THE GROWTH RANGE EQUIVALENCE PROBLEM FOR DOL SYSTEMS IS DECIDABLE

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DAIMI PB-50
August 1975

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SUMMARY

In Nielsen (1974), the decidability of various equivalence problems for DOL systems was studied. One of the questions left open in this paper, was the decidability of what one might call the growth range equivalence problem. Two DOL systems are said to be growth range equivalent iff the ranges of their growth functions coincide. This problem is proved to be decidable in this paper.

NOTATION

Let S be a set, then $|S|$ denotes the cardinality of S. Let x be a string of symbols from some alphabet, then $|x|$ denotes the length of x.

N denotes the set of nonnegative integers.

l denotes the letter lexicographically between k and m, and $\setminus$ denotes the set difference-operator.

DEFINITIONS AND BASIC LEMMAS

Definition 1. A DOL system is an ordered triple $H = \langle \Sigma, \delta, x \rangle$ where Σ is a finite nonempty, ordered set of symbols, $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{|\Sigma|}\}$, (the alphabet of the system), δ is a homomorphism in Σ^*, and x is a nonempty string of symbols from Σ, $x \in \Sigma^+$, (the axiom of the system).

Definition 2. The growth function associated with a DOL system H as in Definition 1, h, is the function mapping nonnegative integers to nonnegative integers, defined by

$$h(n) = |\delta^n(x)|, \quad \text{for all } n \in N,$$

(where $\delta^0(x) = x$). The growth range generated by H, $\mathfrak{R}(H)$, is defined as the range of h, i.e.,

$$\mathfrak{R}(H) = \{ |\delta^n(x)| \mid n \geq 0 \}.$$

Definition 3. Let H and G be two DOL systems with growth functions h and g respectively. H and G are said to be growth equivalent iff their growth functions coincide, i.e., iff

$$h(n) = g(n), \qquad \text{for all } n \in N .$$

H and G are said to be growth range equivalent iff $\mathcal{R}(H) = \mathcal{R}(G)$. The growth (range) equivalence problem is the problem of deciding for any two DOL systems whether or not they are growth (range) equivalent.

Definition 4. Let H be a DOL system as in Definition 1, and let y be any string over its alphabet. Then $\pi(y)$ denotes the Parikh-vector of y, i.e. the vector

$$\pi(y) = (\pi_1(y), \pi_2(y), \ldots, \pi_{|\Sigma|}(y)),$$

where $\pi_i(y)$ is the number of occurrences of the i'th element of Σ, σ_i, in y. The growth matrix of H, M_H, is defined as an $|\Sigma| \times |\Sigma|$ matrix where the (i, j)'th element, $m_{i,j}$, is given by

$$m_{ij} = \pi_j(\delta(\sigma_i)) \qquad \text{for } 1 \le i, j \le |\Sigma| .$$

Let η denote the $|\Sigma|$-dimensional coloumn-vector with all entries equal to 1, then using the notation introduced in Definition 4 you get the following equations:

$$\pi(\delta^n(x)) = \pi(x) \cdot M_H^n$$

(1) $$h(n) = |\delta^n(x)| = \pi(x) \cdot M_H^n \cdot \eta$$

for all $n \in N$.

From the representation (1) of the growth function, h, of a DOL system, and the fact that the growth matrix M_H satisfies its characteristic polynomial of degree less than or equal to $|\Sigma|$, you get that h satisfies a linear, homogenous recurrence relation of length less than or equal to $|\Sigma|$. Consider the shortest recurrence relation satisfied by h,

(2) $$h(n+k) = \sum_{j=1}^{k} c_j \cdot h(n+k-j) \qquad \text{for all } n \in N,\ n \ge l,$$

where $k, c_1, c_2, \ldots, c_k$ and l are computable integers. From this and from classical results on linear recurrence relations it follows that h can be represented in the form

(3) $$h(n) = \sum_{i=1}^{s} p_i(n) \cdot \alpha_i^n \qquad \text{for all } n \in N,\ n \ge l,$$

where the α_i's are the distinct roots of the characteristic polynomial of (2), and the p_i's are polynomials of degree the multiplicity of the corresponding α_i's minus one. In particular $s \le k \le |\Sigma|$.

It is well known that one can distinguish between polynomially bounded and exponential growth functions, in the sense that a growth function h is polynomially bounded iff there exists a polynomial p such that

$$h(n) \le p(n) \qquad \text{for all } n \in N ,$$

and correspondingly h is exponential iff there exist constants $\alpha > 1$ and $n_0 \in N$ such that

$$h(n) \ge \alpha^n \qquad \text{for all } n > n_0 .$$

Note that this property does not follow directly from (3), but is a particular property of DOL growth functions (for a detailed discussion, see Berstel (1975)). A DOL system is called polynomially bounded (exponential) iff its growth function is polynomially bounded (exponential).

