THE GROWTH RANGE EQUIVALENCE PROBLEM
FOR DOL SYSTEMS IS DECIDABLE

by

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SUMMARY

In Nielsen (1974), the decidability of various equivalence problems for
DOL systems was studied. One of the questions left open in this paper, was the
decidability of what one might call the growth range equivalence problem. Two
DOL systems are said to be growth range equivalent iff the ranges of their growth
functions coincide. This problem is proved to be decidable in this paper.

NOTATION

Let $S$ be a set, then $|S|$ denotes the cardinality of $S$. Let $x$ be a
string of symbols from some alphabet, then $|x|$ denotes the length of $x$.

$N$ denotes the set of nonnegative integers.

$\cdot$ denotes the letter lexicographically between $k$ and $m$, and $\setminus$ denotes
the set difference-operator.

DEFINITIONS AND BASIC LEMMAS

Definition 1. A DOL system is an ordered triple $H = \langle S, \delta, x \rangle$ where $S$
is a finite nonempty, ordered set of symbols, $S = \{s_1, s_2, \ldots, s_n \mid \Sigma \}$, (the alphabet
of the system), $\delta$ is a homomorphism in $S^*$, and $x$ is a nonempty string of
symbols from $S$, $x \in S^+$, (the axiom of the system).

Definition 2. The growth function associated with a DOL system $H$ as in
Definition 1, $h$, is the function mapping nonnegative integers to nonnegative integers,
defined by

$$h(n) = \left| \delta^n(x) \right|,$$

for all $n \in N$,

(where $\delta^0(x) = x$). The growth range generated by $H$, $R(H)$, is defined as the range
of $h$, i.e.,

$$R(H) = \{ \left| \delta^n(x) \right| \mid n \geq 0 \}.$$
Definition 3. Let $H$ and $G$ be two DOL systems with growth functions $h$ and $g$ respectively. $H$ and $G$ are said to be growth equivalent iff their growth functions coincide, i.e., iff
\[ h(n) = g(n), \quad \text{for all } n \in \mathbb{N}. \]

$H$ and $G$ are said to be growth range equivalent iff $\mathcal{R}(H) = \mathcal{R}(G)$. The growth (range) equivalence problem is the problem of deciding for any two DOL systems whether or not they are growth (range) equivalent.

Definition 4. Let $H$ be a DOL system as in Definition 1, and let $y$ be any string over its alphabet. Then $\pi(y)$ denotes the Parikh-vector of $y$, i.e. the vector
\[ \pi(y) = (\pi_1(y), \pi_2(y), \ldots, \pi_{|\Sigma|}(y)), \]
where $\pi_i(y)$ is the number of occurrences of the $i$th element of $\Sigma$, $\sigma_i$, in $y$.

The growth matrix of $H, M_H$, is defined as an $|\Sigma| \times |\Sigma|$ matrix where the $(i,j)$th element, $m_{i,j}$, is given by
\[ m_{i,j} = \pi_j(\delta(\sigma_i)) \quad \text{for } 1 \leq i, j \leq |\Sigma|. \]

Let $\eta$ denote the $|\Sigma|$-dimensional columnn-vector with all entries equal to 1, then using the notation introduced in Definition 4 you get the following equations:
\begin{align*}
\pi(\delta^n(x)) &= \pi(x) \cdot M^n_H \\
h(n) &= |\delta^n(x)| = \pi(x) \cdot M^n_H \cdot \eta
\end{align*}
for all $n \in \mathbb{N}$.

From the representation (1) of the growth function, $h$, of a DOL system, and the fact that the growth matrix $M_H$ satisfies its characteristic polynomial of degree less than or equal to $|\Sigma|$, you get that $h$ satisfies a linear, homogenous recurrence relation of length less than or equal to $|\Sigma|$. Consider the shortest recurrence relation satisfied by $h$,
\begin{align*}
h(n+k) &= \sum_{j=1}^{k} c_j \cdot h(n+k-j) \quad \text{for all } n \in \mathbb{N}, \ n \geq 1,
\end{align*}
where $k, c_1, c_2, \ldots, c_k$ and $l$ are computable integers. From this and from classical results on linear recurrence relations it follows that $h$ can be represented in the form
\begin{align*}
h(n) &= \sum_{i=1}^{s} p_i(n) \cdot \alpha_i^n \quad \text{for all } n \in \mathbb{N}, \ n \geq 1,
\end{align*}
where the $\alpha_i$'s are the distinct roots of the characteristic polynomial of (2), and
the $p_i$'s are polynomials of degree the multiplicity of the corresponding $\alpha_i$'s minus one. In particular $s \leq k \leq |\Sigma|$. 

It is well known that one can distinguish between polynomially bounded and exponential growth functions, in the sense that a growth function $h$ is polynomially bounded iff there exists a polynomial $p$ such that

$$h(n) \leq p(n) \quad \text{for all } n \in \mathbb{N},$$

and correspondingly $h$ is exponential iff there exist constants $\alpha > 1$ and $n_0 \in \mathbb{N}$ such that

$$h(n) \geq \alpha^n \quad \text{for all } n > n_0.$$

Note that this property does not follow directly from (3), but is a particular property of DOL growth functions (for a detailed discussion, see Berstel (1975)). A DOL system is called polynomially bounded (exponential) iff its growth function is polynomially bounded (exponential).

