a very basic language over the abstract set of action symbols, Δ, containing a combinator for internal nondeterminism beside combinators for sequencing and parallelism with auto-parallelism (but without communication).

$BL\mathcal{R}_\Omega^{rec}$ will also contain refinement combinators, which for each atomic action states how it should be implemented by a basic process expression. So intuitively such a process should behave as if the refinements were substituted in advance.

Finally $BL\mathcal{R}_\Omega^{rec}$ has the usual constructors for recursion, $rec\,x.\,\_$, where x is a member of a fixed countable infinite set of variables, X.

So $BL\mathcal{R}_\Omega^{rec}$ consists of the closed expressions of $BL\mathcal{R}_\Omega^{rec}(X)$, which in turn is the least set closed under expressions of the form (Δ) – (rec):

(Δ)	a	individual process labelled $a \in \Delta$
$(;)$	$E_0\,;E_1$	sequential composition of E_0 and E_1
$(\oplus)$	$E_0 \oplus E_1$	internal nondeterministic composition of E_0 and E_1
$(\parallel)$	$E_0 \parallel E_1$	parallel composition of E_0 and E_1
$(\mathcal{R})$	$E[\varrho]$	action refinement of E according to BL-refinement ϱ
(Ω)	Ω	the completely undefined process
(X)	x	process variable $x \in X$
(rec)	$rec\,x.\,E$	the process, E, recursive in $x \in X$

where a BL-*refinement* is defined to be a mapping $\varrho : \Delta \longrightarrow BL$ and BL is the least set closed under expressions of the form (Δ) – $(\parallel)$ above; e.g., if $E_0, E_1 \in BL$ then $E_0\,;E_1 \in BL$.

It will be convenient to define different sublanguages of $BL\mathcal{R}_\Omega^{rec}$; $BL\mathcal{R}$ is obtained from (Δ) – $(\mathcal{R})$, BL_Ω from (Δ) – $(\parallel)$, (Ω) etc.. These will be used in open versions too; e.g., $BL_\Omega(X)$ is obtained from (Δ) – $(\parallel)$, (Ω), (X).

It will turn out that the binary combinators are associative, a fact we shall make use of together with an assumption of the combinator precedence: unary combinators, ;, $\parallel$, $\oplus$—unary binding strongest.

4 Operational Semantics

Central to our idea of process behaviour will be the notion of a process performing a sequence of actions. What actions a process can perform will be given by an action relation, $\Rightarrow$, holding through an $a \in \Delta$ between configurations, with each $BL\mathcal{R}_\Omega^{rec}$-expression being a possible start configuration. Configurations are expressions from $CL\mathcal{R}_\Omega^{rec}$, which is almost like $BL\mathcal{R}_{\Omega}^{rec}$ with Δ extended with $\dagger$ (a symbol distinct from those of Δ). Intuitively $\dagger$ represents the extinct action and thereby indicates how far control has reached. Formally $CL\mathcal{R}_{\Omega}^{rec}$ is (for technical reasons) defined as the closed expressions of $CL\mathcal{R}_{\Omega}^{rec}(X)$ which is the least set, C, satisfying:

$$\begin{array}{rl} \dagger \in C & \\ BL\mathcal{R}_{\Omega}^{rec}(X) \subseteq C & \\ E_0 \,;\, E_1 \in C & \text{if } E_0 \in C \text{ and } E_1 \in BL\mathcal{R}_{\Omega}^{rec}(X) \\ E_0 \parallel E_1 \in C & \text{if } E_0, E_1 \in C \end{array}$$

$CL\mathcal{R}_{\Omega}(X)$, CL etc. will be considered as $CL\mathcal{R}_{\Omega}^{rec}(X)$ restricted to configurations corresponding to the appropriate sublanguages $BL\mathcal{R}_{\Omega}(X)$, BL etc. E.g., $a \parallel (\dagger \,;\, b) \in CL$ but $\dagger \oplus a \notin CL$ and $a \,;\, (\dagger \,;\, b) \notin CL$.

The construction of $CL\mathcal{R}_{\Omega}^{rec}$ reflects the idea that control cannot pass ; before all previous actions are extinct.

So $\Rightarrow$ will actually be a subset of $CL\mathcal{R}_{\Omega}^{rec} \times \Delta \times CL\mathcal{R}_{\Omega}^{rec}$. If $\langle E, a, E' \rangle \in \Rightarrow$ we write this as $E \stackrel{a}{\Rightarrow} E'$. One can think of this as E can evolve to E' under the (external observable) action a. We shall follow DeNicola [Nic87] and Hennessy [Hen88a] when defining $\Rightarrow$. Hennessy does this in an extended labelled transition system by means of a relation $\rightarrowtail$, which reflects the step of an internal computation, and by a relation $\longrightarrow$ for an external computation step corresponding to an observable action. The slight deviation from Hennessy in defining the relation, $\rightarrowtail$, for internal steps are manily due to differences in the languages considered.

Here the internal steps serves a fourfold purpose:

1) resolve internal nondeterministic choices
2) remove extinct actions
3) substitute action refinements (in a lazy fashion)
4) unfold recursive definitions

The action relation, $\stackrel{a}{\Rightarrow}$, is defined as $\rightarrowtail^{*} \stackrel{a}{\longrightarrow} \rightarrowtail^{*}$, where $\rightarrowtail \subseteq CL\mathcal{R}_{\Omega}^{rec\,2}$ and $\longrightarrow \subseteq CL\mathcal{R}_{\Omega}^{rec} \times \Delta \times CL\mathcal{R}_{\Omega}^{rec}$ are defined as the least relations satisfying the following axioms and inference rules.

$$
\begin{array}{ll}
a \xrightarrow{a} \dagger & \dfrac{E_0 \xrightarrow{a} E_0'}{E_0\,;E_1 \xrightarrow{a} E_0'\,;E_1} \\[2ex]
 & \dfrac{E_0 \xrightarrow{a} E_0'}{\begin{array}{l} E_0 \parallel E_1 \xrightarrow{a} E_0' \parallel E_1 \\ E_1 \parallel E_0 \xrightarrow{a} E_1 \parallel E_0' \end{array}} \\[2ex]
\dagger\,; E \rightarrowtail E & \dfrac{E_0 \rightarrowtail E_0'}{E_0\,;E_1 \rightarrowtail E_0'\,;E_1} \\[2ex]
\begin{array}{l} E_0 \oplus E_1 \rightarrowtail E_0 \\ E_0 \oplus E_1 \rightarrowtail E_1 \end{array} & \\[2ex]
\begin{array}{l} \dagger \parallel E \rightarrowtail E \\ E \parallel \dagger \rightarrowtail E \end{array} & \dfrac{E_0 \rightarrowtail E_0'}{\begin{array}{l} E_0 \parallel E_1 \rightarrowtail E_0' \parallel E_1 \\ E_1 \parallel E_0 \rightarrowtail E_1 \parallel E_0' \end{array}} \\[2ex]
\begin{array}{l} a[\varrho] \rightarrowtail \varrho(a) \\ (E_0\,;E_1)[\varrho] \rightarrowtail E_0[\varrho]\,;E_1[\varrho] \\ (E_0 \oplus E_1)[\varrho] \rightarrowtail E_0[\varrho] \oplus E_1[\varrho] \\ (E_0 \parallel E_1)[\varrho] \rightarrowtail E_0[\varrho] \parallel E_1[\varrho] \end{array} & \dfrac{E \rightarrowtail E'}{E[\varrho] \rightarrowtail E'[\varrho]} \\[2ex]
\begin{array}{l} \Omega \rightarrowtail \Omega \\ \Omega[\varrho] \rightarrowtail \Omega \\ rec\,x.\,E \rightarrowtail E[rec\,x.\,E/x] \end{array} &
\end{array}
$$

Example: For BL-refinements, ϱ' and ϱ, with $\varrho'(b) = c;d$ and $\varrho(c) = e$ we get:

$$
\begin{array}{rl}
a\,;b \parallel a & \xrightarrow{a} \dagger\,;b \parallel a \\
 & \xrightarrow{a} \dagger\,;b \parallel \dagger \\
 & \rightarrowtail b \parallel \dagger \\
 & \xrightarrow{b} \dagger \parallel \dagger \\
 & \rightarrowtail \dagger
\end{array}
\qquad
\begin{array}{rl}
(a \parallel b)[\varrho'][\varrho] & \rightarrowtail (a[\varrho'] \parallel b[\varrho'])[\varrho] \\
 & \rightarrowtail (a[\varrho'] \parallel c\,;d)[\varrho] \\
 & \rightarrowtail a[\varrho'][\varrho] \parallel (c\,;d)[\varrho] \\
 & \rightarrowtail a[\varrho'][\varrho] \parallel c[\varrho]\,;d[\varrho] \\
 & \rightarrowtail a[\varrho'][\varrho] \parallel e\,;d[\varrho] \xrightarrow{e} \ldots
\end{array}
$$

Example: The scenarios below show possible evolvements of $F = (rec\,x.\,E)[\varrho]$ and $F' = rec\,x.\,(E[\varrho])$:

$$
\begin{array}{lll}
F & \rightarrowtail (a \oplus a \,;\, rec\,x.\, E)[\varrho] & \rightarrowtail a[\varrho] \\
 & \rightarrowtail (a \,;\, rec\,x.\, E)[\varrho] & \rightarrowtail b \\
 & \rightarrowtail a[\varrho] \,;\, F & \\
 & \rightarrowtail b \,;\, F & \\
 & \overset{b}{\Rightarrow} F \cdots & \\
 & \quad \vdots &
\end{array}
\qquad
\begin{array}{llll}
F' & \rightarrowtail & (a \oplus a \,;\, F')[\varrho] & \rightarrowtail \; a[\varrho] \\
 & \rightarrowtail & (a \,;\, F')[\varrho] & \rightarrowtail \; b \\
 & \rightarrowtail^* & (b \,;\, F'[\varrho]) & \\
 & \overset{b}{\Rightarrow} & F'[\varrho] & \rightarrowtail^* b[\varrho] \\
 & \rightarrowtail^* & (b \,;\, F'[\varrho])[\varrho] & \rightarrowtail \; a \\
 & \rightarrowtail^* & (a \,;\, F'[\varrho][\varrho]) & \\
 & \overset{a}{\Rightarrow} & (F'[\varrho])[\varrho] & \rightarrowtail^* a[\varrho] \\
 & & \vdots & \rightarrowtail^* b
\end{array}
$$

where $E = a \oplus a \,;\, x$ and ϱ is a BL-refinement such that $\varrho(a) = b$ and $\varrho(b) = a$.

So informally F can perform a finite sequence $s \in \Delta^*$, *iff* $s \in b^*$ and similar F', *iff* $s \in (ba)^* \cup (ba)^* b$.

The behavioural equivalences of processes we shall use will be very simple: two process are equivalent if they can perform the same sequences of observable actions. However it remains to determine the sort of sequences to be used. Suppose only maximal sequences (in the sense that the process cannot do any actions afterwards) are considered and denote the associated equivalence by $\underset{\sim}{\times}$. $\underset{\sim}{\times}$ will be able to distinguish recursive processes like:

$$rec\,x.\,(a \oplus b \,;\, x) \quad \text{and} \quad rec\,x.\,(c \oplus b \,;\, x)$$

because they obviously can do different maximal sequences. On the other hand there will be no way to distinguish the processes:

$$rec\,x.\,(a \,;\, x) \quad \text{and} \quad rec\,x.\,(b \,;\, x) \tag{6}$$

This is satisfactory if nontermination is viewed as unimportant and only termination matters. Taking the opposite point of view, disregarding termination, they must be distinguished. Denote the equivalence arising when considering prefixes of (possibly maximal) sequences by $\underset{\sim}{\sqcup}$. Then $\underset{\sim}{\sqcup}$ will be able to distinguish the processes of (6) but in return identify

$$rec\,x.\,(b \oplus b \,;\, x) \quad \text{and} \quad rec\,x.\,(b \,;\, x)$$

which on the contrary would not be identified by $\underset{\sim}{\times}$. The appropriate equivalence depends on what view is taken. However there is the serious drawback of $\underset{\sim}{\sqcup}$ that it is not a congruence—not even on BL:

$$a \oplus a \,;\, b \underset{\sim}{\sqcup} a \,;\, b \text{ but } (a \oplus a \,;\, b) \,;\, c \not\underset{\sim}{\sqcup} (a \,;\, b) \,;\, c$$

Therefore $\underset{\approx}{\sqsubset}^{\{;\}}$ would be more appropriate to study, where in general for a set Σ, of combinators, we use $\sqsubseteq^{\Sigma}$ to denote the largest Σ-congruence contained in the equivalence $\sqsubseteq$. Though $\underset{\approx}{\sqsubset}^{\{;\}}$ and $\lesseqgtr$ are congruences w.r.t. ;, $\oplus$ and $\|$ they are not preserved in $[\varrho]$-contexts:

Example 4.1 Suppose e.g., $\varrho(a) = a\,;a \oplus b$, and $E_0 = a \parallel a$, $E_1 = a\,;a$. Then $E_0 \lesseqgtr, \lesseqgtr^{\{;\}} E_1$ but $E_0[\varrho] \not\lesseqgtr, \not\underset{\approx}{\sqsubset}^{\{;\}} E_1[\varrho]$ because $E_0[\varrho] \overset{aba}{\Longrightarrow} \dagger$ and $E_1[\varrho] \overset{aba}{\not\Longrightarrow}$.

So we shall rather be interested in $\lesseqgtr^c$ and $\underset{\approx}{\sqsubset}^c$, where c is all combinators—the recursive inclusive. Actually the congruences will be induced from corresponding preorders $\lesssim$ and $\underset{\approx}{\sqsubset}$ respectively. Formally define for a finite sequence $s \in \Delta^*$ and $E, E' \in CL\mathcal{R}_{\Omega}^{rec}$:

$$\begin{array}{l} E \overset{s}{\Rightarrow} E', s = a_1 a_2 \ldots a_n \in \Delta^* \textit{ iff} \\ \exists E_1, \ldots, E_n \in CL \exists a_1, \ldots, a_n \in \Delta, n \geq 0. \\ E \overset{a_1}{\Longrightarrow} E_1 \overset{a_2}{\Longrightarrow} \ldots \overset{a_n}{\Longrightarrow} E_n = E' \end{array}$$

where the case $n = 0$ means $E \rightarrowtail^* E'$.

Definition 4.2 $\lesssim, \underset{\approx}{\sqsubset} \subseteq BL\mathcal{R}_{\Omega}^{rec} \times BL\mathcal{R}_{\Omega}^{rec}$ are then defined:

$$\begin{array}{lll} E_0 \lesssim E_1 & \textit{iff} & \forall s \in \Delta^*.\ E_0 \overset{s}{\Rightarrow} \dagger \text{ implies } E_1 \overset{s}{\Rightarrow} \dagger \\ E_0 \underset{\approx}{\sqsubset} E_1 & \textit{iff} & \forall s \in \Delta^*.\ E_0 \overset{s}{\Rightarrow} \text{ implies } E_1 \overset{s}{\Rightarrow} \end{array}$$

$\lesseqgtr$ is induced from $\lesssim$ by $E_0 \lesseqgtr E_1$ *iff* $E_0 \lesssim E_1$ *and* $E_1 \lesssim E_0$. Similar for $\underset{\approx}{\sqsubset}$. □

Notice that as expected $\lesseqgtr$ as well as $\underset{\approx}{\sqsubset}$ identifies $a\,;(b \oplus c)$ and $a\,;b \oplus a\,;c$.

The Finite Sublanguage

We shall now elaborate on the previous comment that $E[\varrho]$ behaves as if the refinements were substituted in E in advance. This could be done for $E \in RBL_{\Omega}^{rec}$, but for developments in the sequel it will suffice with $E \in BL\mathcal{R}_{\Omega}$. To this end we formalize substitution as mapping $\sigma : CL\mathcal{R}_{\Omega} \longrightarrow CL_{\Omega}$, using $\{\varrho\} : BL_{\Omega} \longrightarrow BL_{\Omega}$ which performs a single substitution in a refinement free expression. Because of their syntactic nature we write them postfix. The definitions of σ and $\{\varrho\}$ are in full:

$$
\begin{aligned}
\dagger\sigma &= \dagger & & \\
\Omega\sigma &= \Omega & \Omega\{\varrho\} &= \Omega \\
a\sigma &= a & a\{\varrho\} &= \varrho(a) \\
(E_0\,;E_1)\sigma &= E_0\sigma\,;E_1\sigma & (E_0\,;E_1)\{\varrho\} &= E_0\{\varrho\}\,;E_1\{\varrho\} \\
(E_0 \oplus E_1)\sigma &= E_0\sigma \oplus E_1\sigma & (E_0 \oplus E_1)\{\varrho\} &= E_0\{\varrho\} \oplus E_1\{\varrho\} \\
(E_0 \parallel E_1)\sigma &= E_0\sigma \parallel E_1\sigma & (E_0 \parallel E_1)\{\varrho\} &= E_0\{\varrho\} \parallel E_1\{\varrho\} \\
E[\varrho]\sigma &= (E\sigma)\{\varrho\} & &
\end{aligned}
$$

Notice that $E[\varrho]$ only if $E \in BL\mathcal{R}_\Omega$ and σ when restricted to $BL\mathcal{R}_\Omega$ yield a map $\sigma : BL\mathcal{R}_\Omega \longrightarrow BL_\Omega$.

Proposition 4.3 For $E \in BL\mathcal{R}_\Omega$ we have $E \lessapprox E\sigma$ and $E \sqsubseteq\!\!\!\!\approx E\sigma$

Proof Since $E''\sigma = \dagger$ *iff* $E'' = \dagger$ the proposition follows from

$E \overset{s}{\Rightarrow} E'$ implies $E\sigma \overset{s}{\Rightarrow} E'\sigma$

$E\sigma \overset{s}{\Rightarrow} E'$ implies $\exists E'' \in CL\mathcal{R}_\Omega, E \overset{s}{\Rightarrow} E'', E''\sigma = E'$

where $s \in \Delta^*$ and E now is supposed to be from $CL\mathcal{R}_\Omega$. Each implication is proven by induction on the "length" of $\overset{s}{\Rightarrow}$ where the inductive steps essentially consists of proofs of similar propositions for $\rightarrowtail$ and $\xrightarrow{a}$, but using induction on the structure of E. □

5 Denotational Semantics

This section is devoted the motivation and introduction of two denotational models M_{or} and M_{or}^p intended to characterize $\lesssim^c$ and $\sqsubseteq\!\!\!\!\sim^c$ respectively. The models are best motivated by considering how a model for $\lesssim^c$ w.r.t. process expressions of $BL\mathcal{R}$ should look like and then generalize this to $\sqsubseteq\!\!\!\!\sim^c$ and the rest of the language.

In general a denotational model, M, for a (behavioural) preorder will consist of a partial ordered domain, A, together with a denotational map, $A[\![\_]\!]$, which for each process expression yields an element of A.

Now to get an idea of how the denotational map, $A_{or}[\![\_]\!]$, of M_{or} should be first recall that an $E \in BL\mathcal{R}$ behaves as if the refinement where substituted in advance, i.e., as $E\sigma$, so it is fair to expect $A_{or}[\![E]\!] = A_{or}[\![E\sigma]\!]$.