Lemma 1. Let $H = \langle \Sigma_H, \delta_H, x_H \rangle$ and $G = \langle \Sigma_G, \delta_G, x_G \rangle$ be two DOL systems with growth functions h and g respectively. If there exist constants m and n , such that $m \le n$ and

$$h(m+i) = g(n+i) \qquad \text{for all } i ,\ 0 \le i < |\Sigma_H| + |\Sigma_G| ,$$

then

$$h(i) = g(n-m+i) \qquad \text{for all } i ,\ |\Sigma_H| + |\Sigma_G| \le i .$$

Proof. Concider the function f defined by

$$f(i) = h(i) - g(n-m+i) \qquad \text{for all } i \in N .$$

f is represented in the form

$$f(i) = \pi' \cdot \begin{bmatrix} M_H & 0 \\ 0 & M_G \end{bmatrix}^i \cdot \eta' , \quad \text{for all } i \in N ,$$

where π' is the $(|\Sigma_H| + |\Sigma_G|)$-dimensional row-vector with the first $|\Sigma_H|$ entries equal to the vector $\pi(x_H)$ and the last $|\Sigma_G|$ equal to the vector $\pi(x_G) \cdot M_G^{n-m}$, and η' is the $(|\Sigma_H| + |\Sigma_G|)$-dimensional coloumn-vector with all its $|\Sigma_H|$ first entries equal to 1 and all its $|\Sigma_G|$ last entries equal to -1 . The matrix of the representation of f above satisfies its characteristic polynomial, which implies that f satisfies a linear, homogenous recurrence relation of length less than or equal to the dimension of the matrix, $|\Sigma_H| + |\Sigma_G|$. This proves the lemma. ∎

Definition 5. A DOL system H and its growth function h are said to be growing iff h is strictly increasing, i.e., iff $h(n) < h(n+1)$ for all $n \in N$.

Definition 6. For any DOL system $H = \langle \Sigma, \delta, x \rangle$ and any two nonnegative integers n and m, the DOL system SUB(H, n, m) is defined by

$$\mathrm{SUB}(H, n, m) = \langle \Sigma, \delta^n, \delta^m(x) \rangle .$$

n is called the period of the SUB-system.

Definition 7. Let $\mu = (m_1, \ldots, m_k)$ and $\nu = (n_1, \ldots, n_k)$ be two k-dimensional vectors with entries from N. The relations $\leq, =$ and $<$ on N^k are defined as follows:

$$\begin{array}{lll} \mu \leq \nu & \text{iff} & m_i \leq n_i \quad \text{for all } i, \ 1 \leq i \leq k ; \\ \mu = \nu & \text{iff} & m_i = n_i \quad \text{for all } i, \ 1 \leq i \leq k ; \\ \mu < \nu & \text{iff} & \mu \leq \nu \text{ and } \mu \neq \nu ; \end{array}$$

Lemma 2. Let $H = \langle \Sigma, \delta, x \rangle$ be a DOL system, and let R be one of the relations $\leq, \geq$ or $=$ on $N^{|\Sigma|}$. If for some $n, m \in N$

$$\pi(\delta^m(x)) \quad R \quad \pi(\delta^{m+n}(x))$$

then

$$\pi(\delta^{m+ni+j}(x)) \quad R \quad \pi(\delta^{m+n(i+1)+j}(x))$$
for all i, $0 \leq i$, and all j, $0 \leq j \leq n$.

Proof. See Nielsen (1974). ∎

Lemma 3. It is decidable for any DOL system H whether $\mathfrak{R}(H)$ is finite or infinite, and if $\mathfrak{R}(H)$ is finite, then $\mathfrak{R}(H)$ can be constructed.

Proof. See Nielsen (1974). ∎

Lemma 4. Let $H = \langle \Sigma, \delta, x \rangle$ be a DOL system for which $\mathfrak{R}(H)$ is infinite. Then m and n, $m, n \in N$, can be computed such that the DOL systems SUB(H, n, m+i), $0 \leq i < n$, are all growing.

Proof. Note first that $\mathfrak{R}(H)$ is infinite iff $\{\pi(\delta^n(x)) \mid n \geq 0\}$ is infinite. This, the assumption of the lemma and Lemma 2 imply that

(4) $\quad$ if $\pi(\delta^i(x)) = \pi(\delta^j(x))$ $\quad$ then $i = j$.

So, compute m and n such that $\pi(\delta^m(x))$ and $\pi(\delta^{m+n}(x))$ are comparable. It follows from Konig (1959) that any infinite sequence of π-values contains at least two comparable elements, hence m and n satisfying this requirement can be computed. Assume that $\pi(\delta^m(x)) \geq \pi(\delta^{m+n}(x))$, this would imply, according to Lemma 2, that $\mathfrak{R}(H)$ is finite, i.e. a contradiction. Hence $\pi(\delta^m(x)) < \pi(\delta^{m+n}(x))$

and from (4) above and Lemma 2 you get that

$$\pi(\delta^{m+ni+j}(x)) < \pi(\delta^{m+n(i+1)+j}(x))$$

for all i, $0 \le i$, and all j, $0 \le j < n$.

This proves the lemma. ■

Lemma 5. It is decidable, given any DOL system H and any $n \in N$, whether or not $n \in \mathfrak{R}(H)$.

Proof. Follows from Lemma 4. ■

Definition 8. Two DOL systems $H = \langle \Sigma_H, \delta_H, x_H \rangle$ and $G = \langle \Sigma_G, \delta_G, x_G \rangle$ with growth functions h and g respectively are said to be ultimately growth equivalent iff there exist constants $m, n \in N$ such that

$$h(m+i) = g(n+i) \qquad \text{for all } i,\ 0 \le i.$$

RESULTS ON THE EQUIVALENCE PROBLEMS

Theorem 1. The growth equivalence problem for DOL systems is decidable.