**Lemma 1.** Let $H = \langle \Sigma_H, \delta_H, x_H \rangle$ and $G = \langle \Sigma_G, \delta_G, x_G \rangle$ be two DOL systems with growth functions $h$ and $g$ respectively. If there exist constants $m$ and $n$, such that $m \leq n$ and

$$h(m+i) = g(n+i) \quad \text{for all } i, \ 0 \leq i \leq |\Sigma_H| + |\Sigma_G|,$$

then

$$h(i) = g(n-m+i) \quad \text{for all } i, \ |\Sigma_H| + |\Sigma_G| \leq i.$$

**Proof.** Consider the function $f$ defined by

$$f(i) = h(i) - g(n-m+i) \quad \text{for all } i \in \mathbb{N}.$$

$f$ is represented in the form

$$f(i) = \pi^i \cdot \begin{bmatrix} M_H & 0 & 0 \\ 0 & M_G \end{bmatrix} \cdot \eta^i \quad \text{for all } i \in \mathbb{N},$$

where $\pi^i$ is the $(|\Sigma_H| + |\Sigma_G|)$-dimensional row-vector with the first $|\Sigma_H|$ entries equal to the vector $\pi(x_H)$ and the last $|\Sigma_G|$ equal to the vector $\pi(x_G) \cdot M_{n-m}^i$, and $\eta^i$ is the $(|\Sigma_H| + |\Sigma_G|)$-dimensional column-vector with all its $|\Sigma_H|$ first entries equal to 1 and all its $|\Sigma_G|$ last entries equal to -1. The matrix of the representation of $f$ above satisfies its characteristic polynomial, which implies that $f$ satisfies a linear, homogenous recurrence relation of length less than or equal to the dimension of the matrix, $|\Sigma_H| + |\Sigma_G|$. This proves the lemma. $\blacksquare$
Definition 5. A DOL system $H$ and its growth function $h$ are said to be growing iff $h$ is strictly increasing, i.e., iff $h(n) < h(n+1)$ for all $n \in \mathbb{N}$.

Definition 6. For any DOL system $H = \langle \Sigma, \delta, x \rangle$ and any two nonnegative integers $n$ and $m$, the DOL system $\text{SUB}(H, n, m)$ is defined by

$$\text{SUB}(H, n, m) = \langle \Sigma, \delta^n, \delta^m(x) \rangle.$$  

$n$ is called the period of the SUB-system.

Definition 7. Let $\mu = (m_1, \ldots, m_k)$ and $\nu = (n_1, \ldots, n_k)$ be two $k$-dimensional vectors with entries from $\mathbb{N}$. The relations $\leq, =, <$ on $\mathbb{N}^k$ are defined as follows:

- $\mu \leq \nu$ iff $m_i \leq n_i$ for all $i$, $1 \leq i \leq k$;
- $\mu = \nu$ iff $m_i = n_i$ for all $i$, $1 \leq i \leq k$;
- $\mu < \nu$ iff $\mu \leq \nu$ and $\mu \neq \nu$;

Lemma 2. Let $H = \langle \Sigma, \delta, x \rangle$ be a DOL system, and let $R$ be one of the relations $\leq, =, <$ on $\mathbb{N}^{|\Sigma|}$. If for some $n, m \in \mathbb{N}$

$$\pi(\delta^m(x)) \quad R \quad \pi(\delta^{m+n}(x))$$

then

$$\pi(\delta^{m+n+i+j}(x)) \quad R \quad \pi(\delta^{m+n+i+1+j}(x))$$

for all $i$, $0 \leq i$, and all $j$, $0 \leq j \leq n$.


Lemma 3. It is decidable for any DOL system $H$ whether $\mathcal{R}(H)$ is finite or infinite, and if $\mathcal{R}(H)$ is finite, then $\mathcal{R}(H)$ can be constructed.


Lemma 4. Let $H = \langle \Sigma, \delta, x \rangle$ be a DOL system for which $\mathcal{R}(H)$ is infinite. Then $m$ and $n$, $m, n \in \mathbb{N}$, can be computed such that the DOL systems $\text{SUB}(H, n, m+i)$, $0 \leq i < n$, are all growing.

Proof. Note first that $\mathcal{R}(H)$ is infinite iff $\{\pi(\delta^i(x)) | n \geq 0\}$ is infinite. This, the assumption of the lemma and Lemma 2 imply that

$$\text{(4)} \quad \text{if } \pi(\delta^i(x)) = \pi(\delta^j(x)) \quad \text{then } i = j.$$  

So, compute $m$ and $n$ such that $\pi(\delta^m(x))$ and $\pi(\delta^{m+n}(x))$ are comparable. It follows from Konig (1959) that any infinite sequence of $\pi$-values contains at least two comparable elements, hence $m$ and $n$ satisfying this requirement can be computed. Assume that $\pi(\delta^m(x)) \geq \pi(\delta^{m+n}(x))$, this would imply, according to Lemma 2, that $\mathcal{R}(H)$ is finite, i.e. a contradiction. Hence $\pi(\delta^m(x)) < \pi(\delta^{m+n}(x))$.
and from (4) above and Lemma 2 you get that
\[ \pi(\delta^{m+n+i}(x)) < \pi(\delta^{m+n(i+1)+j}(x)) \]
for all \( i \), \( 0 \leq i \), and all \( j \), \( 0 \leq j < n \).