Since the examples of the previous section show that $\lesssim^c$ rather is concerned with the nonsequential than the nondeterministic aspects of behaviour, the denotational map should be formed as an abstraction of sets of pomsets in place of an abstraction of computation trees. To this end we define:

Definition 5.1 *Canonical Pomset Association*
The *canonical associated pomsets* of a BL-expression is given by the map $\wp : BL \to \mathcal{P}(\mathbf{P} \setminus \{\varepsilon\}) \setminus \emptyset$ defined compositionally as follows:

$$
\begin{aligned}
\wp(a) &= \{a\} \\
\wp(E_0\,;E_1) &= \wp(E_0) \cdot \wp(E_1) \\
\wp(E_0 \oplus E_1) &= \wp(E_0) \cup \wp(E_1) \\
\wp(E_0 \parallel E_1) &= \wp(E_0) \times \wp(E_1)
\end{aligned}
$$

By analogy to the most abstract computation tree based models our first suggestion for an abstraction might be to take the linearizations of the canonical associated pomsets, i.e., use the map $\delta_{\mathsf{w}}\wp(\_\sigma)$ and for the domain, A_{or}, choose the finite subsets of $\mathbf{P} \setminus \{\varepsilon\}$ ordered under inclusion. But looking at example 4.1 we see that δ_{w} must be rejected as being to abstract. Now δ_{w} can be regarded as abstracting from the nontotal ordered pomsets of δ so a second attempt could be to use $\delta\wp(\_\sigma)$. However from

$$
\begin{aligned}
&E_0' = (a\,;(b \parallel d)) \parallel c \oplus a \parallel (c\,;(b \parallel d)) \text{ and} \\
&E_1' = E_0' \oplus a\,;b \parallel c\,;d
\end{aligned}
$$

follows that δ in return is not abstract enough. Thought by this experience we shall look for a pomset property P_* turning δ_* into a suitable abstraction between δ_{w} and δ. Returning to example 4.1 the reason for our success of distinguishing E_0 and E_1 through $\lesssim^c$ is the ability for each of the "concurrent" actions to choose an appropriate BL-refinement and make actions of these overlap when a sequence of actions is performed, thereby *reflecting* the "concurrency" of E_0. To transfer this idea to pomsets and find a property of pomsets stating when this "overlapping" is possible, we shall temporarily and for sake of argument appeal to the following operational intuition of pomsets: if a is a minimal element it can be performed (corresponds to an atomic action) resulting in the pomset obtained by removing a. Of course image finite $\mathcal{P}(\mathbf{P})$-refinements take over the rôle of the BL-refinements. Now consider the pomset,

$$\begin{matrix} a \to b \\ c \to d \end{matrix}$$

associated with E_1' above. A prerequisite for overlapping is a refinement splitting the actions—a *fission* refinement. I.e., we use a $\mathcal{P}(\mathbf{P})$-refinement, ϱ_f, such that $a \mapsto \{a_S \cdot a_F\}$ and similar for b, c and d. In order to reflect that a is concurrent to c and d, we start of the refinements of a and c: a_S and c_S, in some order. Since a is concurrent to d we want to start of d as well and do that by finishing c (i.e., perform c_F) at first. But then we prevent ourselves from reflecting that c is concurrent to b, since the refinement of c is already finished of before having a chance to get b started. Similar if we finished a at first. This suggest that we by "overlapping" can reflect all the concurrency of a pomset, $\mathbf{p}$, if $\mathbf{p}$ has the following recursively defined property P_{ol}:

p has the property P_{ol} *iff*

- *either* p is empty
- *or* there is a minimal element, x, of p such that
  - a) the remaining minimal elements of p is exactly the elements concurrent to x
  - b) p' has the property P_{ol}, where p' is obtained from p by removing x

This property is easely formalized and proven equivalent to the following alternative and more tractable pomset property:

Definition 5.2 *P_{or}-Property for Pomsets*

A pomset $\mathbf{p}$ is said to have the P_{or}-property, $P_{or}(\mathbf{p})$ *iff* for all x, x', y, y' in X_p we have:

$$\text{if} \quad \begin{matrix} x <_p x' \\ co_p \\ y <_p y' \end{matrix} \quad \text{then} \quad \begin{matrix} x <_p y' \\ or \\ y <_p x' \end{matrix} \qquad \square$$

Notice that by the universal quantification of x and y the P_{or}-property is hereditary and since the concurrent elements of a sequential composition of pomsets must steem from the same pomset it follows that P_{or} is dot synthesizable too.

Example: $\begin{tikzcd} a \arrow[r] \arrow[dr] & b \\ c \arrow[r] & d \end{tikzcd}$ and $\begin{tikzcd} a \arrow[r] & b \\ c & \end{tikzcd}$ has the P_{or}-property, $\begin{tikzcd} a \arrow[r] & b \\ c \arrow[r] & d \end{tikzcd}$ has not.

So we arrive at a model with denotational map $A_{or}[\![\_]\!] = \delta_{or}\wp(\_\sigma)$.

Now w.r.t. $\underset{\sim}{\sqsubset}^c$. From the operational semantics we see that a sequence of $E\,;F$ involving actions of F must contain a maximal sequence, thus getting the hint to incooperate the M_{or} model directly in the model M_{or}^p capturing $\underset{\sim}{\sqsubset}^c$.

The final step consists in extending these ideas to handle recursion too. This is in general (see e.g., [Hen88a]) for a language of recursive expressions, $REC_\Sigma(X)$, over a signature, Σ, with free variables X, done by extending A to an algebraic complete partial order (algebraic cpos for short) and give the denotational map, $A[\![\_]\!]$, by means of environments:

$$A[\![x]\!]\rho_A = \rho_A(x)$$
$$A[\![f(t_1,\ldots,t_k)]\!]\rho_A = f_A(A[\![E_1]\!]\rho_A,\ldots,A[\![E_k]\!]\rho_A),\ f \in \Sigma\ k\text{-ary}$$
$$A[\![rec\,x.\,E]\!]\rho_A = Y\lambda a.\ A[\![E]\!]\rho_A[a/x],\ \text{where}$$

- $\rho_A \in \mathrm{ENV}_A$, the cpo set of A-environments (maps from X to A)
- f_A is a k-ary continuous operator on A associated with f and in the special case of Ω, Ω_A is the constant function yielding the least element $\perp_A$ of A.
- Y is a function yielding the least fixpoint of $\lambda a.\ A[\![E]\!]\rho_A[a/x]$ in A

Hence for each expression, E, $A[\![E]\!]$ gives a continuous map from ENV_A to A, i.e., $A[\![E]\!] \in [\mathrm{ENV}_A \longrightarrow A]$, and if furthermore E is closed (without free variables), $A[\![E]\!]$ is a constant function giving óne element, ambiguously denoted $A[\![E]\!]$, of A.

Giving meaning to expressions in this way has different pleasant consequences like that the induced denotational preorder, $\trianglelefteq_A$, given by

$$E \trianglelefteq_A F \ \textit{iff}\ \forall \rho_A \in \mathrm{ENV}_A.\ A[\![E]\!]\rho_A \leq_A A[\![F]\!]\rho_A$$

is a precongruence w.r.t. all the combinators. Also the meaning of an expression, E, is the limit of its finite approximations $\mathrm{Fin}(E)$. An expression E' is an approximation to E if they are related by the syntactic preorder, $\preceq$, defined to be the least precongruence (w.r.t. to the ordinary combinators) which satisfies:

$$\Omega \preceq E$$
$$E[rec\, x.\, E/x] \preceq rec\, x.\, E$$

So $\mathrm{Fin}(E) = \{E' \in FREC_\Sigma(X) \mid E' \preceq E\}$, where $FREC_\Sigma(X)$ is the set of finite expressions. The term "approximates" for $\preceq$ is justified by the fact $\preceq \subseteq \trianglelefteq_A$.

Having finite approximations the notion of algebraic relations mentioned in the introduction can be introduced formally:

A relation R over REC_Σ (i.e., closed expressions) is *algebraic* if for all $E, F \in REC_\Sigma$:

$$E\ R\ F \ \textit{iff}\ \forall E' \in \mathrm{Fin}(E) \exists F' \in \mathrm{Fin}(F).\, E'\ R\ F'$$

Actually the preorder $\trianglelefteq_A$ (when restricted to REC_Σ) is algebraic provided the denotations of closed finite expressions are compact elements of A (an element is compact if whenever it is dominated by a lub of a directed set then so it is by an element of that set). If on the other hand any compact element is denotable by closed finite expression, $\trianglelefteq_A$ is substitutive, where

a relation R over $REC_\Sigma(X)$ is *substitutive* if for all $E, F \in REC_\Sigma(X)$:

$$E\ R\ F\ \textit{iff}\ \text{ for all closed syntactic substitutions } \rho,\ E\rho\ R\ F\rho$$

Proposition 5.3 If a model is *finitary* (i.e., an element is compact *iff* it is denotable by a closed finite expression) then the denotational induced preorder is substitutive and algebraic.

Using [DNH84] Hennessy in [Hen83] indicate a proof that $\trianglelefteq_A$ is substitutive when every compact element is denotable by a closed finite expression—a detailed proof of the proposition can be found in [Eng89].

Due to the pleasant consequences of having finitary models, the goal will therefore be to extend the domains of the previous models to deal with "infinity" while at the same time enforcing constraints which ensures the reachability of compact elements. The first subgoal is easely attained simply by considering infinite sets of pomsets instead of finite. Recalling that the previous obtained denotational maps were based on the canonical map, $\wp$, we get a clue for the second subgoal. At first we look at what pomsets we can get by $\wp$. Here we shall lean on a result of Grabowski [Gra81] which essentially states that the sets of pomsets generated from the singleton pomsets and ε by sequential and parallel composition exactly are the N-free pomsets.

Definition 5.4 *$P_{N\text{-free}}$-Property for Pomsets*

A pomset $\mathbf{p}$ is said to have the $P_{N\text{-free}}$-property, $P_{N\text{-free}}(\mathbf{p})$ *iff* for all x, x', y, y' in X_p we have:

$$\text{if} \quad \begin{array}{cc} x <_p x' & \\ co_p \quad co_p & \\ y <_p y' & \end{array} \quad \text{and} \quad x <_p y' \quad \text{then} \quad y \leq_p x'$$

If a pomset $\mathbf{p}$ has the $P_{N\text{-free}}$-property we say that $\mathbf{p}$ is N-free. Also we shall say that a $\mathcal{P}(\mathbf{P})$-refinement, ϱ, is N-free *iff* $\mathbf{p}$ is N-free for all $\mathbf{p} \in \varrho(a)$ and $a \in \Delta$. Similar for particular refinements. $\square$

Example: $\begin{tikzcd} a \arrow[r] \arrow[dr] & b \\ c \arrow[r] \arrow[ur] & d \end{tikzcd}$ and $\begin{tikzcd} a \arrow[r] & b \\ c & \end{tikzcd}$ are N-free, but $\begin{tikzcd} a \arrow[r] \arrow[dr] & b \\ c \arrow[r] & d \end{tikzcd}$ is not.

The result of Grabowski can (slightly modified for our set-up) be formulated:

Proposition 5.5 P is a finite and nonempty set of N-free pomsets such that $\varepsilon \notin P$ *iff* $\exists E_P \in BL.\ \wp(E_P) = P$.

On top of the canonical map the δ_{or}-closure were used to obtain the denotation. This suggests to let the elements of A_{or} be sets of pomsets which are obtained as the δ_{or}-closure of a set, t, of N-free nonempty pomsets. As already argued, information of the M_{or}-model must be incorporated when it comes to the M^p_{or}-model for the semantics concerning prefix. Using the π-closure of pomsets to capture the idea of prefixes of sequences it appears that elements of A^p_{or} should be pairs where the second component is an element of A_{or} and the first component is the δ_{or}- and π-closure of a nonempty set, s, of N-free pomsets with the additional constraint that s shall be a superset of the set, t, which the second component is a δ_{or}-closure of. The additional constraint originates in the fact that if a maximal sequence can be recorded then so can any prefix of it. As noticed the P_{or} property is both hereditary and dot synthesizable, so by proposition 2.15 δ_{or} and π then commute and it make sense to talk about the δ_{or}-/ π-closure of a set. Formally

$$\boxed{\begin{aligned} A_{or} &= \{\delta_{or}(t) \mid t \subseteq \mathbf{P}_{N\text{-free}}, \varepsilon \notin t\} \\ A^p_{or} &= \{\langle \delta_{or}\pi(s), \delta(t)\rangle \mid s, t \subseteq \mathbf{P}_{N\text{-free}}, \varepsilon \notin t \subseteq s \neq \emptyset\} \end{aligned}}$$

We shall often make use of the observation that $t \subseteq s \Rightarrow \delta_*(t) \subseteq \delta_*(s) \Rightarrow \delta_*(t) \subseteq \delta_*\pi(s)$ which follows from δ_* being $\subseteq$-monotone and the general fact $\mathbf{p} \in \pi(\mathbf{p})$. With some care concerning the closures it can be shown:

Proposition 5.6 $\langle A_{or}, \subseteq\rangle$ and $\langle A^p_{or}, \subseteq\rangle$ (component wise) are algebraic cpos with least elements $\emptyset$ and $\langle\{\varepsilon\}, \emptyset\rangle$ respectively. The compact elements are those $\delta_{or}(s) \in A_{or}$ and $\langle \delta_{or}\pi(s), \delta_{or}(t)\rangle \in A^p_{or}$ where s and t are finite sets. Every nonempty $D \subseteq A_{or}$ has a lub: $\bigvee_{or} D = \bigcup D \in A_{or}$ and similar every nonempty $D \subseteq A^p_{or}$ has a lub $\bigvee^p_{or} D = \langle \bigcup D_1, \bigcup D_2\rangle \in A^p_{or}$ where $D_i = \{d_i \mid \langle d_1, d_2\rangle \in D\}$ for $i = 0, 1$.

Definition 5.7 Assume $d = \langle P, Q\rangle$ and $d_i = \langle P_i, Q_i\rangle$ for $i = 0, 1$ are elements of A^p_{or}. Then the operators of the M^p_{or} model are defined as follows:

$$\begin{aligned} \Omega^p_{or} &= \langle\{\varepsilon\}, \emptyset\rangle \\ a^p_{or} &= \langle\{\varepsilon, a\}, \{a\}\rangle \\ d_0 \,;^p_{or}\, d_1 &= \langle P_0 \cup Q_0 \cdot P_1, Q_0 \cdot Q_1\rangle \\ d_0 \oplus^p_{or} d_1 &= \langle P_0 \cup P_1, Q_0 \cup Q_1\rangle \\ d_0 \parallel^p_{or} d_1 &= \langle \delta_{or}(P_0 \times P_1), \delta_{or}(Q_0 \times Q_1)\rangle \\ d[\varrho]^p_{or} &= \langle \delta_{or}\pi(P{<}\wp(\varrho){>}), \delta_{or}(Q{<}\wp(\varrho){>})\rangle, \text{ where} \end{aligned}$$

$\wp(\varrho)$ is the ε-free $\mathcal{P}(\mathbf{P})$-refinement $\wp(\varrho)$ given by $(\wp(\varrho))(a) = \wp(\varrho(a))$. The operators of the M_{or} model are derived from those of the M^p_{or} simply by projecting the second component. I.e. if $P_0, P_1 \in A_{or}$ then $P_0 \parallel_{or} P_1$ equals $\delta_{or}(P_0 \times P_1)$.

Proposition 5.8 The operators above are well-defined.

Proof The well-definedness of the A_{or} operators follows similar as for A^p_{or}. We just look at $;^p_{or}$ and $[\varrho]^p_{or}$. Assume $d_0, d_1 \in A^p_{or}$. Then $d_0 = \langle \delta_{or}\pi(s_0), \delta_{or}(t_0)\rangle$ for some $s_0, t_0 \in \mathbf{P}_{N\text{-free}}$ such that $\varepsilon \notin t_0 \subseteq s_0 \neq \emptyset$. Similar for d_1.

From a) of proposition 5.9 below and the distributivity of δ_{or} over $\cdot$ we immediately get: $d_0 ;^p_{or} d_1 = \langle \delta_{or}\pi(s_0 \cup t_0 \cdot s_1), \delta_{or}(t_0 \cdot t_1)\rangle$. By Grabowski $\mathbf{p} \cdot \mathbf{q}$ is N-free when $\mathbf{p}$ and $\mathbf{q}$ are. So $d_0 ;^p_{or} d_1 \in A^p_{or}$ then follows from

$$
\begin{array}{ll}
\varepsilon \notin t_0 \cdot t_1 & \text{because } \varepsilon \notin t_0, t_1 \\
\quad \subseteq s_0 \cup t_0 \cdot s_1 & \text{since } t_1 \subseteq s_1 \\
\quad \neq \emptyset & \text{by } s_0 \neq \emptyset
\end{array}
$$

Now to see that the $[\varrho]^p_{or}$ operator on A^p_{or} is well-defined, let a $d = \langle \delta_{or}\pi(s), \delta_{or}(t)\rangle \in A^p_{or}$ be given. Using d) of proposition 5.9 below for the first component and lemma 5.10 for the second we get
$d[\varrho]^p_{or} = \langle \delta_{or}\pi(s{<}\wp(\varrho){>}), \delta_{or}(t{<}\wp(\varrho){>})\rangle$.
$\varrho(a) \in BL$ for every $a \in \Delta$, so from proposition 5.5 $(\wp(\varrho))(a)$ is a set of N-free nonempty pomsets when $a \in \Delta$. It can then be shown that $s{<}\wp(\varrho){>}$ and $t{<}\wp(\varrho){>}$ are sets of N-free pomsets because s and t are assumed to be N-free too. $\wp(\varrho)$ is ε-free so we conclude that $d[\varrho]^p_{or} \in A^p_{or}$. □

The following proposition is useful not only for the proof of the proposition above but also for other to come.

Proposition 5.9 Let ϱ be an ε-free $\mathcal{P}(\mathbf{P})$-assignment and suppose P, Q and R are sets of pomsets such that $P \supseteq R$. Then

a) $\delta_{or}\pi(P) \cup \delta_{or}(R) \cdot \delta_{or}\pi(Q) = \delta_{or}\pi(P \cup R \cdot Q)$

b) $\delta_{or}\pi(P) \cup \delta_{or}\pi(Q) = \delta_{or}\pi(P \cup Q)$

c) $\delta_{or}(\delta_{or}\pi(P) \times \delta_{or}\pi(Q)) = \delta_{or}\pi(P \times Q)$

d) $\delta_{or}\pi((\delta_{or}\pi(P)){<}\varrho{>}) = \delta_{or}\pi(P{<}\varrho{>})$

Proof For a) notice at first :

$$\pi(\mathbf{p} \cdot \mathbf{q}) = \pi(\mathbf{p}) \cup \{\mathbf{p}\} \cdot \pi(\mathbf{q}) \tag{7}$$

We then get:

$$\begin{array}{ll}
\quad \delta_{or}\pi(P) \cup \delta_{or}(R) \cdot \delta_{or}\pi(Q) & \\
= \delta_{or}(\pi(P) \cup R \cdot \pi(Q)) & \delta_{or} \text{ distributes over } \cdot \text{ and } \cup \\
= \delta_{or}(\pi(P) \cup \pi(R) \cup R \cdot \pi(Q)) & R \subseteq P \text{ and } \pi \text{ is } \subseteq\text{-monotone} \\
= \delta_{or}(\pi(P) \cup \pi(R \cdot Q)) & \text{by (7)} \\
= \delta_{or}\pi(P \cup R \cdot Q) & \pi \text{ distributes over } \cup
\end{array}$$

b) and c) follows from the distributivity of δ_{or} and π over $\cup$, proposition 2.15 and distributivity of π over $\times$.

$$\begin{array}{ll}
\text{d) } \delta_{or}\pi((\delta_{or}\pi(P))<\varrho>) & \\
= \pi\delta_{or}((\delta_{or}\pi(P))<\varrho>) & \delta_{or} \text{ and } \pi \text{ commutes} \\
= \pi\delta_{or}((\pi(P))<\varrho>) & \text{lemma 5.10 } (\varrho \text{ is } \varepsilon\text{-free}) \\
= \delta_{or}\pi((\pi(P))<\varrho>) & \delta_{or} \text{ and } \pi \text{ commutes} \\
= \delta_{or}\pi(P<\varrho>) & \text{lemma 5.11 below}
\end{array}$$

$\square$

Lemma 5.10 Let P be a set of pomsets and ϱ an ε-free $\mathcal{P}(\mathbf{P})$-refinement. Then

$$\delta_{or}((\delta_{or}(P))<\varrho>) = \delta_{or}(P<\varrho>)$$

Proof Clearly it is enough to prove $\delta_{or}((\delta_{or}(\mathbf{p}))<\varrho>) = \delta_{or}(\mathbf{p}<\varrho>)$ for a single pomset $\mathbf{p}$. Each inclusion is proven separately.