Proof. This theorem was proved in Paz and Salomaa (1973). It follows directly from Lemma 1 that two DOL systems H and G as in Lemma 1 are growth equivalent iff

$$|\delta_H^i(x_H)| = |\delta_G^i(x_G)| \qquad \text{for all } i,\ 0 \le i < |\Sigma_H| + |\Sigma_G|.$$

This proves the theorem. ■

Corollary 1. For any two DOL systems H and G as in Definition 8, for which $\mathfrak{R}(H)$ and $\mathfrak{R}(G)$ are both infinite, it is decidable whether or not H and G are ultimately growth equivalent.

Proof. It follows from Lemma 1 that H and G are ultimately growth equivalent iff there exists an $I \in N$ such that

$$\begin{array}{lll} \text{either} & h(i) = g(I+i) & \text{for all } i,\ i \ge |\Sigma_H| + |\Sigma_G|, \\ \text{or} & h(I+i) = g(i) & \text{for all } i,\ i \ge |\Sigma_H| + |\Sigma_G|. \end{array}$$

But the existence of such an I can be decided from Lemma 4 and Theorem 1. ■

The purpose of this paper is to prove that also the growth range equivalence problem for DOL systems is decidable.

So let $H = \langle \Sigma_H, \delta_H, x_H \rangle$ and $\langle \Sigma_G, \delta_G, x_G \rangle$ be two given DOL systems. The first step in an algorithm to decide whether or not H is growth range equivalent to G, is to apply the finiteness-algorithm of Lemma 3 to H and G. If one or both of the systems turn out to be finite, then the question of growth range equivalence is trivially decidable. In the following we shall therefore assume that $\mathfrak{R}(H)$ and $\mathfrak{R}(G)$ are both infinite.

Lemma 6. If H is polynomially bounded and G is exponential, then $\mathcal{R}(H) \neq \mathcal{R}(G)$.

Proof. Since $\mathcal{R}(H)$ and $\mathcal{R}(G)$ are infinite you may decompose both systems according to Lemma 4 into growing subsystems, or, in other words, you may compute constants n_H and m_H (n_G and m_G respectively) satisfying the requirements of Lemma 4. Let us denote the growth functions of $SUB(H, n_H, m_H+i)$ ($SUB(G, n_G, m_G+i)$) by h_i , $0 \leq i < n_H$ (g_i , $0 \leq i < n_G$) . Then from the assumptions of the lemma there exist constants n_0 , $k \geq 1$ and $\alpha > 1$ such that

(5) $$h_i(j) \leq j^k \qquad \text{for all } j,\ j > n_0,\ \text{and all } i,\ 0 \leq i < n_H ;$$

(6) $$g_i(j) \geq \alpha^j \qquad \text{for all } j,\ j > n_0,\ \text{and all } i,\ 0 \leq i < n_G ;$$

This follows almost directly from the fact that $h_i(j) = h(m_H+i+jn_H)$ and correspondingly $g_i(j) = g(m_G+i+jn_G)$.

But (5) and (6) imply

(7) $$|\{m \in \mathcal{R}(H) \mid m \leq n\}| \geq n^{\frac{1}{k}} \qquad \text{for all } n > n_0{}^k ;$$

(8) $$|\{m \in \mathcal{R}(G) \mid m \leq n\}| \leq n_H \cdot \log_\alpha(n) + m_H \qquad \text{for all } n > \alpha^{n_0} .$$

Note that it is used here that the constructed SUB-systems are all growing. (7) and (8) above are obviously contradictions to the assumption that $\mathcal{R}(H) = \mathcal{R}(G)$. This proves the lemma. ■

Lemma 7. For any DOL system H it is decidable whether H is polynomially bounded or exponential.

Proof. See Salomaa (1973). ■

From the two previous lemmas you get that the second step in an algorithm to decide growth range equivalence between two given DOL systems is to apply the algorithm of Lemma 7 to both of the given systems. If their growth types turn out to be different, then you know from Lemma 6 that they are not growth range equivalent.

So, after having applied the suggested two steps to the given systems, you have got your answer to the question of growth range equivalence, except in two cases: the growth ranges of both systems are infinite, and their growth types are the same (either both polynomially bounded or both exponential). These two cases will be treated seperately in the next two sections. The third step in the algorithm to decide growth range equivalence will, however, almost be the same in the two cases. Basically, what you get in both cases, is that H and G are growth range equivalent iff each SUB-system of H with "sufficiently long" period has an ultimately growth equivalent SUB-system of G, and the other way round.