This proves the lemma. \( \blacksquare \)

Lemma 5. It is decidable, given any DOL system \( H \) and any \( n \in \mathbb{N} \), whether or not \( n \in \mathbb{N} \).

Proof. Follows from Lemma 4. \( \blacksquare \)

Definition 8. Two DOL systems \( H = \langle \Sigma_H, \delta_H, x_H \rangle \) and \( G = \langle \Sigma_G, \delta_G, x_G \rangle \) with growth functions \( h \) and \( g \) respectively are said to be ultimately growth equivalent iff there exist constants \( m, n \in \mathbb{N} \) such that
\[ h(m+i) = g(n+i) \]
for all \( i \), \( 0 \leq i \).

RESULTS ON THE EQUIVALENCE PROBLEMS

Theorem 1. The growth equivalence problem for DOL systems is decidable.

Proof. This theorem was proved in Paz and Salomaa (1973). It follows directly from Lemma 1 that two DOL systems \( H \) and \( G \) as in Lemma 1 are growth equivalent iff
\[ |\delta_H^i(x_H)| = |\delta_G^i(x_G)| \] for all \( i \), \( 0 \leq i < |\Sigma_H| + |\Sigma_G| \).

This proves the theorem. \( \blacksquare \)

Corollary 1. For any two DOL systems \( H \) and \( G \) as in Definition 8, for which \( \mathbb{N}(H) \) and \( \mathbb{N}(G) \) are both infinite, it is decidable whether or not \( H \) and \( G \) are ultimately growth equivalent.

Proof. It follows from Lemma 1 that \( H \) and \( G \) are ultimately growth equivalent iff there exists an \( i \in \mathbb{N} \) such that

either \( h(i) = g(i+i) \) for all \( i \), \( i \geq |\Sigma_H| + |\Sigma_G| \),
or \( h(i+i) = g(i) \) for all \( i \), \( i \geq |\Sigma_H| + |\Sigma_G| \).

But the existence of such an \( i \) can be decided from Lemma 4 and Theorem 1. \( \blacksquare \)

The purpose of this paper is to prove that also the growth range equivalence problem for DOL systems is decidable.

So let \( H = \langle \Sigma_H, \delta_H, x_H \rangle \) and \( \langle \Sigma_G, \delta_G, x_G \rangle \) be two given DOL systems. The first step in an algorithm to decide whether or not \( H \) is growth range equivalent to \( G \), is to apply the finiteness-algorithm of Lemma 3 to \( H \) and \( G \). If one or both of the systems turn out to be finite, then the question of growth range equivalence is trivially decidable. In the following we shall therefore assume that \( \mathbb{N}(H) \) and \( \mathbb{N}(G) \) are both infinite.
Lemma 6. If $H$ is polynomially bounded and $G$ is exponential, then $\mathbb{R}(H) \not= \mathbb{R}(G)$.

Proof. Since $\mathbb{R}(H)$ and $\mathbb{R}(G)$ are infinite you may decompose both systems according to Lemma 4 into growing subsystems, or, in other words, you may compute constants $n_H$ and $m_H$ ($n_G$ and $m_G$ respectively) satisfying the requirements of Lemma 4. Let us denote the growth functions of $\text{SUB}(H, n_H, m_H + i)$ ($\text{SUB}(G, n_G, m_G + i)$) by $h_i$, $0 \leq i < n_H$ ($g_i$, $0 \leq i < n_G$). Then from the assumptions of the lemma there exist constants $n_0$, $k \geq 1$ and $\alpha > 1$ such that

\begin{align}
(5) & \quad h_i(j) \leq j^k \quad \text{for all } j, j > n_0, \text{ and all } i, \quad 0 \leq i < n_H; \\
(6) & \quad g_i(j) \geq \alpha^j \quad \text{for all } j, j > n_0, \text{ and all } i, \quad 0 \leq i < n_G;
\end{align}

This follows almost directly from the fact that $h_i(j) = h(m_H + i + jn_H)$ and correspondingly $g_i(j) = g(m_G + i + jn_G)$.

But (5) and (6) imply

\begin{align}
(7) & \quad \left| \{ m \in \mathbb{R}(H) \mid m \leq n \} \right| \leq n^k \quad \text{for all } n > n_0^k; \\
(8) & \quad \left| \{ m \in \mathbb{R}(G) \mid m \leq n \} \right| \leq n_{H\cdot \log_{\alpha}(n)} + m_H \quad \text{for all } n > \alpha^{n_0}.
\end{align}

Note that it is used here that the constructed SUB-systems are all growing. (7) and (8) above are obviously contradictions to the assumption that $\mathbb{R}(H) = \mathbb{R}(G)$. This proves the lemma. □

Lemma 7. For any DOL system $H$ it is decidable whether $H$ is polynomially bounded or exponential.


From the two previous lemmas you get that the second step in an algorithm to decide growth range equivalence between two given DOL systems is to apply the algorithm of Lemma 7 to both of the given systems. If their growth types turn out to be different, then you know from Lemma 6 that they are not growth range equivalent.