To see $\delta_{or}((\delta_{or}(\mathbf{p}))<\varrho>) \subseteq \delta_{or}(\mathbf{p}<\varrho>)$ let $\mathbf{q} \in \delta_{or}((\delta_{or}(\mathbf{p}))<\varrho>)$. Then $P_{or}(\mathbf{q})$ and there exists a $\mathbf{q}' \in (\delta_{or}(\mathbf{p}))<\varrho>$ such that $\mathbf{q} \preceq \mathbf{q}'$. Therefore $\mathbf{q}' \in \mathbf{p}'<\varrho>$ for some $\mathbf{p}' \in \delta_{or}(\mathbf{p})$ and we have $\mathbf{p}' \preceq \mathbf{p}$. But by the nature of $<\varrho>$ this implies $\forall \mathbf{r}' \in \mathbf{p}'<\varrho> \exists \mathbf{r} \in \mathbf{p}<\varrho>. \, \mathbf{r}' \preceq \mathbf{r}$. Hence there exists a $\mathbf{r} \in \mathbf{p}<\varrho>$ such that $\mathbf{q} \preceq \mathbf{q}' \preceq \mathbf{r}$. Since $P_{or}(\mathbf{q})$ we have $\mathbf{q} \in \delta_{or}(\mathbf{p}<\varrho>)$.

$\delta_{or}((\delta_{or}(\mathbf{p}))<\varrho>) \supseteq \delta_{or}(\mathbf{p}<\varrho>)$: Suppose $\mathbf{q} \in \delta_{or}(\mathbf{p}<\varrho>)$. This means $P_{or}(\mathbf{q})$ and $\mathbf{q} \preceq [p<\pi_p>]$, where $<\pi_p>$ is a ϱ-consistent particular refinement for a representative, p, of $\mathbf{p}$. So it is enough to find an $\mathbf{p}' \in \delta_{or}(\mathbf{p})$ such that $\mathbf{q} \preceq [p'<\pi_{p'}>]$, where $\pi_{p'}$ also is consistent with ϱ.

By proposition 2.8 $\mathbf{q} \preceq [p<\pi_p>]$ implies the existence of a representative, q, of $\mathbf{q}$ such that $q = \langle X_{p<\pi_p>}, \leq_q, \ell_{p<\pi_p>} \rangle$ and $\leq_q \supseteq \leq_{p<\pi_p>}$.

Define $p' := \langle X_p, \leq_{p'}, \ell_p \rangle$, where $\leq_{p'}$ is the reflexive closure of $<_{p'} \subseteq {X_p}^2$ defined by:

(8) $\qquad x <_{p'} y \text{ iff } \forall \langle x, x' \rangle, \langle y, y' \rangle \in X_q. \, \langle x, x' \rangle <_q \langle y, y' \rangle$

That is, we order elements x, y in p' if and only if all elements from $\pi_p(y)$ are causally dependent on all elements $\pi_p(x)$ in q.

To see that p' in fact is an lpo notice that $\leq_{p'}$ by definition is reflexive, clearly also transitive and the antisymmetry is seen from (8), the ε-freeness of π_p (a consequence of ϱ being ε-free) and the antisymmetry of $\leq_q$.

$X_p = X_{p'}$ and $\ell_p = \ell_{p'}$ so $\mathbf{p'} \preceq \mathbf{p}$ follows by proving $\leq_{p'} \supseteq \leq_p$. By definition $x \leq_{p'} x$. If $x <_p y$ then $x \neq y$, so by the construction of $p{<}\pi_p{>}$ we have $\forall \langle x, x' \rangle, \langle y, y' \rangle \in X_{p<\pi_p>}. \langle x, x' \rangle <_{p<\pi_p>} \langle y, y' \rangle$ and from $\leq_q \supseteq \leq_{p<\pi_p>}$ this implies $\forall \langle x, x' \rangle, \langle y, y' \rangle \in X_q. \langle x, x' \rangle <_q \langle y, y' \rangle$. By definition of $<_{p'}$ then $x <_{p'} y$.

If p' have the P_{or}-property it then follows that $\mathbf{p'} \in \delta_{or}(\mathbf{p})$.

Assume that p' does not have the P_{or}-property. That is $X_{p'}$ contain elements x_1, x_2, y_1, y_2 such that:

$$(9) \quad \begin{array}{l} x_1 <_{p'} y_1 \\ co_{p'} \\ x_2 <_{p'} y_2 \end{array} \qquad\qquad (10) \begin{array}{l} x_1 \not<_{p'} y_2 \\ and \\ x_2 \not<_{p'} y_1 \end{array}$$

From the definition of p', the ε-freeness of ϱ and (10) it then follows that there exists x_1', x_2', y_1', y_2' such that (11) below holds. From (9) then also (12):

$$(11) \quad \begin{array}{l} \langle x_1, x_1' \rangle \not<_q \langle y_2, y_2' \rangle \\ and \\ \langle x_2, x_2' \rangle \not<_q \langle y_1, y_1' \rangle \end{array} \qquad\qquad (12) \quad \begin{array}{l} \langle x_1, x_1' \rangle <_q \langle y_1, y_1' \rangle \\ \\ \langle x_2, x_2' \rangle <_q \langle y_2, y_2' \rangle \end{array}$$

But from (11) and (12) it follows that:

$$\langle x_1, x_1' \rangle \; co_q \; \langle x_2, x_2' \rangle$$

and we have a contradiction to the fact that q has the P_{or}-property.

It remains to prove $\mathbf{q} \preceq [p'{<}\pi_{p'}{>}]$ for some ϱ-consistent p. ref., $\pi_{p'}$, for p'. Since $X_p = X_{p'}$, π_p is also a p. ref. for p' and we know that it is ϱ-consistent. For the same reason $X_{p'<\pi_p>} = X_{p<\pi_p>} = X_q$ and similarly $\ell_{p'<\pi_p>} = \ell_q$.

Next we show $\leq_q \supseteq \leq_{p'<\pi_p>}$. Assume $\langle x, x'\rangle \leq_{p'<\pi_p>} \langle y, y'\rangle$. By construction of $p'<\pi_p>$ this implies $x <_{p'} y$ or $(x = y, x' \leq_{\pi_p(x)} y')$. In the former case (8) directly gives $\langle x, x'\rangle <_q \langle y, y'\rangle$ and in the latter case we have $\langle x, x'\rangle <_{p<\pi_p>} \langle x, y'\rangle$ from the construction of $p<\pi_p>$. Since $\leq_q \supseteq \leq_{p<\pi_p>}$ this implies $\langle x, x'\rangle <_q \langle x, y'\rangle$. Hence $\leq_q \supseteq \leq_{p'<\pi_p>}$.
Collecting the facts we can use proposition 2.8 again to conclude $\mathbf{q} \preceq [p'<\pi_p>]$ as desired. $\square$

Lemma 5.11 Let P be a set of pomset and ϱ a $\mathcal{P}(\mathbf{P})$-refinement. Then

$$\pi((\pi(P))<\varrho>) = \pi(P<\varrho>)$$

Proof π is a natural extension to sets of pomsets so it will do to show: $\pi((\pi(\mathbf{p}))<\varrho>) = \pi(\mathbf{p}<\varrho>)$. $\supseteq$: Immediate from $\mathbf{p} \in \pi(\mathbf{p})$.

$\subseteq$: Let a $\mathbf{q} \in \pi((\pi(\mathbf{p}))<\varrho>)$ be given. Then $\mathbf{q} \sqsubseteq \mathbf{r}$ for some $\mathbf{r} \in \mathbf{s}<\varrho>$ where $\mathbf{s} \sqsubseteq \mathbf{p}$. By definition of $\_<\varrho>$, $\mathbf{r} \in \mathbf{s}<\varrho>$ implies there is a ϱ-consistent p. ref. π_s for s with $\mathbf{r} = [s<\pi_s>]$. Since $\mathbf{s} \sqsubseteq \mathbf{p}$ we can by the alternative characterization of $\sqsubseteq$ find a representative p' of $\mathbf{p}$ such that $s = p'|_{X_s}$ and X_s is $\leq_{p'}$-downwards closed. $X_s \subseteq X_{p'}$ so we can extend π_s to a ϱ-consistent p. ref. $\pi_{p'}$ for p'. Because $s = p'|_{X_s}$ and $\pi_{p'}$ equals π_s on X_s we see $s<\pi_s> = p'<\pi_{p'}>|_{X_{s<\pi_s>}}$.

We now show that $X_{s<\pi_s>}$ is $\leq_{p'<\pi_{p'}>}$-downwards closed. Suppose $\langle x, x'\rangle \leq_{p'<\pi_{p'}>} \langle y, y'\rangle$ and $\langle y, y'\rangle \in X_{s<\pi_s>}$. By construction of $p'<\pi_{p'}>$ the former implies $x \leq_{p'} y$. The latter similarly implies $y \in X_s$. Since X_s is $\leq_{p'}$-downwards closed then $x \in X_s$. Now $x' \in X_{\pi_{p'}(x)}$ so because $\pi_{p'}$ equals π_s on X_s we also have $x' \in X_{\pi_s(x)}$. Hence $\langle x, x'\rangle \in X_{s<\pi_s>}$.

Using the alternative characterization of $\sqsubseteq$ again we deduce $[s<\pi_s>] \sqsubseteq [p'<\pi_{p'}>]$. From the transitivity of $\sqsubseteq$, $\mathbf{q} \sqsubseteq \mathbf{r} = [s<\pi_s>]$ and $[p'<\pi_{p'}>] \in \mathbf{p}'<\varrho> = \mathbf{p}<\varrho>$ we then get $\mathbf{q} \in \pi(\mathbf{p}<\varrho>)$ as desired. $\square$

Proposition 5.12 The operators of A^p_{or} and A_{or} are continuous.

Proof The continuity of the A_{or}-operators is derived from the continuity of the A^p_{or}-operators which easely are checked. E.g., to see that $;^p_{or}$ is right continuous let D' be a nonempty subset of A^p_{or} and suppose $\langle P, Q\rangle$ is a member of A^p_{or}.

Let $D = \langle P, Q\rangle \,;^p_{or}\, D' = \{\langle P \cup Q \cdot P', Q \cdot Q'\rangle \mid \langle P', Q'\rangle \in D\}$. Then $D_1 = \{P \cup Q \cdot P' \mid \langle P', Q'\rangle \in D'\} = \{P \cup Q \cdot P'_1 \mid P'_1 \in D'_1\} = P \cup Q \cdot D'_1$ where the last equation follows from $D'_1 \neq \emptyset$ which in turn is a consequence of $D' \neq \emptyset$. Also $D_2 = \{Q \cdot Q' \mid \langle P', Q'\rangle \in D'\} = Q \cdot D'_2$. We then have: $\bigvee^p_{or}(\langle P, Q\rangle \,;^p_{or}\, D') = \langle \bigcup D_1, \bigcup D_2\rangle = \langle \bigcup(P \cup Q \cdot D'_1), \bigcup(Q \cdot D'_2)\rangle = \langle P \cup Q \cdot (\bigcup D'_1), Q \cdot (\bigcup D'_2)\rangle = \langle P, Q\rangle \,;^p_{or} \bigvee^p_{or} D'$.

$\square$

Now were we have showed that A_{or} and A^p_{or} are algebraic cpos and seen that the different operators are continuous on the respective domains, we for $BL\mathcal{R}^{rec}_{\Omega}(X)$ get the denotational maps:

$$\boxed{\begin{aligned} A_{or}[\![\_]\!] &: BL\mathcal{R}^{rec}_{\Omega}(X) \longrightarrow [\mathrm{ENV}_{A_{or}} \longrightarrow A_{or}] \\ A^p_{or}[\![\_]\!] &: BL\mathcal{R}^{rec}_{\Omega}(X) \longrightarrow [\mathrm{ENV}_{A^p_{or}} \longrightarrow A^p_{or}] \end{aligned}}$$

$A^p_{or}[\![\_]\!]_1$ and $A^p_{or}[\![\_]\!]_2$ will be used to refer to the first and second component of $A^p_{or}[\![\_]\!]$ respectively. Notice that if E is a closed expression then $A_{or}[\![E]\!] = A^p_{or}[\![E]\!]_2 \subseteq A^p_{or}[\![E]\!]_1$.

The preorders induced by $A_{or}[\![\_]\!]$ and $A^p_{or}[\![\_]\!]$ will be denoted $\trianglelefteq_{or}$ and $\trianglelefteq^p_{or}$ respectively.

The Finite Sublanguage

In this subsection we shall prove some of the claims (like the alternative characterization $A_{or}[\![\_]\!] = \delta_{or}\wp(\_\sigma))$ stated in the motivation of the models, not only for expressions of $BL\mathcal{R}$, but for the hole finite sublanguage $BL\mathcal{R}_{\Omega}$.

To this end we extend the canonical map, $\wp$, to BL_{Ω} by deriving it from $\wp^p$ needed for the M^p_{or} model.

Definition 5.13 The map $\wp^p : BL_\Omega \longrightarrow \mathcal{P}(\mathbf{P}) \times \mathcal{P}(\mathbf{P})$ is defined inductively:

$$\begin{array}{ll}
\wp^p(\Omega) & = \langle\{\varepsilon\}, \emptyset\rangle \\
\wp^p(a) & = \langle\{\varepsilon, a\}, \{a\}\rangle \\
\wp^p(E_0\,;E_1) & = \langle\wp_1^p(E_0) \cup \wp_2^p(E_0) \cdot \wp_1^p(E_1), \wp_2^p(E_0) \cdot \wp_2^p(E_1)\rangle \\
\wp^p(E_0 \oplus E_1) & = \langle\wp_1^p(E_0) \cup \wp_1^p(E_1), \wp_2^p(E_0) \cup \wp_2^p(E_1)\rangle \\
\wp^p(E_0 \parallel E_1) & = \langle\wp_1^p(E_0) \times \wp_1^p(E_1), \wp_2^p(E_0) \times \wp_2^p(E_1)\rangle
\end{array}$$

where as usual $\wp_1^p(E) = P$ and $\wp_2^p(E) = Q$ if $\wp^p(E) = \langle P, Q\rangle$.
The ordinary canonical map $\wp$ is extended to BL_Ω by $\wp = \wp_2^p$.

Observe that $\forall E \in BL_\Omega.\ \wp_2^p(E) \subseteq \wp_1^p(E)$.

Example: From $\wp^p(\Omega;d) = \langle\{\varepsilon\}, \emptyset\rangle$ it can be seen that $\wp^p((a;b);(\Omega;d\oplus c)) = \langle\{\varepsilon, a, a\rightarrow b, a\rightarrow b\rightarrow c\}, \{a\rightarrow b\rightarrow c\}\rangle$ and $\wp^p((a\,;(\Omega\,;d)\oplus b)\,;c) = \langle\{\varepsilon, a, b, b\rightarrow c\}, \{b\rightarrow c\}\rangle$

One can think of $\wp_1^p$ as the canonical association of pomset prefixes of an expression:

Proposition 5.14

a) If $E \in BL$ then $\wp_1^p(E) = \pi(\wp(E))$.

b) If $E \in BL_\Omega$ then $\wp_1^p(E) = \pi(\wp_1^p(E))$.

Proof By structural induction on E using (7) and $\wp_2^p = \wp$ in the case of $E = E_0\,;E_1$. $\square$

Clearly the definition of $\wp^p$ is designed with the denotational map of the M_{or}^p-model in mind. An easy structural induction in fact shows:

Proposition 5.15 Given an $E \in BL_\Omega$ then $\wp^p(E) = \langle P, Q\rangle$ implies $\varepsilon \notin Q \subseteq P \neq \emptyset$ and P, Q are finite subsets of $\mathbf{P}_{N\text{-free}}$.

As a first step we prove the alternative characterizations of the denotational maps for BL_Ω.

Proposition 5.16 For any $E \in BL_\Omega$:

a) $A_{or}[\![E]\!] = \delta_{or}(\wp(E))$

b) $A^p_{or}[\![E]\!]_i = \delta_{or}(\wp^p_i(E))$ for $i = 1, 2$

Proof a) follows directly by induction on the structure of E using the properties of δ_{or} and in b) we use the fact that $\wp = \wp^p_2$ and $A_{or}[\![E]\!]$ equals $A^p_{or}[\![E]\!]_2$ to see that a) also reads

$$A^p_{or}[\![E]\!]_2 = \delta_{or}(\wp^p_2(E)) \tag{13}$$

Then b) follows from $A^p_{or}[\![E]\!]_1 = \delta_{or}(\wp^p_1(E))$ which also is proven by induction on the structure of E. Here we just show the case $E = E_0\,; E_1$:

$$\begin{aligned}
A^p_{or}[\![E]\!]_1 &= A^p_{or}[\![E_0]\!]_1 \cup A^p_{or}[\![E_0]\!]_2 \cdot A^p_{or}[\![E_1]\!]_1 && \text{definition of } A^p_{or}[\![_]\!] \\
&= \delta_{or}(\wp^p_1(E_0)) \cup \delta_{or}(\wp^p_2(E_0)) \cdot \delta_{or}(\wp^p_1(E_1)) && \text{induction and (13)} \\
&= \delta_{or}(\wp^p_1(E_0) \cup \wp^p_2(E_0) \cdot \wp^p_1(E_1)) && \text{distributivity of } \delta_{or} \\
&= \delta_{or}(\wp^p_1(E_0\,;E_1)) = \delta_{or}(\wp^p_1(E)) && \text{definition of } \wp^p \quad \square
\end{aligned}$$

A simple consequence of this proposition and proposition 5.14 is:

Corollary 5.17 $A^p_{or}[\![E]\!] = \langle \delta_{or}\pi(\wp(E)), \delta_{or}(\wp(E))\rangle$ for every $E \in BL$

With the results obtained so far we are already able to show that the models are surjective.

Proposition 5.18 Every compact element of A^p_{or} and A_{or} is the denotation of a finite expression.