POLYNOMIALLY BOUNDED CASE

Lemma 8. Let H be a DOL system with polynomially bounded growth function for which $\mathcal{R}(H)$ is infinite. Then m and n, $m, n \in N$, can be computed such that the growth functions h_i of $SUB(H, n, m+i)$, $0 \le i < n$, are all growing, and furthermore there exist polynomials p_i, $0 \le i < n$, all of the same degree such that

$$h_i(j) = p_i(j) \qquad \text{for all } j,\ 0 \le j, \text{ and all } i,\ 0 \le i < n.$$

Proof. It was proved in Ruohonen (1975) that there exist computable constants m_1 and n_1 such that the growth functions of the systems $SUB(H, n_1, m_1+i)$, $0 \le i < n_1$, are all polynomials of the same degree. From Lemma 4 it follows that you can also compute m_2 and n_2 such that the DOL systems $SUB(H, n_2, m_2+i)$, $0 \le i < n_2$, are all growing. But then obviously $m = \max(m_1, m_2)$ and $n = n_1 \cdot n_2$ will satisfy the requirements of the lemma. ■

Lemma 9. Let H and G be two growth range equivalent DOL systems, both of them with polynomially bounded growth functions. Furthermore, let m_H, n_H and m_G, n_G be constants satisfying the conditions of Lemma 8 for systems H and G respectively. Let k denote the common degree of the polynomial growths of $SUB(H, n_H, m_H+i)$, h_i, $0 \le i < n_H$, and let l correspondingly denote the common degree of the polynomial growths of $SUB(G, n_G, m_G+i)$, g_i, $0 \le i < n_G$. Then $k = l$.

Proof. The proof follows essentially the same lines as the proof of Lemma 6. Assume, e.g., that $l < k$. Then there exist constants $n_0 \in N$, $a > 0$ and b such that

$$h_i(j) \ge a\, j^k \qquad \text{for all } j,\ n_0 < j, \text{ and all } i,\ 0 \le i < n_H;$$

$$g_i(j) \le b\, j^l \qquad \text{for all } j,\ n_0 < j, \text{ and all } i,\ 0 \le i < n_G.$$

But these relations imply

$$|\{m \in \mathcal{R}(H) \mid m \le n\}| \le n_H (n/a)^{\frac{1}{k}} + m_H \qquad \text{for all } n, n > a\, n_0^{\,k};$$

$$|\{m \in \mathcal{R}(G) \mid m \le n\}| \ge (n/b)^{\frac{1}{l}} \qquad \text{for all } n, n > b\, n_0^{\,l}.$$

Again the fact that all SUB-systems are growing is used, and again you reach a contradiction to the assumption that H is growth range equivalent to G. ■

Now, take any two DOL systems H and G and constants m_H, n_H, m_G and n_G satisfying the conditions of Lemma 9, and take any of the systems $SUB(H, n_H, m_H+i_1)$ from Lemma 9. Let us denote the growth function of this system by p,

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0 .$$

All the values of p occur, from the assumption that H and G are growth range equivalent as values of the growth function of G. That is, you may define a function γ mapping N to the set $\{0, 1, \ldots, n_G\}$ satisfying

(9) if $\gamma(j) = i$ for some i, $0 \le i < n_G$ then $p(j)$ occurs as a value of the growth function of $SUB(G, n_G, m_G+i)$, g_i;

if $\gamma(j) = n_G$ then $p(j)$ occurs as one of the first m_G-1 values of the growth function of G, i.e., $p(j) = g(n)$ for some $n < m_G$;

Note that γ is not necessarily unique, but at least one function γ satisfies (9), and when referring to γ in the following we shall mean one fixed γ satisfying (9).

Obviously, there exists an $n_0 \in N$ such that for all $j > n_0$, $\gamma(j) < n_G$. Secondly, applying the theorem of van der Waerden as stated in Chintschin (1951) to γ, you get that there exists a computable function, W, mapping N to N, such that for any $l \in N$, any set of $W(l)$ consecutive elements of N contains an arithmetic progression of length l on which γ assumes the same value. W is defined recursively as follows:

$$W(l) = U(n_G, l) ;$$

$$U(x, y) = \text{if } y \le 2 \text{ then } x+1 \text{ otherwise } V(x, y-1) ;$$

$$V(x, y) = \text{if } x = 0 \text{ then } 1 \text{ otherwise } 2 \cdot U(x^{V(x-1, y)}, y) \cdot V(x-1, y) ;$$

It should be stressed that the theorem of van der Waerden does not give you an infinite arithmetic progression on which γ assumes the same value. The application of the theorem gives you, however, the following lemma.

<u>Lemma 10</u>. There exist an i_0, $0 \le i_0 < n_G$, and a constant $r \in N$, $r < W(|\Sigma_H| + |\Sigma_G| - 1)$, such that for arbitrary big n-values

$$i_0 = \gamma(n) = \gamma(n+r) = \ldots = \gamma(n+r(|\Sigma_H| + |\Sigma_G| - 1)) .$$

Proof. Let N_j denote the set of consecutive elements of N from $j \cdot W(|\Sigma_H| + |\Sigma_G| - 1)$ to $(j+1) \cdot W(|\Sigma_H| + |\Sigma_G| - 1) - 1$ for all $j \in N$. If you apply the theorem of van der Waerden as stated above for $l = |\Sigma_H| + |\Sigma_G| - 1$ to N_j for every $j \in N$, you get that every N_j contains an arithmetic progression of length $|\Sigma_H| + |\Sigma_G| - 1$ on which γ assumes the same value. But since the periods of these arithmetic progressions are obviously bounded by $W(|\Sigma_H| + |\Sigma_G| - 1)/(|\Sigma_H| + |\Sigma_G| - 1)$, and since γ assumes only finitely many values, the lemma follows directly. ■

Let us denote the growth function of $SUB(G, n_G, m_G+i_0)$ by q, where i_0 satisfies Lemma 10,

$$q(n) = b_k n^k + b_{k-1} n^{k-1} + \ldots + b_0 ;$$

Let N_1 denote the infinite subset of N for which γ assumes the value i_0. Define φ as the function mapping N_1 to reals, satisfying

$$\begin{aligned} (10) \qquad p(n) &= q((a_k/b_k)^{1/k} \cdot n + \varphi(n)) \\ &= q(c \cdot n + \varphi(n)) \qquad \text{for all } n \in N_1 . \end{aligned}$$

Note that φ is unique from the fact that all SUB-systems are growing.