So, after having applied the suggested two steps to the given systems, you have got your answer to the question of growth range equivalence, except in two cases: the growth ranges of both systems are infinite, and their growth types are the same (either both polynomially bounded or both exponential). These two cases will be treated seperately in the next two sections. The third step in the algorithm to decide growth range equivalence will, however, almost be the same in the two cases. Basically, what you get in both cases, is that $H$ and $G$ are growth range equivalent iff each SUB-system of $H$ with "sufficiently long"
period has an ultimately growth equivalent SUB-system of $G$, and the other way round.

**POLYNOMIALLY BOUNDED CASE**

**Lemma 8.** Let $H$ be a DOL system with polynomially bounded growth function for which $\mathbb{R}(H)$ is infinite. Then $m$ and $n$, $m, n \in N$, can be computed such that the growth functions $h_i$ of $\text{SUB}(H, n, m+i)$, $0 \leq i < n$, are all growing, and furthermore there exist polynomials $p_i$, $0 \leq i < n$, all of the same degree such that

$$h_i(j) = p_i(j) \quad \text{for all } j, 0 \leq j, \text{ and all } i, 0 \leq i < n.$$

**Proof.** It was proved in Ruohonen (1975) that there exist computable constants $m_1$ and $n_1$ such that the growth functions of the systems $\text{SUB}(H, n_1, m_1+i)$, $0 \leq i < n_1$, are all polynomials of the same degree. From Lemma 4 it follows that you can also compute $m_2$ and $n_2$ such that the DOL systems $\text{SUB}(H, n_2, m_2+i)$, $0 \leq i < n_2$, are all growing. But then obviously $m = \max(m_1, m_2)$ and $n = n_1$, $n_2$ will satisfy the requirements of the lemma.

**Lemma 9.** Let $H$ and $G$ be two growth range equivalent DOL systems, both of them with polynomially bounded growth functions. Furthermore, let $m_H, n_H$ and $m_G, n_G$ be constants satisfying the conditions of Lemma 8 for systems $H$ and $G$ respectively. Let $k$ denote the common degree of the polynomial growths of $\text{SUB}(H, n_H, m_H+i)$, $h_i$, $0 \leq i < n_H$, and let $l$ correspondingly denote the common degree of the polynomial growths of $\text{SUB}(G, n_G, m_G+i)$, $g_i$, $0 \leq i < n_G$. Then $k = l$.

**Proof.** The proof follows essentially the same lines as the proof of Lemma 6. Assume, e.g., that $l < k$. Then there exist constants $n_0 \in N$, $a > 0$ and $b$ such that

$$h_i(j) \geq a j^k \quad \text{for all } j, n_0 < j, \text{ and all } i, 0 \leq i < n_H;$$

$$g_i(j) \geq b j^l \quad \text{for all } j, n_0 < j, \text{ and all } i, 0 \leq i < n_G.$$  

But these relations imply

$$\left\{ m \in \mathbb{R}(H) \mid m \leq n \right\} \leq n_H(n/a)^{1/k} + m_H \quad \text{for all } n, n > a n_0^k;$$

$$\left\{ m \in \mathbb{R}(G) \mid m \leq n \right\} \geq (n/b)^{1/l} \quad \text{for all } n, n > b n_0^l.$$

Again the fact that all SUB-systems are growing is used, and again you reach a contradiction to the assumption that $H$ is growth range equivalent to $G$. 

Now, take any two DOL systems \( H \) and \( G \) and constants \( m_H, n_H, m_G \) and \( n_G \) satisfying the conditions of Lemma 9, and take any of the systems \( \text{SUB}(H, n_H, m_{H+i_1}) \) from Lemma 9. Let us denote the growth function of this system by \( p \),

\[
p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0.
\]

All the values of \( p \) occur, from the assumption that \( H \) and \( G \) are growth range equivalent as values of the growth function of \( G \). That is, you may define a function \( \gamma \) mapping \( \mathbb{N} \) to the set \( \{0, 1, \ldots, n_G\} \) satisfying

\[
\text{if } \gamma(j) = i \quad \text{for some } i, \quad 0 \leq i < n_G \quad \text{then } p(j) \text{ occurs as a value of the growth function of } \text{SUB}(G, n_G, m_{G+i}) \text{, } g_i; \]

\[
\quad \text{if } \gamma(j) = n_G \quad \text{then } p(j) \text{ occurs as one of the first } m_{G-1} \text{ values of the growth function of } G \text{, i.e., } p(j) = g(n) \text{ for some } n < m_G;
\]

Note that \( \gamma \) is not necessarily unique, but at least one function \( \gamma \) satisfies (9), and when referring to \( \gamma \) in the following we shall mean one fixed \( \gamma \) satisfying (9).

Obviously, there exists an \( n_0 \in \mathbb{N} \) such that for all \( j > n_0 \), \( \gamma(j) < n_G \). Secondly, applying the theorem of van der Waerden as stated in Chintischin (1951) to \( \gamma \), you get that there exists a computable function, \( W \), mapping \( \mathbb{N} \) to \( \mathbb{N} \), such that for any \( l \in \mathbb{N} \), any set of \( W(l) \) consecutive elements of \( \mathbb{N} \) contains an arithmetic progression of length \( l \) on which \( \gamma \) assumes the same value. \( W \) is defined recursively as follows:

\[
W(l) = \cup(n_G, l) ;
\]

\[
\cup(x, y) = \begin{cases} x + 1 & \text{if } y \leq 2 \\ \upsilon(x, y-1) & \text{otherwise} \end{cases} ;
\]

\[
\upsilon(x, y) = \begin{cases} 1 & \text{if } x = 0 \\ 2 \cdot \cup(x, \upsilon(x-1, y), y) \cdot \upsilon(x-1, y) & \text{otherwise} \end{cases} ;
\]

It should be stressed that the theorem of van der Waerden does not give you an infinite arithmetic progression on which \( \gamma \) assumes the same value. The application of the theorem gives you, however, the following lemma.