Proof Again the result for A_{or} is easely derived from the corresponding proof for A^p_{or}. To see this we for a given compact element $a \in A^p_{or}$ just find an expression $E \in BL_\Omega \subseteq BL\mathcal{R}_\Omega$ such that $A_{or}[\![E]\!] = a$. Recall at first that a is an element of A^p_{or} in the M^p_{or} model when

$$a = \langle \delta_{or}\pi(s), \delta_{or}(t)\rangle \tag{14}$$

where s and t are two finite sets of N-free pomsets such that $\varepsilon \notin t \subseteq s \neq \emptyset$.

Now if u is an arbitrary finite and nonempty set of N-free pomset such that $\varepsilon \notin u$ we from the last corollary and proposition 5.5 deduce there exists an $E_u \in BL$ with

$$A^p_{or}[\![E_u]\!] = \langle \delta_{or}\pi(u), \delta_{or}(u)\rangle \tag{15}$$

Now let a compact element a like (14) be given. We deal with different cases of s and t:

$\varepsilon \notin s$ and $t = \emptyset$: Take $E = E_s \,; \Omega \in BL_\Omega$, where E_s is a BL-expression fulfilling (15).

$\varepsilon \notin s$ and $t \neq \emptyset$: Because $\varepsilon \notin t$ and $s \neq \emptyset$ we can then find $E_s, E_t \in BL$ fulfilling (15) and $E = (E_s \,; \Omega) \oplus E_t \in BL_\Omega$ can be used.

$s = \{\varepsilon\}$: From $t \subseteq s$ and $\varepsilon \notin t$ follows $t = \emptyset$ so $E = \Omega$ will do.

$\varepsilon \in s$ and $s \setminus \{\varepsilon\} \neq \emptyset$: Then no matter whether $t = \emptyset$ or $t \neq \emptyset$ we can as above find an $E' \in BL_\Omega$ such that $A^p_{or}[\![E']\!] = \langle \delta_{or}\pi(s \setminus \{\varepsilon\}), \delta_{or}(t)\rangle$. Letting $E = \Omega \oplus E'$ we get $A^p_{or}[\![E]\!] = \langle \delta_{or}\pi(\{\varepsilon\} \cup (s \setminus \{\varepsilon\})), \delta_{or}(\emptyset \cup t)\rangle = \langle \delta_{or}\pi(s), \delta_{or}(t)\rangle$.

Inspecting how s and t can be for compact elements of A^p_{or} like (14) we see that all cases are covered. $\square$

Proposition 5.19 For every $E \in BL\mathcal{R}_\Omega$ we have:

a) $A_{or}[\![E]\!] = A_{or}[\![E\sigma]\!]$

b) $A^p_{or}[\![E]\!] = A^p_{or}[\![E\sigma]\!]$

Proof a) goes as b) which is a simple induction on the structure of E except for the case $E = F[\varrho]$ which goes as follows:

$A^p_{or}[\![E]\!] = (A^p_{or}[\![F]\!])[\varrho]^p_{or}$	definition of $A^p_{or}[\![\_]\!]$
$\quad = (A^p_{or}[\![F\sigma]\!])[\varrho]^p_{or}$	induction
$\quad = (A^p_{or}[\![(F\sigma)\{\varrho\}]\!])$	lemma 5.20 and $F\sigma \in BL_\Omega$
$\quad = (A^p_{or}[\![(F[\varrho]\sigma)]\!]) = A^p_{or}[\![E]\!]$	definition of σ $\square$

Lemma 5.20 If $E \in BL_\Omega$ then

a) $A_{or}[\![E\{\varrho\}]\!] = (A_{or}[\![E]\!])[\varrho]_{or}$

b) $A^p_{or}[\![E\{\varrho\}]\!] = (A^p_{or}[\![E]\!])[\varrho]^p_{or}$

Proof At first a) is proven by structural induction (following the same line as b) but without the complication of an extra component).

From a) and the definition of $[\varrho]^p_{or}$ we as usual deduce

$$(16) \qquad A^p_{or}[\![E\{\varrho\}]\!]_2 = \delta_{or}(A^p_{or}[\![E]\!]_2{<}\wp(\varrho){>})$$

With this we then by induction on the structure of $E \in BL_\Omega$ prove

$$A^p_{or}[\![E\{\varrho\}]\!]_1 = \delta_{or}\pi(A^p_{or}[\![E]\!]_1{<}\wp(\varrho){>})$$

from which b) then follows using (16). We just show the cases $E = a$ and $E = E_0\,;E_1$:

$E = a$ Then:

$$\begin{array}{ll}
\quad A^p_{or}[\![a\{\varrho\}]\!]_1 = A^p_{or}[\![\varrho(a)]\!]_1 & \text{definition of } \{\varrho\} \\
= \delta_{or}(\wp^p_1(\varrho(a))) & \varrho(a) \in BL \text{ and proposition 5.16} \\
= \delta_{or}\pi(\wp(\varrho(a))) & \wp^p_1 = \pi \circ \wp \text{ and proposition 5.14} \\
= \delta_{or}\pi((\wp(\varrho))(a)) & \text{definition of } \wp(\varrho) \\
= \delta_{or}\pi(\{\varepsilon, a\}{<}\wp(\varrho){>}) & \text{proposition 2.6} \\
= \delta_{or}\pi(A^p_{or}[\![a]\!]_1{<}\wp(\varrho){>}) &
\end{array}$$

$E = E_0\,;E_1$ We get:

$$\begin{array}{ll}
\quad A^p_{or}[\![E\{\varrho\}]\!]_1 = A^p_{or}[\![E_0\{\varrho\}\,;E_1\{\varrho\}]\!]_1 & \text{definition of } \{\varrho\} \\
= A^p_{or}[\![E_0\{\varrho\}]\!]_1 \cup A^p_{or}[\![E_0\{\varrho\}]\!]_2 \cdot A^p_{or}[\![E_1\{\varrho\}]\!]_1 & \text{definition of } A^p_{or}[\![_]\!] \\
= \delta_{or}\pi(A^p_{or}[\![E_0]\!]_1{<}\wp(\varrho){>}) \cup & \\
\quad \delta_{or}(A^p_{or}[\![E_0]\!]_2{<}\wp(\varrho){>}) \cdot \delta_{or}\pi(A^p_{or}[\![E_1]\!]_1{<}\wp(\varrho){>}) & \text{induction and (16)} \\
= \delta_{or}\pi(A^p_{or}[\![E_0]\!]_1{<}\wp(\varrho){>} \cup & \text{proposition 5.9 and} \\
\quad A^p_{or}[\![E_0]\!]_2{<}\wp(\varrho){>} \cdot A^p_{or}[\![E_1]\!]_1{<}\wp(\varrho){>}) & A^p_{or}[\![E_0]\!]_2 \subseteq A^p_{or}[\![E_0]\!]_1 \\
= \delta_{or}\pi((A^p_{or}[\![E_0]\!]_1 \cup A^p_{or}[\![E_0]\!]_2 \cdot A^p_{or}[\![E_1]\!]_1){<}\wp(\varrho){>}) & \text{proposition 2.6} \\
= \delta_{or}\pi(A^p_{or}[\![E_0\,;E_1]\!]_1) & \text{definition of } A^p_{or}[\![_]\!]
\end{array}$$

$\Box$

Proposition 5.21 The denotation of a finite expression is a compact element.

Proof The proof for the M^p_{or} model is exemplary for the corresponding for the M_{or} model. Suppose $E \in BL_\Omega$. Then

$$\begin{array}{ll}
A^p_{or}[\![E]\!] = \langle \delta_{or}(\wp^P_1(E)), \delta_{or}(\wp(^p_2(E)) \rangle & \text{proposition 5.16} \\
\qquad = \langle \delta_{or}\pi(\wp^P_1(E)), \delta_{or}(\wp(^p_2(E)) \rangle & \text{proposition 5.14}
\end{array}$$

By proposition 5.15 it then follows that $A^p_{or}[\![E]\!] \in \mathrm{Fin}(A^p_{or})$. Now if $E \in BL\mathcal{R}_\Omega$ then by proposition 5.19 $A^p_{or}[\![E]\!] = A^p_{or}[\![E\sigma]\!]$ and because $E\sigma \in BL_\Omega$ it follows that $A^p_{or}[\![E]\!]$ denotes a compact element in A^p_{or}. $\square$

6 Full Abstractness

In this section we connect the denotational semantics with the operational through full abstractness results which are obtained by lifting via algebraicity of the involved preorders the corresponding results for the finite sublanguage.

As mentioned in the motivation of the behavioural process equivalenceses in section 4 we are after the largest precongruence contained in the relevant preorder. Of course we want the obtained preorder to be a precongruence not only w.r.t. the ordinary combinators but also w.r.t. to the recursive combinators. If this shall be nontrivial the operational preorders have to be extended to open expressions. This is usually done in what might be called the substitutive way:

$$E_0 \lesssim E_1 \textit{ iff } \text{ for every closed syntactic substitution } \rho,\ E_0\rho \lesssim E_1\rho$$

Similar for $\sqsubseteq\!\!\!\!\sim$. As for equivalences, we for a preorder, $\sqsubseteq$, use $\sqsubseteq^\Sigma$ to denote the largest Σ-precongruence contained in $\sqsubseteq$.

Theorem 6.1 The following denotations are fully abstract:

a) $A_{or}[\![\_]\!]$ on $BL\mathcal{R}^{rec}_\Omega(X)$ w.r.t. $\lesssim^c$

b) $A^p_{or}[\![\_]\!]$ on $BL\mathcal{R}^{rec}_\Omega(X)$ w.r.t. $\sqsubseteq\!\!\!\!\sim^c$

Proof At first we draw the attention to the easely derivable general fact (see [Eng89]) that if $\Sigma \subseteq \Sigma'$ and $\sqsubseteq^\Sigma$ agrees with some Σ'-precongruence then $\sqsubseteq^\Sigma = \sqsubseteq^{\Sigma'}$. Because the denotational preorders qua induced by the denotational maps, are precongruences w.r.t. all the combinators (the recursion combinators inclusive), it is then enough to show the theorem to hold where the operational precongruences now are understood to be the largest w.r.t. the ordinary combinators. These shall in the sequel ambiguously be denoted $\lesssim^c$ and $\sqsubseteq\!\!\!\!\sim^c$.

Since the domains are finitary (proposition 5.18 and 5.21) the associated denotational induced preorders are by proposition 5.3 then substitutive as well as algebraic. The different operational preorders are by definition substitutive from which it follows that the associated precongruences are substitutive too, so if we can manage to show that the involved operational precongruences are algebraic and agrees with the denotational preorders on the closed finite sublanguage the theorem clearly follows.

From theorem 6.4 we know that $\lesssim$ and $\sqsubseteq\!\!\!\sim$ are algebraic over $BL\mathcal{R}^{rec}_{\Omega}$. Since theorem 6.14 gives the corresponding full abstractness results for the finite sublanguage it only remains to show the operational precongruences (w.r.t. the ordinary combinators) are algebraic. Both $\lesssim$ and $\sqsubseteq\!\!\!\sim$ are algebraic and by theorem 6.15 $BL\mathcal{R}_{\Omega}$ is expressive w.r.t. both preorders (restricted to $BL\mathcal{R}_{\Omega}$). Theorem 6.3 then gives us that $\lesssim^c$ and $\sqsubseteq\!\!\!\sim^c$ are algebraic. $\square$

We shall now prove all the propositions we used to get the full abstractness results. In order to introduce the idea of a language being expressive we need the notion of contexts.

When considering a language a context, $\mathcal{C}$, is normally thought of as an expression with zero or more "holes", to be filled by some other expression of the language. Strictly speaking $\mathcal{C}$ is not an expression of the language, but if we think of a "hole" as a special constant symbol, e.g., $[\,]$, a context will be an expression of the language extended with this constant and the filling, $\mathcal{C}[E]$, of a context $\mathcal{C}$ with an expression E, is obtained by replacing the special constant with E. This allows us to use the syntactic precongruence $\preceq$ on contexts just as we do on ordinary expressions and for example prove that if $\mathcal{C}$ and $\mathcal{C}'$ are $FREC_{\Sigma}(X)$-contexts and E, F are $REC_{\Sigma}(X)$ expressions then

$$E \preceq F \textit{ implies } \mathcal{C}[E] \preceq \mathcal{C}[F] \tag{17}$$

$$\mathcal{C} \preceq \mathcal{C}' \textit{ implies } \mathcal{C}[E] \preceq \mathcal{C}'[E] \tag{18}$$

With contexts we are also able to to give an alternative characterization of $\sqsubseteq^{\Sigma}$:

$$E \sqsubseteq^{\Sigma} F \textit{ iff } \forall \mathcal{L}_{\Sigma}\text{-contexts } \mathcal{C}.\, \mathcal{C}[E] \sqsubseteq \mathcal{C}[F], \tag{19}$$

where E, F belongs to a language $\mathcal{L}$ and $\mathcal{L}_{\Sigma} \subseteq \mathcal{L}$ is the language obtained from the signature Σ.

Definition 6.2 Given a preorder, $\sqsubseteq$, over a language $\mathcal{L}$ and a subset $A \subseteq \mathcal{L}$. $\mathcal{L}$ is said to be A-*expressive* w.r.t. $\sqsubseteq$ *iff* for every $E \in \mathcal{L}$ there exists a *characteristic* $\mathcal{L}$-*context* $\mathcal{C}_E[\,]$ such that

$$\forall F \in A.\, E \sqsubseteq^c F \text{ iff } \mathcal{C}_E[E] \sqsubseteq \mathcal{C}_E[F]$$

where c is the combinators of $\mathcal{L}$. If $A = \mathcal{L}$ then $\mathcal{L}$ is simply said to be expressive w.r.t. $\sqsubseteq$.

Theorem 6.3 Let $\sqsubseteq$ be an algebraic preorder over REC_Σ containing the syntactic preorder $\preceq$. If $FREC_\Sigma$ is $\mathrm{Fin}(E)$-expressive w.r.t $\sqsubseteq$ (restricted to $FREC_\Sigma$) for every $E \in REC_\Sigma$ then $\sqsubseteq^\Sigma$ is algebraic too.

Proof Given $E, F \in REC_\Sigma$ we show

$$E \sqsubseteq^\Sigma F \Leftrightarrow \forall E' \in \mathrm{Fin}(E) \exists F' \in \mathrm{Fin}(F).\, E' \sqsubseteq^\Sigma F'$$

$\Leftarrow$: Assume $\forall E' \in \mathrm{Fin}(E) \exists F' \in \mathrm{Fin}(F).\, E' \sqsubseteq^\Sigma F'$. By (19) it is enough to show $\mathcal{C}[E] \sqsubseteq \mathcal{C}[F]$ for any given $FREC_\Sigma$-context $\mathcal{C}$. So suppose $\mathcal{C}$ is such a context. Let an $E'' \in \mathrm{Fin}(\mathcal{C}[E])$ be given. Then there is an $FREC_\Sigma$-context $\mathcal{C}' \preceq \mathcal{C}$ and an $E' \in \mathrm{Fin}(E)$ such that $E'' \preceq \mathcal{C}'[E']$. By assumption there is an $F' \in \mathrm{Fin}(F)$ with $E' \sqsubseteq^\Sigma F'$ and so also $\mathcal{C}'[E'] \sqsubseteq \mathcal{C}'[F']$. Clearly $\mathcal{C}'[F'] \in FREC_\Sigma$ and from $F' \preceq F$ it follows by (17) and (18) that $\mathcal{C}'[F'] \preceq \mathcal{C}[F'] \preceq \mathcal{C}[F]$ so we actually have $\mathcal{C}'[F'] \in \mathrm{Fin}(\mathcal{C}[F])$. $\preceq \subseteq \sqsubseteq$ and the transitivity of $\sqsubseteq$ gives $E'' \sqsubseteq \mathcal{C}'[F'] =: F''$. Hence for every $E'' \in \mathrm{Fin}(\mathcal{C}[E])$ we have found an $F'' \in \mathrm{Fin}(\mathcal{C}[F])$ such that $E'' \sqsubseteq F''$. Because $\sqsubseteq$ is algebraic this implies $\mathcal{C}[E] \sqsubseteq \mathcal{C}[F]$ as we wanted.

$\Rightarrow$: Assume $E \sqsubseteq^\Sigma F$ and let an $E' \in \mathrm{Fin}(E)$ be given. We shall find an $F' \in \mathrm{Fin}(F)$ such that $E' \sqsubseteq^\Sigma F'$. Since $E' \in FREC_\Sigma$ and $FREC_\Sigma$ is $\mathrm{Fin}(F)$-expressive there for (this E') is an $FREC_\Sigma$ context, $\mathcal{C}$, such that for all $F' \in \mathrm{Fin}(F)$

$$\mathcal{C}[E'] \sqsubseteq \mathcal{C}[F'] \text{ iff } E' \sqsubseteq^\Sigma F'$$

Let $\mathcal{C}$ be such a characteristic context for E'. We then just have to find a $F' \in \mathrm{Fin}(F)$ such that $\mathcal{C}[E'] \sqsubseteq \mathcal{C}[F']$. Since $E' \preceq E$ we by (17) have $\mathcal{C}[E'] \preceq \mathcal{C}[E]$ and because $\mathcal{C}$ is an $FREC_\Sigma$-context this gives $\mathcal{C}[E'] \in \mathrm{Fin}(\mathcal{C}[E])$. $E \sqsubseteq^\Sigma F$ implies $\mathcal{C}[E] \sqsubseteq \mathcal{C}[F]$ and by the algebraicity of $\sqsubseteq$ we deduce there must be an $F'' \in \mathrm{Fin}(\mathcal{C}[F])$ such that $\mathcal{C}[E'] \sqsubseteq F''$.

Because $F'' \in \mathrm{Fin}(\mathcal{C}[F])$ we can then find a $\mathcal{C}' \preceq \mathcal{C}$ and an $F' \in \mathrm{Fin}(F)$ with $F'' \preceq \mathcal{C}'[F']$. By (18) $F'' \preceq \mathcal{C}'[F'] \preceq \mathcal{C}[F']$ and from $\preceq \subseteq \sqsubseteq$ and transitivity of $\sqsubseteq$ we obtain $\mathcal{C}[E'] \sqsubseteq \mathcal{C}[F']$ as desired. $\Box$

Algebraicity of the Operational Preoders

In order to prove the algebraicity of the operational preoders we extend the syntactic preorder, $\preceq$, to $BL\mathcal{R}_\Omega^{rec}(X)$ in the obvious way. I.e., $\preceq$ is extended to $CL\mathcal{R}_\Omega^{rec}(X)$ simply by letting $\preceq$ be the least relation over $CL\mathcal{R}_\Omega^{rec}(X)$ which satisfies the rules below:

$$E \preceq E \qquad \Omega \preceq E \qquad E[rec\,x.\,E/x] \preceq E$$

$$\frac{E \preceq F, F \preceq G}{E \preceq G} \qquad \frac{E_0 \preceq F_0, E_1 \preceq F_1}{\begin{array}{c} E_0 \,;\, E_1 \preceq F_0 \,;\, F_1 \\ E_0 \oplus E_1 \preceq F_0 \oplus F_1 \\ E_0 \parallel E_1 \preceq F_0 \parallel F_1 \end{array}} \qquad \frac{E \preceq F}{E[\varrho] \preceq F[\varrho]}$$

Notice that in this way we may only have $E \preceq F[\varrho]$ if E and F comes from $BL\mathcal{R}_\Omega^{rec}(X)$. It is also important to notice that $\dagger \preceq E$ implies $E = \dagger$ and that $\preceq$ contains the old precongruence over $BL\mathcal{R}_\Omega^{rec}(X)$.