Lemma 11. $\varphi(n)/n \to 0$ for $n \to \infty$ in N_1.

Proof. From (10) you get (dividing by n^k)

$$a_k + \frac{a_{k-1}}{n} + \ldots + \frac{a_0}{n^k} = \frac{b_k(cn + \varphi(n))^k}{n^k} + \ldots + \frac{b_0}{n^k} .$$

The lefthand side of this equation converges on N_1 to a_k for n going to infinity. But since clearly $(cn + \varphi(n))$ goes to infinity with n, you have that the quotient between any of the k last termes on the righthand side and the first term on the righthand side converge to 0 for n going to infinity in N_1. From this you get

$$\frac{b_k(cn + \varphi(n))^k}{n^k} \to a_k \qquad \text{for } n \to \infty \text{ in } N_1$$

which implies

$$\frac{cn + \varphi(n)}{n} \to \left(\frac{a_k}{b_k}\right)^{1/k} \qquad \text{for } n \to \infty \text{ in } N_1$$

and this proves the lemma. ■

Lemma 12.

$$\varphi(n) \to \frac{a_{k-1} - b_{k-1} \cdot c^{k-1}}{k \cdot b_k \cdot c^{k-1}} = d \quad \text{for } n \to \infty \text{ in } N_1 .$$

Proof. From (10) you get that for all $n \in N_1$

$$\sum_{i=0}^{k} a_i n^i = \sum_{i=0}^{k} b_i \cdot \left(\sum_{j=0}^{i} \binom{i}{j} (cn)^{i-j} (\varphi(n))^j \right),$$

which implies

$$\sum_{i=0}^{k} (a_i - b_i c^i) n^i = \varphi(n) \cdot \sum_{i=0}^{k} b_i \cdot \left(\sum_{j=1}^{i} \binom{i}{j} (cn)^{i-j} (\varphi(n))^{j-1} \right) . \tag{11}$$

Observe that the coefficient of the n^k-term on the lefthand side of (11) is equal to zero, so if you divide the equation (11) by n^{k-1} you get that the resulting left-hand side converges to $(a_{k-1} - b_{k-1} c^{k-1})$ for n going to infinity in N_1. Furthermore, you get from Lemma 11 that if you divide the sum-expression of the righthand side of (11) by n^{k-1}, the resulting sumexpression will converge to $b_k \cdot k \cdot c^{k-1}$ for n going to infinity in N_1. This proves the lemma. ■

Lemma 13. Let r be the constant of Lemma 10, $r \in N$, then $c = \frac{s}{r}$, for some $s \in N$.

Proof. Let N_2 denote the following subset of N_1

$$N_2 = \{ n \in N_1 \mid i_0 = \gamma(n) = \gamma(n+r) \} .$$

It then follows from Lemma 10 that N_2 is infinite. Consider the function ψ mapping N_2 for N defined by

$$\psi(n) = (c(n+r) + \varphi(n+r)) - (cn + \varphi(n))$$

$$= cr + \varphi(n+r) - \varphi(n) \qquad \text{for all } n \in N_2 .$$

From Lemma 12 you get

$$\psi(n) \to cr + d - d = cr \qquad \text{for } n \to \infty \text{ in } N_2 .$$

But since $\psi(n)$ is an integer-function this means that cr is an integer, and this proves the lemma. ■

Lemma 14. $d = \frac{t}{r}$ for some $t \in N$, and there exists a constant $n_0 \in N_1$ such that

$$\varphi(n) = d \qquad \text{for all } n > n_0 ,\ n \in N_1 .$$

Proof. It follows from the definition of φ that $(cn+\varphi(n))$ is an integer-function, i.e. (Lemma 13) $r\varphi(n)$ is an integer-function converging (Lemma 12) to rd for n going to infinity in N_1. This implies that rd is an integer (which proves the first part of the lemma), and secondly it implies the existence of $n_0 \in N_1$ such that $r\varphi(n) = rd$ for all $n > n_0$, $n \in N_1$ (which proves the second part of the lemma). ∎

Combining Lemma 10 and Lemma 14 you get that there exists a constant $n_1 \geq n_0$, such that

$$p(n_1+jr) = q(c(n_1+jr)+d) \quad \text{for all } j,\ 0 \leq j < |\Sigma_H| + |\Sigma_G| - 1 .$$

This implies, however, from Lemma 1 that $SUB(H, rn_H, m_H+i_1+n_1n_H)$ is growth equivalent to $SUB(G, sn_G, m_G+i_0+(d+cn_1)n_G)$.

Now take any constant $p_1 \in N$ and any SUB-system, H_i, of the form

$$H_i = SUB(H, p_1rn_H, m_H+i_1+in_H) \qquad 0 \leq i < p_1 r$$

for which $(n_1 \text{ modulo } p_1r) \neq i$.