\textbf{Lemma 10.} There exist an \( i_0 \), \( 0 \leq i_0 < n_G \), and a constant \( r \in \mathbb{N} \), \( r < W(|\Sigma_H| + |\Sigma_G| - 1) \), such that for arbitrary big \( n \)-values
\[ i_0 = \gamma(n) = \gamma(n+r) = \ldots = \gamma(n+r(|\Sigma_H| + |\Sigma_G| - 1)) \]

**Proof.** Let \( N_j \) denote the set of consecutive elements of \( N \) from \( j \cdot W(|\Sigma_H| + |\Sigma_G| - 1) \) to \((j+1) \cdot W(|\Sigma_H| + |\Sigma_G| - 1) - 1 \) for all \( j \in N \). If you apply the theorem of van der Waerden as stated above for \( I = |\Sigma_H| + |\Sigma_G| - 1 \) to \( N_j \) for every \( j \in N \), you get that every \( N_j \) contains an arithmetic progression of length \( |\Sigma_H| + |\Sigma_G| - 1 \) on which \( \gamma \) assumes the same value. But since the periods of these arithmetic progressions are obviously bounded by \( W(|\Sigma_H| + |\Sigma_G| - 1)/(|\Sigma_H| + |\Sigma_G| - 1) \), and since \( \gamma \) assumes only finitely many values, the lemma follows directly. \( \square \)

Let us denote the growth function of \( \text{SUB}(G, n_G, m_G + i_0) \) by \( q \), where \( i_0 \) satisfies Lemma 10,

\[ q(n) = b_k n^k + b_{k-1} n^{k-1} + \ldots + b_0 \; ; \]

Let \( N_1 \) denote the infinite subset of \( N \) for which \( \gamma \) assumes the value \( i_0 \). Define \( \varphi \) as the function mapping \( N_1 \) to reals, satisfying

\[
(10) \quad p(n) = q((a_k/b_k)^{1/k} \cdot n + \varphi(n))
= q(c \cdot n + \varphi(n)) \quad \text{for all } n \in N_1.
\]

Note that \( \varphi \) is unique from the fact that all \( \text{SUB} \)-systems are growing.

**Lemma 11.** \( \varphi(n)/n \to 0 \) for \( n \to \infty \) in \( N_1 \).

**Proof.** From (10) you get (dividing by \( n^k \))

\[
a_k + \frac{a_{k-1}}{n} + \ldots + \frac{a_0}{n^k} = \frac{b_k (cn + \varphi(n))^k}{n^k} + \ldots + \frac{b_0}{n^k}.
\]

The lefthand side of this equation converges on \( N_1 \) to \( a_k \) for \( n \) going to infinity. But since clearly \( (cn + \varphi(n)) \) goes to infinity with \( n \), you have that the quotient between any of the \( k \) last terms on the righthand side and the first term on the righthand side converge to 0 for \( n \) going to infinity in \( N_1 \). From this you get

\[
\frac{b_k (cn + \varphi(n))^k}{n^k} \to a_k \quad \text{for } n \to \infty \text{ in } N_1
\]

which implies

\[
\frac{cn + \varphi(n)}{n} \to \left(\frac{a_k}{b_k}\right)^{1/k} \quad \text{for } n \to \infty \text{ in } N_1
\]

and this proves the lemma. \( \square \)
Lemma 12.

\[ \varphi(n) \rightarrow \frac{a_{k-1} - b_{k-1} \cdot c^{k-1}}{k \cdot b_k \cdot c^{k-1}} = d \quad \text{for} \quad n \to \infty \quad \text{in} \quad \mathbb{N}_1. \]

**Proof.** From (10) you get that for all \( n \in \mathbb{N}_1 \)

\[
\sum_{i=0}^{k} a_i n^i = \sum_{i=0}^{k} b_i \left( \sum_{j=0}^{i} \binom{i}{j} (cn)^{i-j} (\varphi(n))^{j} \right),
\]

which implies

\[
\sum_{i=0}^{k} (a_i - b_i c^i) n^i = \varphi(n) \cdot \sum_{i=0}^{k} b_i \left( \sum_{j=1}^{i} \binom{i}{j} (cn)^{i-j} (\varphi(n))^{j} \right).
\]

Observe that the coefficient of the \( n^k \)-term on the lefthand side of (11) is equal to zero, so if you divide the equation (11) by \( n^{k-1} \) you get that the resulting left-hand side converges to \((a_{k-1} - b_{k-1} c^{k-1})\) for \( n \) going to infinity in \( \mathbb{N}_1 \).