Theorem 6.4 The preorders $\lesssim$ and $\sqsubseteq\!\!\!\sim$ over $BL\mathcal{R}_\Omega^{rec}$ are algebraic.

Proof The preorder $\lesssim$ is proved algebraic in exactly the same way as we now will prove $\sqsubseteq\!\!\!\sim$ algebraic. For $\sqsubseteq\!\!\!\sim$ we shall prove: $E \sqsubseteq\!\!\!\sim F$ *iff* $\forall E' \in \mathrm{Fin}(E) \exists F' \in \mathrm{Fin}(F).\, E' \sqsubseteq\!\!\!\sim F'$

if: Assume the right hand side holds and let an $s \in \Delta^*$ be given such that $E \overset{s}{\Rrightarrow}$. We prove $F \overset{s}{\Rrightarrow}$. Proposition 6.5 below gives an $E' \in \mathrm{Fin}(E)$ with $E' \overset{s}{\Rrightarrow}$. By assumption there is also an $F' \in \mathrm{Fin}(F)$ such that $E' \sqsubseteq\!\!\!\sim F'$. Hence $F' \overset{s}{\Rrightarrow}$ and using the same proposition again then $F \overset{s}{\Rrightarrow}$.

only if: Assume $E \sqsubseteq\!\!\!\sim F$ and let an $E' \in \mathrm{Fin}(E)$ be given.
Similar as in the *if*-part we can use the assumption and proposition 6.5 to show that for each $s \in \Delta^*$ such that $E' \overset{s}{\Rrightarrow}$ we can pick an $F_s \in \mathrm{Fin}(F)$ with $F_s \overset{s}{\Rrightarrow}$.
Now for any $H \in BL_\Omega$ it is an easy matter to prove by induction on the structure of H that $\{s \in \Delta^* \mid H \overset{s}{\Rrightarrow}\}$ is finite. By proposition 4.3 we

have $\{s \in \Delta^* \mid E' \overset{s}{\Rightarrow}\} = \{s \in \Delta^* \mid E'\sigma \overset{s}{\Rightarrow}\}$, so because $E'\sigma \in BL_\Omega$ we conclude $\{F_s \in \mathrm{Fin}(F) \mid E' \overset{s}{\Rightarrow}\}$ is finite too.
$\mathrm{Fin}(F)$ is directed w.r.t. $\preceq$ wherefore there is an ub $F' \in \mathrm{Fin}(F)$ for $\{F_s \mid E' \overset{s}{\Rightarrow}\}$. By proposition 6.6 $\preceq \subseteq \sqsubsetsim$ this therefore means that for every F_s, F' can perform s. But there is exactly one F_s for each $E' \overset{s}{\Rightarrow}$ wherefore we conclude $E' \sqsubsetsim F'$. $\square$

Proposition 6.5 Given $E \in BL\mathcal{R}_\Omega^{rec}$. Then

a) $E \overset{s}{\Rightarrow} \dagger$ *iff* $\exists E' \in \mathrm{Fin}(E).\, E' \overset{s}{\Rightarrow} \dagger$

b) $E \overset{s}{\Rightarrow}$ *iff* $\exists E' \in \mathrm{Fin}(E).\, E' \overset{s}{\Rightarrow}$

Proof $E' \in \mathrm{Fin}(E)$ means $E' \preceq E$ and $E' \in BL\mathcal{R}_\Omega$, so the *if*-part of a) and b) are just special cases of the following proposition. *only if*:

a) Suppose $E \overset{s}{\Rightarrow} \dagger$. Because $\dagger \succeq \dagger \in CL\mathcal{R}_\Omega$ we can use lemma 6.7 to find $E', F' \in CL\mathcal{R}_\Omega$ such that $E \succeq E' \overset{s}{\Rightarrow} F' \succeq \dagger$. $\dagger \preceq F'$ only if $F' = \dagger$ so this means $E \succeq E' \overset{s}{\Rightarrow} \dagger$. Now $E' \preceq E \in BL\mathcal{R}_\Omega^{rec}$ clearly implies $E' \in BL\mathcal{R}_\Omega^{rec}$ wherefore we from $E' \in CL\mathcal{R}_\Omega$ deduce $E' \in BL\mathcal{R}_\Omega$ and thus $E' \in \mathrm{Fin}(E)$.

b) Suppose $E \overset{s}{\Rightarrow}$. This means $E \overset{s}{\Rightarrow} F$ for some $F \in CL\mathcal{R}_\Omega^{rec}$. Using $F \succeq \Omega$ the rest goes as under a). $\square$

Proposition 6.6 $\lesssim$ and $\sqsubsetsim$ extends $\preceq$ on $BL\mathcal{R}_\Omega^{rec}$.

Proof We shall show that when $\preceq$ is restricted to $BL\mathcal{R}_\Omega^{rec}$ then $\preceq \subseteq \lesssim$ and $\preceq \subseteq \sqsubsetsim$. So let $E, F \in BL\mathcal{R}_\Omega^{rec}$ be given such that $E \preceq F$. $\sqsubsetsim$ is immediate from lemma 6.10 and for $\lesssim$ assume $E \overset{s}{\Rightarrow} \dagger$. By lemma 6.10 there is an F' such that $F \overset{s}{\Rightarrow} F' \succeq \dagger$. Since $\dagger \preceq F'$ only if $F' = \dagger$ we are done. $\square$

We now show that if a (possible recursive) process is able to perform a sequence, then there is a finite approximation which also can do this sequence.

Lemma 6.7 Suppose $E \in CL\mathcal{R}_\Omega^{rec}$. Then

$$E \overset{s}{\Rightarrow} F \succeq F'' \in CL\mathcal{R}_\Omega \text{ implies } \exists E', F' \in CL\mathcal{R}_\Omega.\, E \succeq E' \overset{s}{\Rightarrow} F' \succeq F''$$

Proof By induction on the size of $\stackrel{s}{\Rightarrow}$. In the basic case we have $E = F$ and can choose $E' = F' = F''$. In the inductive step there are two main cases:

$E \rightarrowtail G \stackrel{s}{\Rightarrow}' F \succeq F''$: (where $\stackrel{s}{\Rightarrow} = \rightarrowtail\stackrel{s}{\Rightarrow}'$ and the length of $\stackrel{s}{\Rightarrow}'$ is less than that of $\stackrel{s}{\Rightarrow}$) By hypothesis of induction there are $G', H \in CL\mathcal{R}_\Omega$ such that $G \succeq G' \stackrel{s}{\Rightarrow} H \succeq F''$. Now $E \rightarrowtail G \succeq G'$ implies by lemma 6.8 below the existence of $E', G'' \in CL\mathcal{R}_\Omega$ with $E \succeq E' \rightarrowtail^* G'' \succeq G'$. We can then use lemma 6.10 on $G'' \succeq G' \stackrel{s}{\Rightarrow} H$ to find an F' which fulfills $G'' \stackrel{s}{\Rightarrow} F' \succeq H$. Collecting the facts so far we have $E \succeq E' \rightarrowtail^* G'' \stackrel{s}{\Rightarrow} F' \succeq H \succeq F''$ and so $E \succeq E' \stackrel{s}{\Rightarrow} F' \succeq F''$. For $E' \in CL\mathcal{R}_\Omega$ we easely prove $E' \stackrel{s}{\Rightarrow} F'$ implies $F' \in CL\mathcal{R}_\Omega$ so this case is settled.

$E \xrightarrow{a} G \stackrel{s'}{\Rightarrow} F \succeq F''$: Similar but using lemma 6.9 in place of lemma 6.8. $\square$

Lemma 6.8 If $E \in CL\mathcal{R}_\Omega^{rec}$ then

$$E \rightarrowtail F \succeq F'' \in CL\mathcal{R}_\Omega \text{ implies } \exists E', F' \in CL\mathcal{R}_\Omega.\ E \succeq E' \rightarrowtail^* F' \succeq F''$$

Proof If $F'' = \Omega$ the lemma follows by choosing $E' = F' = \Omega \in CL\mathcal{R}_\Omega$. Hence we do not have to consider cases where $F'' = \Omega$ when we prove the lemma by induction on the size, m, of the internal step $E \rightarrowtail_m F$. This means there is a proof of $E \rightarrowtail F$ from the rules of $\rightarrowtail$ with no more than m stages. See [Win85] for the details. Since $\rightarrowtail_0 = \emptyset$ the basic case is trivial.

We now assume the lemma holds for m when proving it for $m + 1$ by considering the different rules.

$E = \Omega \rightarrowtail_{m+1} \Omega = F \succeq F''$: Not considered.

$E = E_0 \,;\, E_1 \rightarrowtail_{m+1} F \succeq F''$: There are two subcases:

$E_0 = \dagger$ and $F = E_1$: Let $E' = \dagger \,;\, F'' \in CL\mathcal{R}_\Omega$ and $F' = F''$.

$F = F_0 \,;\, E_1$ where $E_0 \rightarrowtail_m F_0$: When $F'' \neq \Omega$ it can then be argued that $F'' \preceq F_0 \,;\, E_1$ implies $F'' = F_0'' \,;\, E_1''$ for some $F_0'' \preceq F_0$ and $E_1'' \preceq E_1$. By hypothesis of induction there are $E_0', F_0' \in CL\mathcal{R}_\Omega$ with $E_0 \succeq E_0 \rightarrowtail^* F_0' \succeq F_0''$. Because $F'' \in CL\mathcal{R}_\Omega$ implies $E_1'' \in BL\mathcal{R}_\Omega$ we then have $E' := E_0' \,;\, E_1'' \in CL\mathcal{R}_\Omega$ and $F' := F_0' \,;\, E_1'' \in CL\mathcal{R}_\Omega$. Also $E' \rightarrowtail^* F'$ and $E' = E_0' ; E_1'' \preceq E_0 ; E_1'' \preceq E_0 ; E_1$ so as $F'' = F_0'' \,;\, E_1'' \preceq F_0' \,;\, E_1'' = F'$.

$E = E_0 \oplus E_1, E_0 \parallel E_1 \rightarrowtail_{m+1} F \succeq F''$: Similar.

$E = G[\varrho] \rightarrowtail_{m+1} F \succeq F''$: There are six subcases to be dealt with.

$G = \Omega$ and $F = \Omega$: Not considered since $\Omega \succeq F''$ only if $F'' = \Omega$.

$G = a$ and $F = \varrho(a)$: Then $E, F \in BL_\Omega$. Chose $E' = E$ and $F' = F$.

$G = G_0\,;G_1$ and $F = G_0[\varrho]\,;G_1[\varrho]$: When F'' is different from Ω we can from $F \succeq F'' \in BL\mathcal{R}_\Omega$ deduce $F'' = F''_0\,;F''_1$ where for $i = 0, 1$ either $(F''_i = \Omega)$ or $(F''_i = G''_i[\varrho]$ and $G_i \succeq G''_i \in BL\mathcal{R}_\Omega)$. There are actually four subcases to consider, but we just treat $F''_0 = G''_0[\varrho]$ and$F''_1 = \Omega$ because the other follow in the same way. Choose $E' = (G''_0;\Omega)[\varrho] \in BL\mathcal{R}_\Omega$ and $F' = G''_0[\varrho];\Omega[\varrho] \in BL\mathcal{R}_\Omega$. Then clearly $E' \rightarrowtail F'$ and $E' \preceq (G_0\,;\Omega)[\varrho] \preceq (G_0\,;G_1)[\varrho] = E$ and also $F'' = G'_0[\varrho]\,;\Omega \preceq G''_0[\varrho]\,;\Omega[\varrho] = F'$.

$G = G_0 \oplus G_1$ and $G = G_0 \parallel G_1$: Analogous to the last case.

$F = H[\varrho]$ where $G \rightarrowtail_m H$: Now $\Omega \neq F'' \preceq H[\varrho]$ only if $F'' = H''[\varrho]$ for some $H'' \preceq H$. By hypothesis of induction there are $G', H' \in BL\mathcal{R}_\Omega$ such that $G \succeq G' \rightarrowtail^* H' \succeq H''$. Now $G' \succeq G \in BL\mathcal{R}^{rec}_\Omega$ and $G' \in CL\mathcal{R}_\Omega$ implies $G' \in BL\mathcal{R}_\Omega$ and similar for H' so we obtain $E' := G'[\varrho] \in BL\mathcal{R}_\Omega$, $F' := H'[\varrho] \in BL\mathcal{R}_\Omega$ and $E' \rightarrowtail^* F'$. Clearly $E' \preceq E$ and $F'' = H''[\varrho] \preceq H'[\varrho] = F'$.

$E = rec\,x.\,G \rightarrowtail_{m+1} G[rec\,x.\,G/x] = F \succeq F''$: Choose $E' = F' = F'' \in CL\mathcal{R}_\Omega$. Then of course $E' \rightarrowtail^0 F' \succeq F' = F''$ and because $E' \preceq F = G[rec\,x.\,G/x] \preceq rec\,x.\,G = E$ we also have $E' \preceq E$. □

Lemma 6.9 If $E \in CL\mathcal{R}^{rec}_\Omega$ and $a \in \Delta$ then

$$E \xrightarrow{a} F \succeq F'' \in CL\mathcal{R}_\Omega \text{ implies } \exists E', F' \in CL\mathcal{R}_\Omega.\, E \succeq E' \xrightarrow{a} F' \succeq F''$$

Proof At first the lemma is proven for the case $F'' \neq \Omega$. This will be done by induction on the size, m, of $E \xrightarrow{a}_m F$. Only the inductive step needs attention. We consider each rule in turn under the assumption $F'' \neq \Omega$ and that the lemma holds for m.

$E = a \xrightarrow{a}_{m+1} \dagger = F \succeq F''$: Clearly $A = a$ and $F'' = \dagger$. Choose $E' = a$ and $F' = \dagger$.

$E = E_0\,;E_1 \xrightarrow{a}_{m+1} F_0\,;E_1 = F$ where $E_0 \xrightarrow{a}_m F_0$: $\Omega \neq F'' \preceq F_0\,;E_1$ implies $F'' = F''_0\,;E''_1$ where $F_0 \succeq F''_0 \in CL\mathcal{R}_\Omega$ and $E_1 \succeq E''_1 \in BL\mathcal{R}_\Omega$.

By induction then $\exists E_0' \in CL\mathcal{R}_\Omega.\ E_0 \succeq E_0' \xrightarrow{a} F_0' \succeq F_0''$. Letting $E' = E_0' \,;\, E_1''$ we have $E' \in CL\mathcal{R}_\Omega$ and $E' \preceq E_0' \,;\, E_1 \preceq E$ and using the same inference rule finally $E' = E_0' \,;\, E_1'' \xrightarrow{a} F_0' \,;\, E_1'' =: F'$ and also $F'' = F_0'' \,;\, E_1'' \preceq F'$.

$E = E_0 \parallel E_1 \xrightarrow{a}_{m+1} F_0 \parallel F_1 = F \succeq F''$: Similar/ symmetric.

Now from the rules of $\xrightarrow{a}$ obviously $E \xrightarrow{a} F$ only if $\dagger$ occurs in F. By structural induction on F an $F''' \in CL\mathcal{R}_\Omega$ can then be found such that $F \succeq F''' \neq \Omega$. As above appropriate $E', F' \in CL\mathcal{R}_\Omega$ are found for F'''. When $F'' = \Omega$ we have $F''' \succeq F''$ so this case is dealt with. $\square$

Up til now we have showed that if a process is able to perform a sequence, then there is a finite approximation which also can do this sequence. Now we take the opposite angel and show that if E' is an approximation of E then E can do all the sequences E' can. A stronger formulation of this is:

Lemma 6.10 Suppose $E, E' \in CL\mathcal{R}_\Omega^{rec}$. Then

$$E \succeq E' \overset{s}{\Rightarrow} F' \text{ implies } \exists F.\ E \overset{s}{\Rightarrow} F \succeq F'$$

Proof As usual by induction on the size of $\overset{s}{\Rightarrow}$ using the analogous for single steps, namely that given $E, E' \in CL\mathcal{R}_\Omega^{rec}$ we have:

a) $E \succeq E' \rightarrowtail F'$ implies $\exists F.\ E \rightarrowtail^* F \succeq F'$

b) $E \succeq E' \xrightarrow{a} F'$ implies $\exists F.\ E \overset{a}{\Rightarrow} F \succeq F'$

If $E' \preceq E$ and E' by a single step evolves to F' we cannot expect that E immediately by a similar step can evolve into F with $F \succeq F'$. This is because $E' \preceq E$ can imply that some of the recursive subexpressions of E have been "unwound" by $\preceq$ in order to obtain an expression equal to E' up to Ω at some places in E'. However by the recursion rule for $\rightarrowtail$ it is possible to do one unwinding, so given $E' \preceq E$ we would ideally like to unwind E soley by internal unwinding steps, $\overset{u}{\rightarrowtail}$, to an E'' which equals E' up to Ω. Then we could be sure that whatever single step E' could do, E'' would be able to do similarly (perhaps with some extra internal steps). There is however the snag about it that the definition of $\rightarrowtail$ *does not* open up for unwinding in the right hand argument of the ;-combinator and neither in the arguments of the $\oplus$-combinator. We shall therefore introduce $E'' \preceq^u E'$ to mean that except for such unwindings E'' is equal to E' up to Ω (both $\preceq^u$ and $\stackrel{u}{\rightarrowtail} \subseteq \rightarrowtail$ are formalized immediately after the proof). With this we then both for a) and b) at first use lemma 6.11 to find an $E'' \succeq^u E'$ such that $E \stackrel{u}{\rightarrowtail}^* E''$. Finally we use lemma 6.13 to find an $F \succeq^u F'$ such that in case of a) $E'' \rightarrowtail^* F$ and in case of b) $E'' \xrightarrow{a} F$. $\square$

We define the subpreorder, $\preceq^u \subseteq \preceq$ as the least relation over $CL\mathcal{R}^{rec}_{\Omega}(X)$ which can be inferred from the rules:

$$\begin{array}{c} E \preceq^u E \\ \Omega \preceq^u E \end{array} \qquad \frac{E \preceq^u F, F \preceq^u G}{E \preceq^u G} \qquad \frac{E \preceq^u F}{E[\varrho] \preceq^u F[\varrho]}$$

$$\frac{E_0 \preceq^u F_0, E_1 \preceq F_1}{E_0 \,;\, E_1 \preceq^u F_0 \,;\, F_1} \qquad \frac{E_0 \preceq F_0, E_1 \preceq F_1}{E_0 \oplus E_1 \preceq^u F_0 \oplus F_1} \qquad \frac{E_0 \preceq^u F_0, E_1 \preceq^u F_1}{E_0 \parallel E_1 \preceq^u F_0 \parallel F_1}$$

Example: $(rec\, y.\, E)\,;(a \parallel rec\, x.\,(a \parallel x)) \preceq^u (rec\, y.\, E)\,; rec\, x.\, a \parallel x$ but $(a \parallel rec\, x.\,(a \parallel x))\,; rec\, y.\, E \not\preceq^u (rec\, x.\, a \parallel x)\,; rec\, y.\, E$

This definition of $\preceq^u$ deserves some remarks. The preorder $\preceq$ is used in the premisses of the ;- and $\oplus$-inference rule just in order to capture the unwindings which cannot be done by internal steps. There is no rule for $rec\, x.$. This reflects that the expressions are equal up to Ω (except of course in connection with ; and $\oplus$). Another consequence is that, as opposed to $\preceq$, if $E \preceq^u F$ and $E \neq \Omega$ we can conclude F is on the same form as E with components related according to the rules of $\preceq^u$. E.g., $E_0\,;E_1 \preceq^u F$ implies $F = F_0\,;F_1$, $E_0 \preceq^u F_0$ and $E_1 \preceq F_1$. Also $rec\, x.\, E \preceq^u F$ implies $F = rec\, x.\, E$.