If you apply the arguments of this section to this SUB-system you get that there exist constants $p_2, n_2 \in N$, $p_2 < W(|\Sigma_H| + |\Sigma_G| - 1)$, such that $SUB(H, p_2p_1rn_H, m_H+i_1+in_H+n_2p_1rn_H)$ is either

(12) growth equivalent to some SUB-system of $SUB(G, n_G, m_G+j)$ for $j \neq i_0$, or

(13) growth equivalent to a SUB-system of $SUB(G, n_G, m_G+i_0)$ with period p_2p_1s.

Note first that (12) and (13) are not necessarily exclusive, and secondly that (13) follows from the fact that the leading coefficient of the polynomial growth of a SUB-system of H_i with period p_2 is equal to $a_k(rp_1p_2)^k$, whereas the leading coefficient of the polynomial growth of a SUB-system of $SUB(G, n_G, m_G+i_0)$ with period x is equal to b_kx^k. So from the assumption that the two SUB-systems are growth equivalent you get

$$a_k(rp_1p_2)^k = b_kx^k$$

which implies

$$x = p_2p_1s .$$

If we denote the growth function of the chosen SUB-system H_i by h_i, and correspondingly the growth-function of $SUB(G, sn_G, m_G+i_0+jn_G)$ by g_j, $0 \leq j < s$, then (13) can be stated in the following form.

(13') there exist $j, l \in N$, $0 \le j < s$, such that the function

$$f(n) = h_i(n+n_2) - g_j(p_1 n+l)$$

assumes an infinite number of zero-values for arguments forming an arithmetic progression with period p_2.

Remember, however, that the function f of (13') satisfies a linear, homogenous recurrence relation of length less than or equal to $(|\Sigma_H| + |\Sigma_G|)$, i.e., you can apply the following lemma, which was proven in Berstel and Mignotte (1974).

<u>Lemma 15</u>. Let f be a function mapping N to integers, satisfying a linear, homogenous recurrence relation of length l, and assuming an infinite number of zero-values. Then the arguments for which f assumes the value zero, form, apart from a finite number of them, a finite union of arithmetic progressions for which the least common multiple of the periods is bounded by $B(l-1)$, where

$$B(x) = \exp(4x\sqrt{10 \log x}),$$

From this you get the following lemma.

<u>Lemma 16</u>. Define $u_1 = W(|\Sigma_H| + |\Sigma_G| - 1)$, $u_2 = B(|\Sigma_H| + |\Sigma_G| - 1)$, $v = (\prod_{i=1}^{u_1} i) \cdot (\prod_{i=1}^{u_2} i)$, and let H_i denote the SUB-system

$$SUB(H, n_H \cdot v, m_H+i) \qquad \text{for all } 0 \le i < n_H \cdot v .$$

Then the assumption that $\mathcal{R}(H) = \mathcal{R}(G)$ implies that for each i, there exists an $n_i \in N$, such that $SUB(H, n_H \cdot v, m_H+i + n_i n_H v)$ is growth equivalent to some subsystem of G. Furthermore, if you denote the leading coefficient of the polynomial growth of $SUB(H, n_H, m_H+i)$ by a_i, $0 \le i < n_H$, and correspondingly the leading coefficient of the polynomial growth of $SUB(G, n_G, m_G+j)$ by b_j, $0 \le j < n_G$, and define

$$y = \max\left\{\left(\frac{a_i}{b_j}\right)^{\frac{1}{k}} \;\middle|\; 0 \le i < n_H,\ 0 \le j < n_G\right\} ,$$

then the period of this subsystem of G is bounded by $n_G \cdot v \cdot y$.

<u>Proof</u>. Follows directly from the arguments of this section, and the fact that you may assume that the period of any arithmetic progression as in (13') is a divisor in v by Lemma 15. ∎

Theorem 2. Let H and G be two DOL systems both of them with polynomially bounded growth functions, for which $\mathcal{R}(H)$ and $\mathcal{R}(G)$ are both infinite sets. It is then decidable whether or not $\mathcal{R}(H) = \mathcal{R}(G)$.

Proof. The algorithm to decide whether $\mathcal{R}(H) = \mathcal{R}(G)$ under the assumptions of the theorem, goes as follows:

Compute n_H, m_H, n_G, m_G, v and y satisfying Lemma 9 and Lemma 16.

For each i , $0 \leq i < n_H v$, you decide whether $SUB(H, n_H v, m_H+i)$ is ultimately growth equivalent to some SUB-system of the form $SUB(G, k, m_G+j)$ where $0 \leq j < k$, and $0 < k < n_G vy$. This can be done by a finite number of applications of the algorithm of Corollary 1. If this is not the case you know from Lemma 16 that $\mathcal{R}(H)$ is not equal to $\mathcal{R}(G)$; if it is the case, you get from the application of the algorithm of Corollary 1 an n_i satisfying Lemma 16. If you do find an ultimately growth equivalent SUB-system of G for all i , $0 \leq i < n_H v$, you know that $\mathcal{R}(H) \subseteq \mathcal{R}(G)$ iff

$$\{ |\delta_H^n(x_H)| \mid 0 \leq n < m_H + \max_{0 \leq i < n_H v} (n_i n_H v)\} \subseteq \mathcal{R}(G) .$$

But this relation can easily be decided by a finite number of applications of the membershipalgorithm of Lemma 5.

Finally, you interchange the roles of systems H and G , and repeat the steps described above.