Furthermore, you get from Lemma 11 that if you divide the sum-expression of the righthand side of (11) by \( n^{k-1} \), the resulting sum-expression will converge to \( b_k \cdot k \cdot c^{k-1} \) for \( n \) going to infinity in \( \mathbb{N}_1 \). This proves the lemma. \( \blacksquare \)

Lemma 13. Let \( r \) be the constant of Lemma 10, \( r \in \mathbb{N}, \) then \( c = \frac{s}{r} \), for some \( s \in \mathbb{N} \).

**Proof.** Let \( \mathbb{N}_2 \) denote the following subset of \( \mathbb{N}_1 \)

\[ \mathbb{N}_2 = \{ n \in \mathbb{N}_1 \mid i_0 = \gamma(n) = \gamma(n+r) \} . \]

It then follows from Lemma 10 that \( \mathbb{N}_2 \) is infinite. Consider the function \( \psi \) mapping \( \mathbb{N}_2 \) for \( N \) defined by

\[ \psi(n) = (c(n+r) + \varphi(n+r)) - (cn + \varphi(n)) \]

\[ = cr + \varphi(n+r) - \varphi(n) \quad \text{for all} \quad n \in \mathbb{N}_2 . \]

From Lemma 12 you get

\[ \psi(n) \rightarrow cr + d - d = cr \quad \text{for} \quad n \to \infty \quad \text{in} \quad \mathbb{N}_2 . \]

But since \( \psi(n) \) is an integer-function this means that \( cr \) is an integer, and this proves the lemma. \( \blacksquare \)

Lemma 14. \( d = \frac{t}{r} \) for some \( t \in \mathbb{N}, \) and there exists a constant \( n_0 \in \mathbb{N}_1 \) such that

\[ \varphi(n) = d \quad \text{for all} \quad n > n_0, \quad n \in \mathbb{N}_1 . \]
Proof. It follows from the definition of $\varphi$ that $(cn + \varphi(n))$ is an integer-function, i.e. (Lemma 13) $\varphi(n)$ is an integer-function converging (Lemma 12) to $\varphi$ for $n$ going to infinity in $N$. This implies that $\varphi$ is an integer (which proves the first part of the lemma), and secondly it implies the existence of $n_0 \in \mathbb{N}$ such that $\varphi(n) = \varphi$ for all $n > n_0$, $n \in \mathbb{N}$ (which proves the second part of the lemma).

Combining Lemma 10 and Lemma 14 you get that there exists a constant $n_1 \geq n_0$, such that

$$p(n_1 + jr) = q(c(n_1 + jr) + d) \text{ for all } j, \ 0 \leq j < |\Sigma_H| + |\Sigma_G| - 1.$$ 

This implies, however, from Lemma 1 that $\text{SUB}(H, r_{n_1}, m_{n_1}, n_{n_1})$ is growth equivalent to $\text{SUB}(G, s_{n_1}, m_{n_1}, n_{n_1})$.

Now take any constant $p_1 \in \mathbb{N}$ and any $\text{SUB}$-system, $H_i$, of the form

$$H_i = \text{SUB}(H, p_1 r_{n_{n_1}}, m_{n_1}, i_{n_1}) \quad 0 \leq i < p_1 r$$

for which $(n_1 \mod p_1 r) \neq i$.

If you apply the arguments of this section to this $\text{SUB}$-system you get that there exist constants $p_2, n_2 \in \mathbb{N}$, $p_2 < W(|\Sigma_H| + |\Sigma_G| - 1)$, such that

$$\text{SUB}(H, p_2 p_1 r_{n_{n_1}}, m_{n_1}, i_{n_1} + n_{n_1})$$

is either

(12) growth equivalent to some $\text{SUB}$-system of $\text{SUB}(G, n_{n_1}, m_{n_1})$ for $j \parallel i_0$, or

(13) growth equivalent to a $\text{SUB}$-system of $\text{SUB}(G, n_{n_1}, m_{n_1})$ with period $p_2 p_1 s$.

Note first that (12) and (13) are not necessarily exclusive, and secondly that (13) follows from the fact that the leading coefficient of the polynomial growth of a $\text{SUB}$-system of $H_i$ with period $p_2$ is equal to $a_k (r p_1 p_2)^k$, whereas the leading coefficient of the polynomial growth of a $\text{SUB}$-system of $\text{SUB}(G, n_{n_1}, m_{n_1})$ with period $x$ is equal to $b_k x^k$. So from the assumption that the two $\text{SUB}$-systems are growth equivalent you get

$$a_k (r p_1 p_2)^k = b_k x^k$$

which implies

$$x = p_2 p_1 s.$$ 

If we denote the growth function of the chosen $\text{SUB}$-system $H_i$ by $h_i$, and correspondingly the growth function of $\text{SUB}(G, s_{n_1}, m_{n_1} + j n_{n_1})$ by $g_j$, $0 \leq j < s$, then (13) can be stated in the following form.
there exist \( j, l \in \mathbb{N}, 0 \leq j < s \), such that the function

\[
f(n) = h_i(n+n_2) - g_j(p_1 n + l)\]

assumes an infinite number of zero-values for arguments forming an arithmetic progression with period \( p_2 \).

Remember, however, that the function \( f \) of (13!) satisfies a linear, homogenous recurrence relation of length less than or equal to \( (|\Sigma_H| + |\Sigma_G|) \), i.e., you can apply the following lemma, which was proven in Berstel and Mignotte (1974).