We write $E \stackrel{u}{\rightarrowtail} F$ for an internal step that solely originate in an unwinding of a recursive subexpression. Formally $\stackrel{u}{\rightarrowtail} \subseteq \rightarrowtail$ is defined to be the least relation over $CL\mathcal{R}^{rec}_{\Omega}$ which can be deduced from $rec\, x.\, E \stackrel{u}{\rightarrowtail} E[rec\, x.\, E/x]$ and the $\stackrel{u}{\rightarrowtail}$ equivalent versions of the $\rightarrowtail$ inference rules.

Lemma 6.11 If $E, E' \in CL\mathcal{R}^{rec}_{\Omega}$ then $E \succeq E'$ implies $\exists F.\, E \stackrel{u}{\rightarrowtail}^* F \succeq^u E'$.

Proof By induction on the number of rules used in the proof of $E' \preceq E$. There are three case in the basis of which the most interesting is: $E' = G[rec\,x.\,G/x] \preceq rec\,x.\,G = E$. By the recursion rule for $\stackrel{u}{\rightarrowtail}$ it is seen that $E \stackrel{u}{\rightarrowtail} G[rec\,x.\,G/x] = E' \succeq^u E'$ so we can choose $F = E'$.

Now for the inductive step there are five ways $E' \preceq E$ could have been obtained.

$E' \preceq E'', E'' \preceq E$: By hypothesis of induction there are F' and F'' such that $E'' \stackrel{u}{\rightarrowtail}^* F' \succeq^u E'$ and $E \stackrel{u}{\rightarrowtail}^* F'' \succeq^u E''$. From lemma 6.12 below we know that $F'' \succeq^u E'' \stackrel{u}{\rightarrowtail}^* F'$ implies the existence of an F such that $F'' \stackrel{u}{\rightarrowtail}^* F \succeq^u F'$. Then we actually have $E \stackrel{u}{\rightarrowtail}^* F'' \stackrel{u}{\rightarrowtail}^* F \succeq^u F' \succeq^u E'$ as we want.

$E' = E_0' \,;\, E_1', E = E_0 \,;\, E_1$ and $E_0' \preceq E_0, E_1' \preceq E_1$: Using the inductive hypothesis on $E_0 \succeq E_0'$ we find an F_0 such that $E_0 \stackrel{u}{\rightarrowtail}^* F_0 \succeq^u E_0'$. Then $E = E_0 \,;\, E_1 \stackrel{u}{\rightarrowtail}^* F_0 \,;\, E_1$ and since $E_1' \preceq E_1$ we by definition of $\preceq^u$ actually have $E' = E_0' \,;\, E_1' \preceq^u F_0 \,;\, E_1$ and we can let $F = F_0 \,;\, E_1$.

$E' = E_0' \oplus E_1', E = E_0 \oplus E_1$ and $E_0' \preceq E_0, E_1' \preceq E_1$: Then also $E \preceq^u E'$ so we can choose $F = E$ because $E \stackrel{u}{\rightarrowtail}^0 F = E \succeq^u E'$.

$E' = E_0' \parallel E_1', E = E_0 \parallel E_1$ and $E_0' \preceq E_0, E_1' \preceq E_1$: By induction there for $i = 0, 1$ exists an F_i such that $E_i \stackrel{u}{\rightarrowtail}^* F_i \succeq^u E_i'$, so we get $E = E_0 \parallel E_1 \stackrel{u}{\rightarrowtail}^* F_0 \parallel F_1$. Letting $F = F_0 \parallel F_1$ we have $F \succeq^u E_0' \parallel E_1' = E'$.

$E' = G'[\varrho], E = G[\varrho]$ and $G' \preceq G$ (and $G, G' \in BL\mathcal{R}_\Omega^{rec}$): Like above we find an H such that $G \stackrel{u}{\rightarrowtail}^* H \succeq^u G'$. By definition of $\preceq^u$ we then have $F := H[\varrho] \succeq^u G'[\varrho] = E'$ and of course $E \stackrel{u}{\rightarrowtail}^* F$. $\square$

Lemma 6.12 If $E, E' \in CL\mathcal{R}_\Omega^{rec}$ then

$$E \succeq^u E' \stackrel{u}{\rightarrowtail}^* F' \text{ implies } \exists F.\, E \stackrel{u}{\rightarrowtail}^* F \succeq^u F'$$

Proof By induction on the number of unwinding steps using:

$$(20) \qquad E \succeq^u E' \stackrel{u}{\rightarrowtail} F' \Rightarrow \exists F.\, E \stackrel{u}{\rightarrowtail} F \succeq^u F'$$

which in turn is proven by induction on the size, m, of $E' \stackrel{u}{\rightarrowtail}_m F'$. We can assume (20) holds for m when proving (20) for $m+1$. The different rules are handled one by one:

$E' = rec\,x.\,G \stackrel{u}{\rightarrowtail}_{m+1} G[rec\,x.\,G/x] = F'$: From $E \succeq^u rec\,x.\,G$ follows $E = rec\,x.\,G$. Let $F = F'$.

$E' = E'_0 \,;\, E'_1 \overset{u}{\rightarrowtail}_{m+1} F'_0 \,;\, E'_1 = F'$ where $E'_0 \overset{u}{\rightarrowtail}_m F'_0$: $E'_0 \,;\, E'_1 \preceq^u E$ implies $E = E_0 \,;\, E_1$ where $E'_0 \preceq^u E_0$ and $E'_1 \preceq E_1$. We can then use the hypothesis of induction to get an F_0 with $E_0 \overset{u}{\rightarrowtail} F_0 \succeq^u F'_0$. Then also $E = E_0 \,;\, E_1 \overset{u}{\rightarrowtail} F_0 \,;\, E_1 \succeq^u F'_0 \,;\, E'_1 = F'$ and we can choose $F = F_0 \,;\, E_1$.

$E' = E'_0 \parallel E'_1 \overset{u}{\rightarrowtail}_{m+1} F'$: Similar/ symmetric as the rule for ;.

$E' = G'[\varrho] \overset{u}{\rightarrowtail}_{m+1} H'[\varrho] = F'$ where $G' \overset{u}{\rightarrowtail}_m H'$: $E \succeq^u G'[\varrho]$ only if $E = G[\varrho]$ and $G' \preceq^u G$, so by induction $G \overset{u}{\rightarrowtail} H \succeq^u H'$ for some H and we get $E \overset{u}{\rightarrowtail} H[\varrho] \succeq^u H'[\varrho] = F'$ as desired.

□

Lemma 6.13 Suppose $E, E' \in CL\mathcal{R}^{rec}_{\Omega}$ and $a \in \Delta$. Then

a) $E \succeq^u E' \rightarrowtail F'$ implies $\exists F.\ E \rightarrowtail^* F \succeq F'$

b) $E \succeq^u E' \xrightarrow{a} F'$ implies $\exists F.\ E \xrightarrow{a} F \succeq F'$

Proof a) By induction on the size, m, of the internal step $E' \rightarrowtail_m F'$. The basic case is trivial and in the inductive case the lemma can be assumed to be true for all internal steps of size m. We now investigate all the rules.
Using the fact that $\preceq^u \subseteq \preceq$ the inference rules are handled exactly as in the proof of lemma 6.12. E.g., $E' = G'[\varrho] \rightarrowtail_{m+1} H'[\varrho] = F'$ where $G' \rightarrowtail_m H'$. $E \succeq^u G'[\varrho]$ only if $E = G[\varrho]$ where $G' \preceq^u G$, so by hypothesis of induction then $G \rightarrowtail^* H$ for some $H \succeq H'$. By definition of $\preceq$ we have $F := H[\varrho] \succeq H'[\varrho] = F'$ and thus also $E = G[\varrho] \rightarrowtail^* H[\varrho] = F$. We will therefore just look at the ordinary rules for $\rightarrowtail$.

$E' = \Omega \rightarrowtail_{m+1} \Omega = F'$: Then $E' = F'$ and we can choose $E = F$. Then $E \rightarrowtail^0 F = E \succeq^u E' = F'$ and since $\preceq^u \subseteq \preceq$ we are done.

$E' = \dagger \,;\, E'_1 \rightarrowtail_{m+1} E'_1 = F'$: $\dagger;E_1 \preceq^u E$ implies $E = \dagger;E_1$ where $E'_1 \preceq E_1$. With $F = E_1$ we then get $E = \dagger \,;\, E_1 \rightarrowtail F = E_1 \succeq E'_1 = F'$.

$E = E'_0 \oplus E'_1 \rightarrowtail_{m+1} F'$: Suppose w.l.o.g. $F' = E'_0$. $E'_0 \oplus E'_1 \preceq^u E$ only if $E = E_0 \oplus E_1$ where $E'_0 \preceq E_0$ and $E'_1 \preceq E_1$. But then also $E \rightarrowtail E_0 \succeq E'_0 = F'$.

$E' = E'_0 \parallel E'_1 \rightarrowtail_{m+1} F'$: Similar/ symmetric as the case with $E' = \dagger \,;\, E'_1$ but with the additional use of $\preceq^u \subseteq \preceq$.

$E' = G'[\varrho] \rightarrowtail_{m+1} F'$: then $E' \preceq^u E$ means $E = G[\varrho]$ where $G' \preceq^u G$. There are five ordinary rules according to the structure of G':

$G' = \Omega$ and $F' = \Omega$: Let $F = E$. Since $E' \preceq^u E$ implies $E' \preceq E$ we then get $E \rightarrowtail^0 F = E \succeq E' = \Omega[\varrho] \succeq \Omega = F'$.

$G' = a$ and $F' = \varrho(a)$: $a \preceq^u G$ only if $G = a$, so we actually have $E = E'$ and one can choose $F = F'$.

$G' = G'_0 \,; G'_1$ and $F' = G'_0[\varrho] \,; G'_1[\varrho]$: $G'_0 \,; G'_1 \preceq^u G$ implies $G = G_0 \,; G_1$ where $G'_0 \preceq^u G_1$ and $G'_1 \preceq G_1$. Again since $\preceq^u \subseteq \preceq$ we by letting $F = G_0[\varrho] ; G_1[\varrho]$ get $F' \preceq F$ and also $E = (G_0 ; G_1)[\varrho] \rightarrowtail G_0[\varrho] \,; G_1[\varrho] = F$.

$G' = G'_0 \oplus G'_1$ and $G' = G'_0 \parallel G'_1$: Similar as last case.

b) By induction on the size of the step $E' \xrightarrow{a} F'$. The proof follows exactly the line of a). □

The Finite Sublanguage

In this subsection we give the full abstractness results for $BL\mathcal{R}_\Omega$ and show the expressiveness of $BL\mathcal{R}_\Omega$ w.r.t. $\lesssim$ and $\precsim$.

Theorem 6.14 The following denotations are fully abstract:

a) $A_{or}[\![\_]\!]$ on $BL\mathcal{R}_\Omega$ w.r.t. $\lesssim^c$

b) $A^p_{or}[\![\_]\!]$ on $BL\mathcal{R}_\Omega$ w.r.t. $\precsim^c$

Proof a) By definition $\lesssim^c \subseteq BL\mathcal{R}_\Omega \times BL\mathcal{R}_\Omega$ is a precongruence w.r.t. the combinators of $BL\mathcal{R}_\Omega$. We then just have to show $\trianglelefteq_{or} = \lesssim^c$. By (19) this follows if we can prove for all $E_0, E_1 \in BL\mathcal{R}_\Omega$

$$E_0 \trianglelefteq_{or} E_1 \textit{ iff } \forall BL\mathcal{R}_\Omega\text{-contexts } \mathcal{C}.\, \mathcal{C}[E_0] \lesssim \mathcal{C}[E_1]$$

only if: Assume $E_0 \trianglelefteq_{or} E_1$ and let a $BL\mathcal{R}_\Omega$-context, $\mathcal{C}$, be given. $\trianglelefteq_{or}$ is a precongruence w.r.t. the combinators of $BL\mathcal{R}_\Omega$ so by structural induction $\mathcal{C}[E_0] \trianglelefteq_{or} \mathcal{C}[E_1]$ or equally $A_{or}[\![\mathcal{C}[E_0]]\!] \subseteq A_{or}[\![\mathcal{C}[E_1]]\!]$. From the $\subseteq$-monotonicity of δ_{w} then $\delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_0]]\!]) \subseteq \delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_1]]\!])$ which by proposition 6.16.a) implies $\mathcal{C}[E_0] \lesssim \mathcal{C}[E_1]$.

if: Assume $E_0 \not\trianglelefteq_{or} E_1$ or equally $A_{or}[\![E_0]\!] \not\subseteq A_{or}[\![E_1]\!]$. From a) of lemma 6.18 we see there is a $BL\mathcal{R}_\Omega$-context, $\mathcal{C}$, such that $\delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_0]]\!]) \not\subseteq \delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_1]]\!])$. Then also $\mathcal{C}[E_0] \not\lesssim \mathcal{C}[E_1]$ by proposition 6.16.a).

b) Similar to a) using b) of proposition 6.16 and lemma 6.18, and in the only if part recalling the definition of $\trianglelefteq^p_{or}$ to deduce $\delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_0]]\!]_1) \subseteq \delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_1]]\!]_1)$ from $\mathcal{C}[E_0] \trianglelefteq^p_{or} \mathcal{C}[E_1]$. $\square$

Theorem 6.15 $BL\mathcal{R}_\Omega$ is expressive w.r.t. both $\lesssim$ and $\underset{\sim}{\sqsubset}$.

Proof $\lesssim$: Suppose $E_0 \in BL\mathcal{R}_\Omega$. Let $\mathcal{C}$ be the $BL\mathcal{R}_\Omega$-context, $[\,][\varrho]$, found by a) of lemma 6.18. Given any $E_1 \in BL\mathcal{R}_\Omega$ we show

$$E_0 \lesssim^c E_1 \textit{ iff } \mathcal{C}[E_0] \lesssim \mathcal{C}[E_1]$$

only if: Since $\lesssim^c$ by definition is a precongruence it follows that $\mathcal{C}[E_0] \lesssim^c \mathcal{C}[E_1]$. Again by definition of $\lesssim^c$ also $\lesssim^c \subseteq \lesssim$.

if: $\mathcal{C}[E_0] \lesssim \mathcal{C}[E_1]$

$\Rightarrow \delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_0]]\!]) \subseteq \delta_{\mathsf{w}}(A_{or}[\![\mathcal{C}[E_1]]\!])$	proposition 6.16.a)
$\Rightarrow A_{or}[\![E_0]\!] \subseteq A_{or}[\![E_1]\!]$	by choice of $\mathcal{C}$
$\Rightarrow E_0 \trianglelefteq_{or} E_1$	definition of $\trianglelefteq_{or}$
$\Rightarrow E_0 \lesssim^c E_1$	by the theorem above

$\underset{\sim}{\sqsubset}$: Similar as for $\lesssim$ but using the $BL\mathcal{R}_\Omega$-context $[\,][\varrho];e$ from b) of lemma 6.18. $\square$

Proposition 6.16 For all $E_0, E_1 \in BL\mathcal{R}_\Omega$:

a) $\delta_{\mathsf{w}}(A_{or}[\![E_0]\!]) \subseteq \delta_{\mathsf{w}}(A_{or}[\![E_1]\!])$ *iff* $E_0 \lesssim E_1$

b) $\delta_{\mathsf{w}}(A^p_{or}[\![E_0]\!]_1) \subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_1]\!]_1)$ *iff* $E_0 \underset{\sim}{\sqsubset} E_1$

Proof a) follows with exactly the same arguments as b) which in return follows from the definition of $\underset{\sim}{\sqsubset}$ and the general deduction ($E \in BL\mathcal{R}_\Omega$)

$\delta_{\mathsf{w}}(A^p_{or}[\![E]\!]_1) = \delta_{\mathsf{w}}(A^p_{or}[\![E\sigma]\!]_1)$	proposition 5.19
$= \delta_{\mathsf{w}}(\wp^p_1(E\sigma))$	proposition 5.16 and $E\sigma \in BL_\Omega$
$= \{w \in W \mid E\sigma \overset{w}{\Rightarrow}\}$	by b) of lemma 6.17 below
$= \{w \in W \mid E \overset{w}{\Rightarrow}\}$	proposition 4.3

$\square$

In reading the following lemma recall our convention to identify Δ^* and W.

Lemma 6.17 Given $E \in BL_\Omega$ and $s \in \Delta^*$. Then

a) $E \stackrel{s}{\Rightarrow} \dagger$ *iff* $\exists \mathbf{p} \in \wp(E).\, s \preceq \mathbf{p}$

b) $E \stackrel{s}{\Rightarrow}$ *iff* $\exists \mathbf{p} \in \wp_1^p(E).\, s \preceq \mathbf{p}$

Proof a) Before we prove each implication notice that $\mathbf{p} \in \wp(E)$ implies $\mathbf{p} \neq \varepsilon$.

if: By induction on the structure of E.

$E = \Omega$: Then $\wp(E) = \emptyset$ and we cannot have $\mathbf{p} \in \wp(E)$.

$E = a$: $\wp(a) = \{a\}$ and we have $\mathbf{p} = a$. Clearly $a \preceq a$ implies $s = a$. The result then follows from $a \stackrel{a}{\Rightarrow} \dagger$.

$E = E_0\,;E_1$: From $\wp(E) = \wp(E_0) \cdot \wp(E_1)$ we see $\mathbf{p} = \mathbf{p}_0 \cdot \mathbf{p}_1$ where $\mathbf{p}_i \in \wp(E_i)$ for $i = 0, 1$. $s \preceq \mathbf{p}_0 \cdot \mathbf{p}_1$ implies the existence of $s_0 \preceq \mathbf{p}_0$ and $s_1 \preceq \mathbf{p}_1$ such that $s = s_0 \cdot s_1$. By hypothesis of induction then $E_0 \stackrel{s_0}{\Rightarrow} \dagger$ and $E_1 \stackrel{s_1}{\Rightarrow} \dagger$ and so $E_0\,;E_1 \stackrel{s_0}{\Rightarrow} \dagger\,;E_1 \rightarrowtail E_1 \stackrel{s_1}{\Rightarrow} \dagger$ as desired.

$E = E_0 \oplus E_1$: $\mathbf{p} \in \wp(E) = \wp(E_0) \cup \wp(E_1)$ implies w.l.o.g. $\mathbf{p} \in \wp(E_0)$. By hypothesis of induction then also $E_0 \oplus E_1 \rightarrowtail E_0 \stackrel{s}{\Rightarrow} \dagger$.

$E = E_0 \parallel E_1$: $\mathbf{p} \in \wp(E) = \wp(E_0) \times \wp(E_1)$ implies $\mathbf{p} = \mathbf{p}_0 \times \mathbf{p}_1$ for some $\mathbf{p}_0 \in \wp(E_0)$ and $\mathbf{p}_1 \in \wp(E_1)$. It can be shown that $s \preceq \mathbf{p}_0 \times \mathbf{p}_1$ implies the existence of $s_0 \preceq \mathbf{p}_0$ and $s_1 \preceq \mathbf{p}_1$ such that s is the interleaving of s_0 and s_1. Hence we can use the hypothesis of induction to see $E_i \stackrel{a_i}{\Rightarrow} \dagger$ for $i = 0, 1$. By appropriate interleaved usage of the $\parallel$-rules for $\stackrel{a}{\longrightarrow}$ we then get $E_0 \parallel E_1 \stackrel{s}{\Rightarrow} \ldots \stackrel{a_n}{\Longrightarrow} \dagger \parallel \dagger \rightarrowtail \dagger$.

only if: Also by induction on the structure of E.