Now, obviously $\mathcal{R}(H) = \mathcal{R}(G)$ iff the algorithms described above end with the results $\mathcal{R}(H) \subseteq \mathcal{R}(G)$ and $\mathcal{R}(G) \subseteq \mathcal{R}(H)$. This proves the theorem. ■

Remark. You should note that the algorithm given in the proof of Theorem 2 does not include an algorithm to decide whether $\mathcal{R}(H) \subseteq \mathcal{R}(G)$ in general. The reason for this is that the first part of the algorithm (the part of finding growth equivalent SUB-systems) is working with the assumption $\mathcal{R}(H) = \mathcal{R}(G)$, i.e. this part may stop with a negative answer to the question of growth range equivalence even though $\mathcal{R}(H) \subseteq \mathcal{R}(G)$. An example of this phenonemon would be H generating the sequence n^4 and G generating n^2 . Then $\mathcal{R}(H) \subseteq \mathcal{R}(G)$, but no SUB-system of H is growth equivalent to any SUB-system of G .

EXPONENTIAL CASE

The proof of the decidability of growth range equivalence in the case where both systems have exponential growth functions follows essentially the same lines as the proof is the polynomially bounded case.

Lemma 17. Let H be a DOL system as in Definition 1 with exponential growth function h satisfying a linear, homogenous recurrence relation (2). Then m and n, $m, n \in N$, can be computed such that the growth functions h_i

of $SUB(H,n,m+i)$, $0 \le i < n$, are all growing, and there exists a real number $\xi > 1$, such that ξ is a root of the characteristic polynomial of the (common) recurrence relation of all the h_i's , and furthermore for any other root α of this polynomial, one has $|\alpha| < \xi$.

Proof. By Lemma 4 integers m_1 and n_1 can be computed such that the growth functions h_i' of $SUB(H,n_1,m_1+i)$, $0 \le i < n_1$, are all growing. By a theorem from Berstel (1975), integers m_2 and n_2 can be computed such that the growth functions h_i'' of $SUB(H,n_2,m_2+i)$, $0 \le i < n_2$, satisfy the second condition of the lemma. Putting now $m = \max(m_1,m_2)$ and $n = n_1 \cdot n_2$ the growth functions h_i of $SUB(H,n,m+i)$, $0 \le i < n$, satisfy both conditions. ∎

Now, take any two growth range equivalent DOL systems H and G , both of them satisfying the conditions of Lemma 17. You can then compute constants $m_H, n_H \in N$ and $\xi > 1$ for system H according to Lemma 17, and correspondingly $m_G, n_G \in N$ and $\eta > 1$ for system G .

Consider then any SUB-system of H of the form $SUB(H,n_H,m_H+i_1)$ where $0 \le i_1 < n_H$. The growth function of this SUB-system can be expressed as (follows from Lemma 17 and (3))

$$p(n) = a\,n^k \xi^n + p_2(n) = p_1(n) + p_2(n) ,$$

for some constants $a > 0$, $k \in N$, and a function p_2 mapping N to reals, for which

$$\frac{p_2(n)}{an^k\xi^n} \to 0 \qquad \text{for } n \to \infty \text{ in } N .$$

Define for this particular SUB-system of H , a function γ exactly like in the previous section (9). Applying van der Waerdens theorem you get that there exist constants $i_0, r \in N$, $0 \le i_0 < n_G$, satisfying the requirements of Lemma 10. The growth function of $SUB(G,n_G,m_G+i_0)$ can be expressed as

$$q(n) = b\,n^l \eta^n + q_2(n) = q_1(n) + q_2(n) .$$

for some constants $b > 0$, $l \in N$, and a function q_2 for which

$$\frac{q_2(n)}{bn^l\eta^n} \to 0 \qquad \text{for } n \to \infty \text{ in } N .$$

Define as in the previous section N_1 as the infinite subset of N for which γ assumes the value i_0 , and define the unique function φ mapping N_1 to reals satisfying

(14) $$p(n) = q(cn + \varphi(n)) \qquad \text{for all } n \in N_1 ,$$

where c is the constant satisfying $\xi = \eta^c$ ($c = \log \xi / \log \eta$). Let λ denote the function defined by

$$\lambda(n) = cn + \varphi(n) \qquad \text{for all } n \in N_1 .$$

Lemma 18. $\varphi(n)/n \to 0$ for $n \to \infty$ in N_1 .

Proof. Since the system $SUB(H, n_H, m_H+i)$ is growing, one has $\lambda(n) \to \infty$ for $n \to \infty$ in N_1 . Now, from (14) one has

$$1 + \frac{p_2(n)}{p_1(n)} = \frac{q_1(\lambda(n))}{p_1(n)} \left(1 + \frac{q_2(\lambda(n))}{p_1(n) \cdot q_1(\lambda(n))}\right)$$

for all $n \in N_1$, and this implies

(15) $$\frac{q_1(\lambda(n))}{p_1(n)} = \frac{b}{a} \frac{(\lambda(n))^l}{n^k} \eta^{\varphi(n)} \to 1 \qquad \text{for } n \to \infty \text{ in } N_1 .$$