**Lemma 15.** Let \( f \) be a function mapping \( \mathbb{N} \) to integers, satisfying a linear, homogenous recurrence relation of length 1, and assuming an infinite number of zero-values. Then the arguments for which \( f \) assumes the value zero, form, apart from a finite number of them, a finite union of arithmetic progressions for which the least common multiple of the periods is bounded by \( B(l-1) \), where

\[
B(x) = \exp \left( 4x \sqrt{10 \log x} \right) ,
\]

From this you get the following lemma.

**Lemma 16.** Define \( u_1 = W(|\Sigma_H| + |\Sigma_G| - 1) \), \( u_2 = B(|\Sigma_H| + |\Sigma_G| - 1) \), \( u_1 \quad u_2 \)

\[
v = (\prod_{i=1}^{u_1} i) \cdot (\prod_{i=1}^{u_2} i),
\]

and let \( H_i \) denote the SUB-system

\[
\text{SUB}(H, n_H \cdot v, m_H \cdot l)
\]

for all \( 0 \leq i < n_H \cdot v \).

Then the assumption that \( R(H) = R(G) \) implies that for each \( i \), there exists an \( n_i \in \mathbb{N} \), such that \( \text{SUB}(H, n_H \cdot v, m_H \cdot l + n_i n_H \cdot v) \) is growth equivalent to some subsystem of \( G \). Furthermore, if you denote the leading coefficient of the polynomial growth of \( \text{SUB}(H, n_H \cdot v, m_H \cdot l) \) by \( a_i \), \( 0 \leq i < n_H \), and correspondingly the leading coefficient of the polynomial growth of \( \text{SUB}(G, n_G, m_G \cdot j) \) by \( b_j \), \( 0 \leq j < n_G \), and define

\[
y = \max \left\{ \left( \frac{a_i}{b_j} \right)^{\frac{1}{k}} \bigg| 0 \leq i < n_H, 0 \leq j < n_G \right\} ,
\]

then the period of this subsystem of \( G \) is bounded by \( n_G \cdot v \cdot y \).

**Proof.** Follows directly from the arguments of this section, and the fact that you may assume that the period of any arithmetic progression as in (13!) is a divisor in \( v \) by Lemma 15. \( \Box \)
Theorem 2. Let \( H \) and \( G \) be two DOL systems both of them with polynomially bounded growth functions, for which \( R(H) \) and \( R(G) \) are both infinite sets. It is then decidable whether or not \( R(H) = R(G) \).

Proof. The algorithm to decide whether \( R(H) = R(G) \) under the assumptions of the theorem, goes as follows:

Compute \( n_H, m_H, n_G, m_G; \) \( v \) and \( y \) satisfying Lemma 9 and Lemma 16.

For each \( i, 0 \leq i \leq n_H \), you decide whether \( \text{SUB}(H, n_H, v, m_H+i) \) is ultimately growth equivalent to some \( \text{SUB} \)-system of the form \( \text{SUB}(G, k, m_G+j) \) where \( 0 \leq j \leq k \), and \( 0 \leq k \leq n_G \).

This can be done by a finite number of applications of the algorithm of Corollary 1. If this is not the case you know from Lemma 16 that \( R(H) \) is not equal to \( R(G) \); if it is the case, you get from the application of the algorithm of Corollary 1 an \( n_i \) satisfying Lemma 16. If you do find an ultimately growth equivalent \( \text{SUB} \)-system of \( G \) for all \( i, 0 \leq i \leq n_H \), you know that \( R(H) \subseteq R(G) \) iff

\[
\{| \delta^n_H(x_H) | 0 \leq n \leq m_H, \max_{0 \leq i \leq n_H} (n_i, n_H, v) \subseteq R(G) \}.
\]

But this relation can easily be decided by a finite number of applications of the membership algorithm of Lemma 5.

Finally, you interchange the roles of systems \( H \) and \( G \), and repeat the steps described above.

Now, obviously \( R(H) = R(G) \) iff the algorithms described above end with the results \( R(H) \subseteq R(G) \) and \( R(G) \subseteq R(H) \). This proves the theorem.

Remark. You should note that the algorithm given in the proof of Theorem 2 does not include an algorithm to decide whether \( R(H) \subseteq R(G) \) in general. The reason for this is that the first part of the algorithm (the part of finding growth equivalent \( \text{SUB} \)-systems) is working with the assumption \( R(H) = R(G) \), i.e. this part may stop with a negative answer to the question of growth range equivalence even though \( R(H) \subseteq R(G) \). An example of this phenomenon would be \( H \) generating the sequence \( n^4 \) and \( G \) generating \( n^2 \). Then \( R(H) \subseteq R(G) \), but no \( \text{SUB} \)-system of \( H \) is growth equivalent to any \( \text{SUB} \)-system of \( G \).

Expontential Case

The proof of the decidability of growth range equivalence in the case where both systems have exponential growth functions follows essentially the same lines as the proof in the polynomially bounded case.

Lemma 17. Let \( H \) be a DOL system as in Definition 1 with exponential growth function \( h \) satisfying a linear, homogenous recurrence relation (2). Then \( m \) and \( n, m, n \in N \), can be computed such that the growth functions \( h \).
of \( \text{SUB}(H, n, m+i) \), \( 0 \leq i < n \), are all growing, and there exists a real number \( \xi > 1 \), such that \( \xi \) is a root of the characteristic polynomial of the (common) recurrence relation of all the \( h_i \)'s, and furthermore for any other root \( \alpha \) of this polynomial, one has \( |\alpha| < \xi \).