$E = \Omega$: Trivial because $\Omega \stackrel{s}{\Rightarrow} F$ only if $s = \varepsilon$.

$E = a$: a can only do the step $a \stackrel{a}{\longrightarrow} \dagger$ so $s = a$. But $s = a \preceq a \in \{a\} = \wp(a)$.

$E = E_0\,;E_1$: The only way a process of the form $E_0\,;E_1$ can evolve to $\dagger$ is if $E_0 \stackrel{s_0}{\Rightarrow} \dagger$ and $E_1 \stackrel{s_1}{\Rightarrow} \dagger$, so we must have $s = s_0 \cdot s_1$. By hypothesis then for $i = 0, 1$ $s_i \preceq \mathbf{p}_i$ where $\mathbf{p}_i \in \wp(E_i)$. By $\preceq$-monotonicity of $\cdot$ then $s = s_0 \cdot s_1 \preceq s_0 \cdot \mathbf{p}_1 \preceq \mathbf{p}_0 \cdot \mathbf{p}_1 \in \wp(E_0) \cdot \wp(E_1)$.

$E = E_0 \oplus E_1$: Inspecting the definition of $\rightarrowtail$ and $\xrightarrow{a}$ one easely sees that $E_0 \oplus E_1 \overset{s}{\Rightarrow} \dagger$ implies $E_0 \oplus E_1 \rightarrowtail F \overset{s}{\Rightarrow} \dagger$ where $F = E_0$ or $F = E_1$. The result then follows from the hypothesis of induction and definition of $\wp$.

$E = E_0 \parallel E_1$: From the $\parallel$-rules $E_0 \parallel E_1 \overset{s}{\Rightarrow} \dagger$ only if $E_0 \parallel E_1 \overset{s}{\Rightarrow} \dagger \parallel \dagger$ and s is the interleaving of some s_0 and s_1 such that $E_i \overset{s_i}{\Rightarrow} \dagger$. Using the hypothesis of induction together with (4), $(\mathbf{p} \times \mathbf{q}) \cdot (\mathbf{p}' \times \mathbf{q}') \preceq (\mathbf{p} \cdot \mathbf{p}') \times (\mathbf{q} \cdot \mathbf{q}')$, the desired result is then obtained similarly as in the case $E = E_0 \,;\, E_1$.

b) Here we use a) and the fact $\wp = \wp_2^p$.

if: By induction on the structure of E.

$E = \Omega$: $\wp_1^p(\Omega) = \{\varepsilon\}$ and we must have $s = \mathbf{p} = \varepsilon$. But $E \rightarrowtail^0 E$.

$E = a$: $\wp_1^p(a) = \{a\}$. There are two possibilities for $\mathbf{p}$—either $\mathbf{p} = \varepsilon$ or $\mathbf{p} = a$. The former case goes as above and the latter as in the corresponding case of a).

$E = E_0 \,;\, E_1$: $\wp_1^p(E) = \wp_1^p(E_0) \cup \wp_2^p(E_0) \cdot \wp_1^p(E_1)$. If $\mathbf{p} \in \wp_1^p(E_0)$ the result follows from hypothesis of induction. Otherwise $\mathbf{p}$ must equal $\mathbf{p}_0 \cdot \mathbf{p}_1$ where $\mathbf{p}_0 \in \wp_2^p(E_0)$ and $\mathbf{p}_1 \in \wp_1^p(E_1)$. From $s \preceq \mathbf{p}_0 \cdot \mathbf{p}_1$ follows $s = s_0 \cdot s_1$ where $s_0 \preceq \mathbf{p}_0$ and $s_1 \preceq \mathbf{p}_1$. Since $\mathbf{p}_0 \in \wp_2^p(E_0)$ we can use a) to get $E_0 \overset{s_0}{\Rightarrow} \dagger$. From $s_1 \preceq \mathbf{p}_1 \in \wp_1^p(E_1)$ we by hypothesis of induction also have $E_1 \overset{s_1}{\Rightarrow}$. Finally we get $E_0 ; E_1 \overset{s_0}{\Rightarrow} \dagger ; E_1 \rightarrowtail E_1 \overset{s_1}{\Rightarrow}$.

$E = E_0 \oplus E_1$ and $E = E_0 \parallel E_1$: On expressions of this form $\wp_1^p$ is defined like $\wp_2^p$ so the arguments are identical to those of a).

only if: Also by induction on the structure of E.

$E = \Omega$: Ω can only perform internal steps wherefore $s = \varepsilon$. But $\varepsilon \preceq \varepsilon \in \{\varepsilon\} = \wp_1^p(\Omega)$.

$E = a$: a can only do the step $a \xrightarrow{a} \dagger$ so either $s = \varepsilon$ or $s = a$. In both cases we have $s \preceq s \in \{\varepsilon, a\} = \wp_1^p(a)$.

$E = E_0 \,;\, E_1$: For $E_0 \,;\, E_1 \overset{s}{\Rightarrow} F$ there are two cases:

$$E_0 \overset{s_0}{\Rightarrow} \dagger,\ E_1 \overset{s_1}{\Rightarrow} F, \text{ where } s = s_0 \cdot s_1,\ \textit{or}$$
$$E_0 \overset{s}{\Rightarrow} F' \text{ for some } F' \text{ such that } F' \,;\, E_1 \rightarrowtail^* F.$$

In the latter case we can apply the hypothesis of induction to find a $\mathbf{p} \in \wp_1^p(E_0)$ such that $s \preceq \mathbf{p}$. As $\wp_1^p(E_0) \subseteq \wp_1^p(E_0 \,;\, E_1)$ this case is settled. In the former case we can use a) to find a $\mathbf{p}_0 \in \wp_2^p(E_0)$ with

$s_0 \preceq \mathbf{p}_0$ and by induction there is a $\mathbf{p}_1 \in \wp_1^p(E_1)$ such that $s_1 \preceq \mathbf{p}_1$. From the $\preceq$-monotonicity of $\cdot$ we then deduce $s = s_0 \cdot s_1 \preceq \mathbf{p}_0 \cdot \mathbf{p}_1 \in \wp_2^p(E_0) \cdot \wp_1^p(E_0) \subseteq \wp_1^p(E_0\,;E_1)$ as we want.

$E = E_0 \oplus E_1$ and $E = E_0 \parallel E_1$: Similar arguments as in a). $\square$

Before proving the lemma giving the characteristic contexts used to show full abstractness and $BL\mathcal{R}_\Omega$ expressive, we need to formalize the notion of *fission* refinement formulated in section 5 when finding the denotational models.

Our notation for fission refinements, which splits an atomic action into two, is inspired by Hennessy [Hen87]. Now let a finite multiplicity function, m, be given and define $n(m) = \max\{k \mid k = 1 \text{ or } \exists a \in \Delta.\, m(a) = k\} \in I\!N^+$. Since Δ is infinite, but countable, there exists an injective function $h : \Delta \times \{S, F\} \times \{1, \ldots, n(m)\} \rightarrow \Delta$. For convenience we shall abbreviate $h(\langle a, S, k\rangle)$ by a_{S_k} and $h(\langle a, F, k\rangle)$ by a_{F_k}.

With such a function we associate a BL-refinement, ϱ, by defining for all $a \in \Delta$:

$$\varrho(a) = a_{S_1}\,; a_{F_1} \oplus \ldots \oplus a_{S_{n(m)}}\,; a_{F_{n(m)}}$$

and call it an m-*fission* refinement.
The corresponding ε-free $\mathcal{P}(\mathbf{P})$-refinement, (ambiguously denoted) ϱ, has

$$\varrho(a) = \{a_{S_1} \cdot a_{F_1}, \ldots, a_{S_{n(m)}} \cdot a_{F_{n(m)}}\}$$

and is also called an m-*fission* refinement.

We shall refer to a_{S_k} and a_{F_k} as a fission *pair* of the m-fission refinement ϱ. I.e., the pair a_{S_k} and a_{F_k} is a fission of a.

With such refinements a ϱ-consistent p. ref., π_p, for an lpo p, corresponds to a certain choice of óne fission pair, a_{S_k} and a_{F_k}, for each $a \in \Delta$ and a-occurrence in p (where an a-occurrence in p is an $x \in X_p$ with $\ell_p(x) = a$). Thus we can define two injective functions, $\text{-}_S^{\pi_p}, \text{-}_F^{\pi_p} : X_p \rightarrow X_{p<\pi_p>}$, which (together) for an $x \in X_p$ yield the occurrence in $X_{p<\pi_p>}$ of the corresponding fission pair. I.e., $x_S^{\pi_p}$ (respectively $x_F^{\pi_p}$) is that element $\langle x, x'\rangle$ where $x' \in X_{\pi_p(x)}$ and $\ell_{\pi_p(x)}(x') = a_{S_k}$ (respectively a_{F_k}) for some $1 \leq k \leq n(m)$, $a = \ell_p(x)$. On the other hand it is clear from the construction of $p{<}\pi_p{>}$ that if $z \in X_{p<\pi_p>}$ is labelled a_{S_k} then there is an unique $x \in X_p$ with $x_S^{\pi_p} = z$. Similar for a_{F_k}. We will drop the superscript, π_p, when it is clear from the context.

However in order to be able to distinguish the fission pairs associated with different a-occurrences certain ϱ-consistent p. ref.'s are of special interest.

Suppose p is an lpo with $m_p \leq m$. Then there clearly are ϱ-consistent p. ref.'s, π_p, injective in the sense:

$$\forall x, y \in X_p.\, x \neq y \Rightarrow [\pi_p(x)] \neq [\pi_p(y)]$$

We call such a π_p for a *distinguishing* ϱ-consistent particular *fission* ref. for p.

We say that an lpo q is *p-reflecting* under the distinguishing p. fission ref., π_p, if and only if any pair of concurrent elements from p have overlapping Start/ Finish (fission pairs) occurrences in q, formally: *iff* $q = \langle X_{p<\pi_p>}, \leq_q, \ell_{p<\pi_p>} \rangle, \leq_q \supseteq \leq_{p<\pi_p>}$ (so $\mathbf{q} \preceq [p{<}\pi_p{>}] \in \mathbf{p}{<}\varrho{>}$) and for all $x, y \in X_p$:

$$\text{if} \quad \begin{matrix} x \\ co_p \\ y \end{matrix} \quad \text{then} \quad \begin{matrix} x_S <_q y_F \\ \textit{and} \\ y_S <_q x_F \end{matrix}$$

With this notation we can then say for pomsets $\mathbf{q}'$ and $\mathbf{p}'$ that $\mathbf{q}'$ is $\mathbf{p}'$-*reflecting* under the fission refinement ϱ *iff* there are representatives p and q of $\mathbf{p}'$ and $\mathbf{q}'$ respectively together with a distinguishing ϱ-consistent p. fission ref., π_p, such that q is p-reflecting under π_p

Lemma 6.18 Given an expression $E_0 \in BL\mathcal{R}_\Omega$. Then there is a refinement combinator, $[\varrho]$,

a) such that for all $E_1 \in BL\mathcal{R}_\Omega$

$$A_{or}[\![E_0]\!] \not\subseteq A_{or}[\![E_1]\!] \Rightarrow \delta_{\mathsf{w}}(A_{or}[\![E_0[\varrho]]\!]) \not\subseteq \delta_{\mathsf{w}}(A_{or}[\![E_1[\varrho]]\!])$$

b) and an action $e \in \Delta$ such that for all $E_1 \in BL\mathcal{R}_\Omega$

$$A^p_{or}[\![E_0]\!] \not\subseteq A^p_{or}[\![E_1]\!] \Rightarrow \delta_{\mathsf{w}}(A^p_{or}[\![E_0[\varrho]\,;e]\!]_1) \not\subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]\,;e]\!]_1)$$

Proof a) Let m be the finite multiplicity function which is the lub for $\{m_{\mathbf{p}} \mid \mathbf{p} \in A_{or}[\![E_0]\!]\}$ (finite set). Choose an m-fission refinement ϱ. The associated refinement combinator, $[\varrho]$, is the one we are after. To see this let an arbitrary $E_1 \in BL\mathcal{R}_\Omega$ be given such that $A_{or}[\![E_0]\!] \not\subseteq A_{or}[\![E_1]\!]$. The proof is by contradiction. Assume on the contrary $\delta_{\mathsf{w}}(A_{or}[\![E_0[\varrho]]\!]) \subseteq \delta_{\mathsf{w}}(A_{or}[\![E_1[\varrho]]\!])$. $A_{or}[\![E_0]\!] \not\subseteq A_{or}[\![E_1]\!]$ only if there is a $\mathbf{p} \in A_{or}[\![E_0]\!]$ such that $\mathbf{p} \notin A_{or}[\![E_1]\!]$. $\mathbf{p} \in A_{or}[\![E_0]\!]$ implies $P_{or}(\mathbf{p})$ and by definition also $m_{\mathbf{p}} \leq m$. By lemma 6.20 there is a $w \in \delta_{\mathsf{w}}(\mathbf{p}{<}\varrho{>})$ which is $\mathbf{p}$-reflecting.

Now $w \in \delta_{\mathsf{w}}(\mathbf{p}{<}\varrho{>})$ and $\mathbf{p} \in A_{or}[\![E_0]\!]$ implies w in $\delta_{\mathsf{w}}(A_{or}[\![E_0]\!]{<}\varrho{>})$ which, because $\delta_{\mathsf{w}} \circ \delta_{or} = \delta_{\mathsf{w}}$, equals $\delta_{\mathsf{w}}(\delta_{or}(A_{or}[\![E_0]\!]{<}\varrho{>}))$. By definition of $\_[\varrho]_{or}$ then also $w \in \delta_{\mathsf{w}}(A_{or}[\![E_0[\varrho]]\!])$ and so $w \in \delta_{\mathsf{w}}(A_{or}[\![E_1[\varrho]]\!])$ by the assumption. Reversing the arguments we find a $\mathbf{q} \in A_{or}[\![E_1]\!]$ such that w is a linearization of a pomset, $\mathbf{r}$, of $\mathbf{q}{<}\varrho{>}$. Because w is $\mathbf{p}$-reflecting we then deduce from lemma 6.19 that $\mathbf{p} \preceq \mathbf{q}$. Since $P_{or}(\mathbf{p})$ and $A_{or}[\![E_1]\!]$ is δ_{or}-closed then $\mathbf{p} \in A_{or}[\![E_1]\!]$—a contradiction.

b) Let $E_0 \in BL\mathcal{R}_\Omega$ be given. As for $A_{or}[\![\_]\!]$ we are after a fission refinement, ϱ, such that any pomset, $\mathbf{p}$, associated with the denotation of E_0 can be reflected in a linearization of $\mathbf{q} \in \mathbf{p}{<}\varrho{>}$, but this time with the additional requirement that e does not occur in any pomset which steems from a ${<}\varrho{>}$-refinement of a pomset associated with the denotation of an arbitrary $E_1 \in BL\mathcal{R}_\Omega$. Since E_1 can be any finite expression there are practical no limitations on what singleton pomsets there may be in a pomset from its denotation. We can therefore just as well pick an arbitrary $e \in \Delta$ and seek a fission refinement ϱ for E_0 such that

$$\forall a \in \Delta.\, e \notin L(\varrho(a)) \tag{21}$$

Let m be the lub of the multiplicity functions of the pomsets of $A^p_{or}[\![E_0]\!]$, i.e., $m = \bigvee\{m_{\mathbf{p}} \mid \mathbf{p} \in A^p_{or}[\![E_0]\!]_1 \cup A^p_{or}[\![E_0]\!]_2\}$ (finite because $E_0 \in BL\mathcal{R}_\Omega$). $\Delta \setminus \{e\}$ is (countable) infinite because Δ is, so similarly as we argued for the existence of fission refinements we can also find an m-fission refinement ϱ with desired property (21). Remember when dealing with fission refinements we use the same symbol for the BL-fission refinement and the $\mathcal{P}(\mathbf{P})$-fission refinement.

Before we continue observe that for any $E \in BL\mathcal{R}_\Omega$:

$$A^p_{or}[\![E[\varrho]\,;e]\!]_1 = A^p_{or}[\![E[\varrho]]\!]_1 \cup A^p_{or}[\![E[\varrho]]\!]_2 \cdot \{e\}$$

Now let any $E_1 \in BL\mathcal{R}_\Omega$ be given and suppose $A^p_{or}[\![E_0]\!] \not\subseteq A^p_{or}[\![E_1]\!]$. Assume on the contrary $\delta_{\mathsf{w}}(A^p_{or}[\![E_0[\varrho]\,;e]\!]_1) \subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]\,;e]\!]_1)$. There are two ways how $A^p_{or}[\![E_0]\!] \not\subseteq A^p_{or}[\![E_1]\!]$ can be:

$A^p_{or}[\![E_0]\!]_2 \not\subseteq A^p_{or}[\![E_1]\!]_2$: Then there is a $\mathbf{p} \in A^p_{or}[\![E_0]\!]_2$ with $\mathbf{p} \notin A^p_{or}[\![E_1]\!]_2$. Since $A^p_{or}[\![E_0]\!]_2$ is δ_{or}-closed $\mathbf{p}$ must have the P_{or}-property. Because ϱ is m-fission refinement and $m_{\mathbf{p}} \leq m$ we can use lemma 6.20 to find a $w \in \delta_{\mathsf{w}}(\mathbf{p}{<}\varrho{>})$ which is $\mathbf{p}$-reflecting. $w \cdot e$ then belongs to:

$$
\begin{array}{ll}
\quad \delta_{\mathsf{w}}(A^p_{or}[\![E_0]\!]_2{<}\varrho{>}) \cdot \{e\} & \\
= \delta_{\mathsf{w}}(\delta_{or}(A^p_{or}[\![E_0]\!]_2{<}\varrho{>})) \cdot \{e\} & \delta_{\mathsf{w}} \circ \delta_{or} = \delta_{\mathsf{w}} \\
= \delta_{\mathsf{w}}(A^p_{or}[\![E_0[\varrho]]\!]_2) \cdot \{e\} & \text{definition of } [\varrho]^p_{or} \\
= \delta_{\mathsf{w}}(A^p_{or}[\![E_0[\varrho]]\!]_2 \cdot \{e\}) & \delta_{\mathsf{w}} \text{ distributes over } \cdot, \delta_{\mathsf{w}}(\{e\}) = \{e\} \\
\subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_0[\varrho] \,;\, e]\!]_1) & \text{observation, } \subseteq \text{ monotonicity of } \delta_{\mathsf{w}} \\
\subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho] \,;\, e]\!]_1) & \text{assumption} \\
= \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_1 \cup A^p_{or}[\![E_1[\varrho]]\!]_2 \cdot \{e\}) & \text{from notice} \\
= \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_1) \cup \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_2) \cdot \{e\} &
\end{array}
$$

Because $A^p_{or}[\![E_1[\varrho]]\!]_1 = \delta_{or}(A^p_{or}[\![E_1]\!]_1{<}\varrho{>})$ we from (21) see that $e \notin L(\delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_1))$. Hence also $w \cdot e \notin \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_1)$ and we are left with $w \cdot e \in \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_2) \cdot \{e\}$. But then $w \in \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_2) = \delta_{\mathsf{w}}(\delta_{or}(A^p_{or}[\![E_1]\!]_2{<}\varrho{>}))$. This means there is a $\mathbf{p}_1 \in A^p_{or}[\![E_1]\!]_2$ and $\mathbf{q} \in \mathbf{p}_1{<}\varrho{>}$ such that $w \preceq \mathbf{q}$. Since w is $\mathbf{p}$-reflecting we by lemma 6.19 get $\mathbf{p} \preceq \mathbf{p}_1$. Because $P_{or}(\mathbf{p})$ and $A^p_{or}[\![E_1]\!]$ is δ_{or}-closed this implies $\mathbf{p} \in A^p_{or}[\![E_1]\!]_2$—a contradiction.