Applying the logarithmic function to this convergence you get

$$\log\left(\frac{b}{a}\right) + l \cdot \log(\lambda(n)) - k \cdot \log(n) + \varphi(n) \cdot \log \eta \to 0$$

$$\text{for } n \to \infty \text{ in } N_1 .$$

Dividing this convergence by cn you get

$$\frac{l \cdot \log(\lambda(n))}{cn} + \frac{\varphi(n) \cdot \log \eta}{cn} \to 0 \qquad \text{for } n \to \infty \text{ in } N_1,$$

i.e.,

$$\frac{l \cdot \log(\lambda(n))}{cn} + \frac{\lambda(n) \cdot \log \eta}{cn} \to \log \eta$$

$$\text{for } n \to \infty \text{ in } N_1 .$$

But now clearly the quotient between the first and the second term of the lefthand side of this convergence converges to zero, which implies

$$\frac{\lambda(n) \log \eta}{cn} \to \log \eta \qquad \text{for } n \to \infty \text{ in } N_1 ,$$

and this proves the lemma. ■

Lemma 19. Let N_2 denote the infinite subset of N_1 defined by

$$N_2 = \{n \in N_1 \mid i_0 = \gamma(n) = \gamma(n+r)\} .$$

Then

$$\varphi(n+r) - \varphi(n) \to 0 \qquad \text{for } n \to \infty \text{ in } N_2 .$$

Proof. From (15) you get

$$\frac{b}{a}\left(c+\frac{\varphi(n)}{n}\right)^{l} n^{l-k}\,\eta^{\varphi(n)} \to 1 \qquad \text{for } n\to\infty \text{ in } N_1 ,$$

but since $\varphi(n)/n$ converges to zero this implies

$$n^{l-k}\,\eta^{\varphi(n)} \to \frac{a}{b\,c^{l}} \qquad \text{for } n\to\infty \text{ in } N_1 .$$

Applying the logarithmic function to this convergence you get

$$(l-k)\cdot\log(n) + \varphi(n)\cdot\log\eta \to \log\left(\frac{a}{b\,c^{l}}\right)$$

$$\text{for } n\to\infty \text{ in } N_1 .$$

From this you get

$$(l-k)\cdot(\log(n+r) - \log(n)) + (\varphi(n+r) - \varphi(n))\cdot\log\eta \to 0$$

$$\text{for } n\to\infty \text{ in } N_2 .$$

Clearly the first term of the above lefthand side expression converges to zero for n going to infinity in N_2, and hence so does the second, and this proves the lemma. ■

Consider the function ψ mapping N_2 to N defined by

$$\psi(n) = \lambda(n+r) - \lambda(n) =$$
$$cr + \varphi(n+r) - \varphi(n) , \qquad \text{for all } n\in N_2 .$$

From Lemma 19 you get that

$$\psi(n) \to cr \qquad \text{for } n\to\infty \text{ in } N_2 .$$

But since $\psi(n)$ is an integer function you get from this that cr is an integer, i.e., you get the following lemma corresponding to Lemma 13.

Lemma 20. $c = \frac{s}{r}$ for some $s \in N$.

Lemma 21. $\exists n_0 \in N \ \forall n \in N_2 ,\ n > n_0 : \varphi(n+r) = \varphi(n)$.

Proof. Notice that $\lambda(n)$ is an integer function, i.e., (from Lemma 20) $\varphi(n) = \tau(n)/r$ for some integer function τ . The lemma follows now directly from Lemma 19. ■

Since γ satisfies the conditions of Lemma 10, Lemma 21 implies that there exists an $n_1 \in N$, $n_1 > n_0$ such that

$$\varphi(n_1) = \varphi(n_1+r) = \ldots = \varphi(n_1 + r(|\Sigma_H| + |\Sigma_G| - 1)) .$$

Applying Lemma 1 this means that $SUB(H, rn_H, m_H + i_1 + n_1 n_H)$ is growth equivalent to $SUB(G, sn_G, m_G + i_0 + (cn_1 + \varphi(n_1))n_G)$.

From this point, the proof goes exactly like the proof in the previous section, and finally you end up with the following theorem.

Theorem 3. Let H and G be two DOL systems both of them with exponential growth functions. It is then decidable whether or not $\mathcal{R}(H) = \mathcal{R}(G)$.

Proof. The proof is identical to the proof of Theorem 2, with the exceptions that in the algorithm to decide whether $\mathcal{R}(H) \subseteq \mathcal{R}(G)$

m_H, n_H, m_G, n_G are computed according to Lemma 17, and

$y = c$.

All other details are left to the reader. ■

Theorem 4. The growth range equivalence problem for DOL-systems is decidable.

Proof. Follows from the remarks just after Lemma 7, Theorem 2 and Theorem 3. ■

Corollary 2. For any two DOL systems H and G it is decidable whether or not there exist finite sets F_H and F_G , $F_H, F_G \subseteq N$, such that $\mathcal{R}(H) \setminus F_H = \mathcal{R}(G) \setminus F_G$.

Proof. Follows from slight modifications of the proof of Theorem 4. The only interesting thing to notice is that the part of the algorithm corresponding to the algorithm of Theorem 2, consists simply of the first part of the algotithm of Theorem 2 (the part of finding growth equivalent SUB-systems) applied to H and G ! And similarly, for the part corresponding to the algorithm of Theorem 3. All details are left to the reader.

Acknowledgment. Part of the work on this paper was done while the first author was visiting the Department of Computer Science of the University of Aarhus. He wishes to express his gratitude for this invitation. Both authors wish to thank professor Arto Salomaa for many helpful discussions.

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