$A^p_{or}[\![E_0]\!]_1 \not\subseteq A^p_{or}[\![E_1]\!]_1$: We see there exists a $\mathbf{p} \in A^p_{or}[\![E_0]\!]_1$ such that $\mathbf{p} \notin A^p_{or}[\![E_1]\!]_1$ and $P_{or}(\mathbf{p})$ because $A^p_{or}[\![E_0]\!]_1$ is δ_{or}-closed (as well as π-closed). We can also here find a $\mathbf{p}$-reflecting linearization $w \in \delta_{\mathsf{w}}(\mathbf{p}{<}\varrho{>})$. Notice that because of (21) we have $e \notin L(w)$. We infer:

$$
\begin{array}{ll}
w \in \delta_{\mathsf{w}}(A^p_{or}[\![E_0]\!]_1{<}\varrho{>}) & \\
\quad \subseteq \delta_{\mathsf{w}}(\pi(A^p_{or}[\![E_0]\!]_1{<}\varrho{>})) & \delta_{\mathsf{w}} \text{ is } \subseteq\text{-monotone and } P \subseteq \pi(P) \\
\quad = \delta_{\mathsf{w}}(\delta_{or}\pi(A^p_{or}[\![E_0]\!]_1{<}\varrho{>})) & \delta_{\mathsf{w}} \circ \delta_{or} = \delta_{\mathsf{w}} \\
\quad \subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_0[\varrho] \,;\, e]\!]_1) & \text{from notice and definition of } ;^p_{or} \\
\quad \subseteq \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho] \,;\, e]\!]_1) & \text{assumption} \\
\quad = \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_1) \cup & \\
\qquad\quad \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_2) \cdot \{e\} & \text{as above}
\end{array}
$$

$e \notin L(w)$ excludes $w \in \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_2) \cdot \{e\}$ so we can deduce that $w \in \delta_{\mathsf{w}}(A^p_{or}[\![E_1[\varrho]]\!]_1) = \delta_{\mathsf{w}}(\delta_{or}\pi((A^p_{or}[\![E_1]\!]_1){<}\varrho{>})) = \delta_{\mathsf{w}}\pi((A^p_{or}[\![E_1]\!]_1){<}\varrho{>})$. Then there must be pomsets such that

$$w \preceq \mathbf{q} \sqsubseteq \mathbf{q}' \in \mathbf{p}_1{<}\varrho{>} \text{ where } \mathbf{p}_1 \in A^p_{or}[\![E_1]\!]_1$$

w is the linearization of some pomset refined by ${<}\varrho{>}$ and therefore must be balanced w.r.t. to the fission pairs of ϱ. Because $w \preceq \mathbf{q}$ they have the same labels and so $\mathbf{q}$ must also be balanced w.r.t. to the fission pairs. With $\mathbf{q} \sqsubseteq \mathbf{q}' \in \mathbf{p}_1{<}\varrho{>}$ we can then use the lemma 6.19 to conclude there is a pomset $\mathbf{p}_1' \sqsubseteq \mathbf{p}_1$ such that $\mathbf{q} \in \mathbf{p}_1'{<}\varrho{>}$. Because $w \preceq \mathbf{q} \in \mathbf{p}_1'{<}\varrho{>}$ and w is $\mathbf{p}$-reflecting we can as in the case above conclude $\mathbf{p} \preceq \mathbf{p}_1'$. $A^p_{or}[\![E_1]\!]_1$ is both δ_{or}- and π-closed, wherefore from $\mathbf{p}_1' \sqsubseteq \mathbf{p}_1 \in A^p_{or}[\![E_1]\!]_1$ and $P_{or}(\mathbf{p})$ we then get $\mathbf{p} \in A^p_{or}[\![E_1]\!]_1$—again a contradiction. $\square$

Lemma 6.19 Suppose $\mathbf{w}'$ is $\mathbf{p}'$-reflecting under the fission refinement ϱ. If $\mathbf{w}' \preceq \mathbf{r} \in \mathbf{q}{<}\varrho{>}$ then $\mathbf{p}' \preceq \mathbf{q}$.

Proof To see $\mathbf{p}' \preceq \mathbf{q}$ we at first elucidate the situation. $\mathbf{w}'$ being $\mathbf{p}'$-reflecting implies there are representatives w of $\mathbf{w}'$ and p of $\mathbf{p}'$ together with a distinguishing ϱ-consistent p. fission ref., π_p, such that $w = \langle X_{p<\pi_p>}, \leq_w, \ell_{p<\pi_p>}\rangle, \leq_w \supseteq \leq_{p<\pi_p>}$.

We also have $\mathbf{w}' \preceq \mathbf{r} \in \mathbf{q}{<}\varrho{>}$. Therefore there is a ϱ-consistent p. ref., π_q, and a morphism of lpos $f : q{<}\pi_q{>} \to w$.

We shall find a morphism of lpos $g : q \to p$. Define

$$g(x) = y \textit{ iff } \exists y \in X_p.\, y_S^{\pi_p} = f(x_S^{\pi_q})$$

(gives sense since $X_q \overset{\pi_q}{\underset{\text{-}S}{\to}} X_{q<\pi_q>} \overset{f}{\to} X_w = X_{p<\pi_p>} \overset{\pi_p}{\underset{\text{-}S}{\leftarrow}} X_p$).

To see this actually defines a function $g : X_q \to X_p$ we prove that there for a given $x \in X_q$ is one and only one $y \in X_p$ such that $y_S = f(x_S)$. π_q is ϱ-consistent, so for each $x \in X_q$, $\ell_{q<\pi_q>}(x_S) = a_{S_k}$ for some a and k. From f being label preserving and $\ell_w = \ell_{p<\pi_p>}$ we get $\ell_{p<\pi_p>}(f(x_S)) = a_{S_k}$ and by definition of $\overset{\pi_p}{\underset{\text{-}S}{}}$ there then exists an unique $y \in X_p$ with $y_S = f(x_S)$.

Before continuing we observe

(22) $f(x_S) = g(x)_S$ (23) $f(x_F) = g(x)_F$

(22) holds by definition of g. For (23) we have $\ell_{p<\pi_p>}(g(x)_S) = a_{S_k}$ for some a and k. As f is label preserving we from (22) get $\ell_{q<\pi_q>}(x_S) = a_{S_k}$. Since π_p and π_q both are ϱ-consistent p. fission ref.'s obviously then $\ell_{p<\pi_p>}(g(x)_F) = a_{F_k} = \ell_{q<\pi_q>}(x_F)$ and again by f being label preserving $\ell_{p<\pi_p>}(f(x_F)) = a_{F_k}$. Now since π_p furthermore is distinguishing there is at most óne element of $p{<}\pi_p{>}$ labelled a_{F_k}. Hence $f(x_F) = g(x)_F$.

As the next step we show g to be bijective.

g injective:
$$
\begin{array}{lll}
x \neq y & \Rightarrow x_S \neq y_S & \overset{\pi_q}{\text{-}S} \text{ injective} \\
 & \Rightarrow f(x_S) \neq f(y_S) & f \text{ injective} \\
 & \Rightarrow g(x)_S \neq g(y)_S & \text{by (22)} \\
 & \Rightarrow g(x) \neq g(y) & \text{because } \overset{\pi_p}{\text{-}S} \text{ is a function}
\end{array}
$$

g surjective: Given $y \in X_p$. Then $\ell_{p<\pi_p>}(y_S) = a_{S_k}$ for some a and k. Since f is surjective and label preserving there is an $z \in X_{q<\pi_q>}$ with $f(z) = y_S$ and $\ell_{q<\pi_q>}(z) = a_{S_k}$. But from the definition of $\overset{\pi_q}{\text{-}S}$ follows that there exists an $x \in X_q$ such that $x_S = z$ and so $f(x_S) = f(z) = y_S$ which implies $g(x) = y$.

It remains to show that g is label and order preserving.

g label preserving: Suppose $x \in X_q$ and $\ell_q(x) = b$. Then $\ell_{q<\pi_q>}(x_S) = b_{S_k}$ for some k, and therefore $b_{S_k} = \ell_{p<\pi_p>}(f(x_S)) = \ell_{p<\pi_p>}(g(x)_S)$ by (22). By definition of $\overset{\pi_p}{\text{-}S}$, $\ell_{p<\pi_p>}(g(x)_S) = b_{S_k}$ can only be because $\ell_p(g(x)) = b$.

g order preserving: Assume $x \leq_q y$. In the case $x = y$ the result follows from the reflexivity of $\leq_p$. In the case $x <_q y$ we have

$$g(x)_F <_w g(y)_S \tag{24}$$

because
$$
\begin{array}{lll}
x <_q y & \Rightarrow x_F <_{q<\pi_q>} y_S & \text{by construction of } q{<}\pi_q{>} \\
 & \Rightarrow f(x_F) <_w f(y_S) & f \text{ is order preserving} \\
 & \Rightarrow g(x)_F <_w g(y)_S & \text{by (22) and (23)}
\end{array}
$$

We cannot have $g(y) <_p g(x)$ since it by construction of $p{<}\pi_p{>}$ would imply $g(y)_S <_{p<\pi_p>} g(x)_F$ which in turn from $\leq_{p<\pi_p>} \subseteq \leq_w$ implies $g(y)_S <_w g(x)_F$—contradicting (24). $g(x)\ co_p\ g(y)$ can also be excluded since we then from the fact that w is p-reflecting would get $g(y)_S <_w g(x)_F$—again contradicting (24). Hence we are left with $g(x) <_p g(y)$ as the only possibility and we are done. $\square$

Lemma 6.20 Let $\mathbf{p}$ be a pomset with the P_{or}-property and $m_{\mathbf{p}} \leq m$, where m is some finite multiplicity function over Δ. Also let ϱ be an m-fission refinement. Then there exists a linearization w of $\mathbf{p}{<}\varrho{>}$ (i.e., $w \in \delta_{\mathsf{w}}(\mathbf{p}{<}\varrho{>})$) which is $\mathbf{p}$-reflecting under ϱ.

Proof If $\mathbf{p} = \varepsilon$ it is trivial that $w = \varepsilon$ will do, so we can assume $\mathbf{p} \neq \varepsilon$ in the following. Since $m_{\mathbf{p}} \leq m$ there is a distinguishing ϱ-consistent p. fission ref., $<\pi_p>$, for p. The result is then a consequence of the corresponding statement for lpos:

> Let π_p be a distinguishing p. fission ref. for $p \neq \varepsilon$. Assume the minimal elements M_p of p listed in some arbitrary order are: $x_1, \ldots, x_n$. Then there exists an p-reflecting linearization w of $p{<}\pi_p{>}$ isomorphic to an lpo of the form:
>
> $$x_{1S} \cdot \ldots \cdot x_{nS} \cdot v$$

In the proof, which is by induction on the size of X_p, we shall use $M_p \subseteq X_p$ to denote the set of minimal elements of p (w.r.t. $\leq_p$).

The basis, X_p a singleton, is clear.
So assume $|X_p| > 1$. p has the P_{or}-property which is equivalent with the P_{ol}-property we saw in section 5, so we can find an element $x_i \in M_p$ such that x_i is dominated in X_p by all successors of M_p. Consider now the lpo, p', obtained by deleting x_i from p.

Notice that $M_p \setminus \{x_i\}$ is a subset of the minimal elements of p', hence we may list $M_{p'}$ as follows:

$$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y_1, \ldots, y_k$$

Clearly $\pi_{p'} = \pi_p|_{X_{p'}}$ is a distinguishing ϱ-consistent p. fission ref. for p', so because the P_{or} property is inherited to p' we can use the inductive hypothesis to find a p'-reflecting linearization w' of $p'{<}\pi_{p'}{>}$ isomorphic to a lpo of the form

$$x_{1S} \cdot \ldots \cdot x_{i-1S} \cdot x_{i+1S} \cdot \ldots \cdot x_{nS} \cdot y_{1S} \cdot \ldots \cdot y_{kS} \cdot v'$$

Since x_i is minimal in p there are no other elements before x_{iS} and x_{iF} in $p{<}\pi_p{>}$, and so $x_{iS} \cdot x_{iF} \cdot w'$ is isomorphic to a possible linearization of $p{<}\pi_p{>}$. By the way x_i was chosen, the elements concurrent to x_i are exactly $M_p \setminus \{x_i\}$. Then x_{iS} and x_{iF} are concurrent to $x_{1S}, \ldots, x_{i-1S}$ and $x_{i+1S}, \ldots, x_{nS}$ in $p{<}\pi_p{>}$, from which it follows that

$$x_{1S} \cdot \ldots \cdot x_{iS} \cdot \ldots \cdot x_{nS} \cdot x_{iF} \cdot y_{1S} \cdot \ldots \cdot y_{kS} \cdot v'$$

must be isomorphic to a linearization, w, of $p{<}\pi_p{>}$, which quite easily is seen to be p-reflecting as desired. $\square$

Lemma 6.21 Let a finite multiplicity function m over Δ be given together with a ε-free m-fission refinement ϱ. Suppose $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are pomsets such that $\mathbf{p} \sqsubseteq \mathbf{q} \in \mathbf{r}{<}\varrho{>}$. If $\mathbf{p}$ is *balanced* w.r.t. to the fission pairs of ϱ in the sense:

$$\forall a \in \Delta, 1 \leq k \leq n(m).\, m_{\mathbf{p}}(a_{S_k}) = m_{\mathbf{p}}(a_{F_k})$$

then there is a pomset $\mathbf{s} \sqsubseteq \mathbf{r}$ such that $\mathbf{p} \in \mathbf{s}{<}\varrho{>}$.

Proof By definition of the refinement operator, $\mathbf{q} \in \mathbf{r}{<}\varrho{>}$ means there is a ϱ-consistent p. ref., π_r, for r such that $\mathbf{q} = [r{<}\pi_r{>}]$. Then also $\mathbf{p} \sqsubseteq [r{<}\pi_r{>}]$.

We illustrate the situation and the idea of the proof by an example. Suppose r is the representative of the pomset

$$\begin{tikzcd} a \arrow[r] & a \\ b \arrow[r] \arrow[ur] & a \end{tikzcd}$$

Then $[r{<}\pi_r{>}]$ typically may look like:

$$\begin{tikzcd} a_{S_2} \arrow[r] & a_{F_2} \arrow[r] & a_{S_1} \arrow[r] & a_{F_1} \\ b_{S_4} \arrow[r] & b_{F_4} \arrow[r] \arrow[ur] & a_{S_2} \arrow[r] & a_{F_2} \end{tikzcd}$$

Evidently no matter how $\mathbf{p}$ is a ($\leq_{r<\pi_r>}$-downwards closed) prefix of $[r{<}\pi_r{>}]$ then for the fission pair a_{S_2}, a_{F_2} the number of times a_{S_2} occur in $\mathbf{p}$ must be greater than or equal the number of times a_{F_2} occur in $\mathbf{p}$. Similar for the other fission pairs. Clearly also if these numbers balance for every fission pair then there can be no element of $\mathbf{p}$ labelled say a_{S_1} without an immediate following element labelled a_{F_1}. By the nature of fission refinement these two elements must originate from the same element in $\mathbf{r}$ and then $\mathbf{p}$ must a refinement of a prefix, $\mathbf{s}$, of $\mathbf{r}$. $\square$

7 Conclusion

To sum up the achievements of the paper one could say that means are brought about to capture concurrency in processes through the trace precongruences and that labelled partial orders in a natural way serve as cornerstones in the associated models. It is in this sense we take the liberty to phrase the paper: "true concurrency can be traced".

Now for possible extensions. At first let us point to the possibility of extending the operational semantics such that process can perform sequences of multisteps. Though the behavioural equivalences would be more discriminating the models fully abstract w.r.t. the associated congruences would remain the same.

However our full abstractness results are obtained at the expense of a simplified process language and an undetailed view on branching. We shall thus discuss a few ideas to redress some of the shortcomings and their impact on the results.

All the combinators of $BL\mathcal{R}^{rec}_{\Omega}$ are quite simple except for the refinement combinator which suffers from an effective way to be specified. As it is now, a refinement is given by a function from the (infinite) set of atomic actions to the process expressions of BL. One way to go would be to introduce the notation $[a_1 \rightsquigarrow E_1, \ldots, a_n \rightsquigarrow E_n]$ for the refinement where all actions remain unrefined except that a_1 is refined to E_1, a_2 to E_2, etc. and only allow such refinements. Then it would not be possible to specify fission refinements as they are formulated now, but a closer look at the proofs, where these refinements are used, shows that refinements which “fission” on a finite set will do and so all the results go through. With the refinement combinator it is possible to imitate relabelling by considering the relabelling functions as a special class of BL-refinements (maps to individual atomic processes). Looking at the way relabelling usually is introduced in transition systems, the relabelling combinator is static in nature in contrast to the more dynamic nature of the refinement combinator, but this difference cannot be uncovered by the equivalences. Inaction (NIL, $SKIP$) seems also easy to include in $BL\mathcal{R}^{rec}_{\Omega}$. The few proofs, where the refinements are assumed not to make actions disappear (ε-freeness), get more complicated. A (maybe unexpected) consequence of adding NIL would be that expressions like a and $a \oplus NIL$ would be distinguished by $\underset{\sim}{\lessgtr}$ and also by the congruence of $\underset{\sim}{\sqcup}$.

The discussed extensions stay so to say within the simplified view on branching. But if we extend the parallel combinator of $BL\mathcal{R}$ such that e.g., synchronization shall happen on all common actions as in TCSP [BHR84] and we look at maximal sequences, we would at once get a finer view, because the possibility of deadlock forces the model to reflect branching structure—see [Pnu85]. We have carried out this work on nonsequentiality “orthogonally” to existing work on branching, but it is an intriguing question, whether such an extension could be modeled by a smooth combination of e.g., the M_{or} model and the broom model of Pnueli—capturing aspects of nonsequentiality as well as branching.

We conclude by a simple example which indicates that such a combination in no way is straightforward to obtain. Suppose

$$E = a \parallel b \quad \text{and} \quad F = a\,;b \oplus b\,;a \oplus a \parallel b$$

Then E and F are identified in both the M_{or} model and the broom model, but $E' = E[a \rightsquigarrow c\,;d]$ and $F' = F[a \rightsquigarrow c\,;d]$ would be distinguishable in a parallel context with $c\,;b\,;d$—c is a possible maximal sequence of $F' \parallel c\,;b\,;d$ whereas this is not the case for $E' \parallel c\,;b\,;d$. Hence a "conjunction" of the two models would be to abstract for the congruence of $\gtreqqless$ w.r.t the two combinators.

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