**Proof.** By Lemma 4 integers \( m_1 \) and \( n_1 \) can be computed such that the growth functions \( h_1 \) of \( \text{SUB}(H, n_1, m_1+i) \), \( 0 \leq i < n_1 \), are all growing. By a theorem from Berstel (1975), integers \( m_2 \) and \( n_2 \) can be computed such that the growth functions \( h_2 \) of \( \text{SUB}(H, n_2, m_2+i) \), \( 0 \leq i < n_2 \), satisfy the second condition of the lemma. Putting now \( m = \max(m_1, m_2) \) and \( n = n_1 \cdot n_2 \) the growth functions \( h_i \) of \( \text{SUB}(H, n, m+i) \), \( 0 \leq i < n \), satisfy both conditions.

Now, take any two growth range equivalent DOL systems \( H \) and \( G \), both of them satisfying the conditions of Lemma 17. You can then compute constants \( m_H, n_H \in \mathbb{N} \) and \( \xi > 1 \) for system \( H \) according to Lemma 17, and correspondingly \( m_G, n_G \in \mathbb{N} \) and \( \eta > 1 \) for system \( G \).

Consider then any JUB-system of \( H \) of the form \( \text{SUB}(H, n_{i_1}, m_{i_1}+i) \) where \( 0 \leq i_1 < n_H \). The growth function of this SUB-system can be expressed as (follows from Lemma 17 and (3))

\[
p(n) = a \cdot n^k \cdot \xi^n + p_2(n) = p_1(n) + p_2(n),
\]

for some constants \( a > 0, \ k \in \mathbb{N}, \) and a function \( p_2 \) mapping \( \mathbb{N} \) to reals, for which

\[
\frac{p_2(n)}{a n^k \cdot \xi^n} \to 0 \quad \text{for } n \to \infty \quad \text{in } \mathbb{N}.
\]

Define for this particular SUB-system of \( H \), a function \( \gamma \) exactly like in the previous section (9). Applying van der Waerden's theorem you get that there exist constants \( i_0, r \in \mathbb{N}, \ 0 \leq i_0 < n_G \), satisfying the requirements of Lemma 10. The growth function of \( \text{SUB}(G, n_{i_0}, m_{i_0}+i_0) \) can be expressed as

\[
q(n) = b \cdot n^l \cdot \eta^n + q_2(n) = q_1(n) + q_2(n),
\]

for some constants \( b > 0, \ l \in \mathbb{N}, \) and a function \( q_2 \) for which

\[
\frac{q_2(n)}{b n^l \cdot \eta^n} \to 0 \quad \text{for } n \to \infty \quad \text{in } \mathbb{N}.
\]

Define as in the previous section \( N_1 \) as the infinite subset of \( \mathbb{N} \) for which \( \gamma \) assumes the value \( i_0 \), and define the unique function \( \varphi \) mapping \( N_1 \) to reals satisfying

\[
p(n) = q(cn + \varphi(n)) \quad \text{for all } n \in N_1,
\]
where \( c \) is the constant satisfying \( \xi = \eta^c \) (\( c = \log \xi / \log \eta \)). Let \( \lambda \) denote the function defined by

\[
\lambda(n) = cn + \varphi(n) \quad \text{for all} \quad n \in N_1.
\]

**Lemma 18.** \( \varphi(n)/n \to 0 \) for \( n \to \infty \) in \( N_1 \).

**Proof.** Since the system \( \text{SUB}(H, n_H, m_H+i) \) is growing, one has \( \lambda(n) \to \infty \) for \( n \to \infty \) in \( N_1 \). Now, from (14) one has

\[
1 + \frac{p_2(n)}{p_1(n)} = \frac{q_1(\lambda(n))}{p_1(n)} (1 + \frac{q_2(\lambda(n))}{p_1(n) \cdot q_1(\lambda(n))})
\]

for all \( n \in N_1 \), and this implies

\[
(15) \quad \frac{q_1(\lambda(n))}{p_1(n)} = \frac{b}{a} \left( \frac{(\lambda(n))}{n^k} \right)^{\varphi(n)} \to 1 \quad \text{for} \quad n \to \infty \quad \text{in} \quad N_1.
\]

Applying the logarithmic function to this convergence you get

\[
\log \left( \frac{b}{a} \right) + \frac{1}{n} \cdot \log(\lambda(n)) - k \cdot \log(n) + \varphi(n) \cdot \log \eta \to 0
\]

for \( n \to \infty \) in \( N_1 \).

Dividing this convergence by \( cn \) you get

\[
\frac{1}{cn} \cdot \frac{\log(\lambda(n))}{\log \eta} + \frac{\varphi(n) \cdot \log \eta}{cn} \to 0 \quad \text{for} \quad n \to \infty \quad \text{in} \quad N_1,
\]

i.e.,

\[
\frac{1}{cn} \cdot \frac{\log(\lambda(n))}{\log \eta} + \frac{\lambda(n) \cdot \log \eta}{cn} \to \log \eta
\]

for \( n \to \infty \) in \( N_1 \).

But now clearly the quotient between the first and the second term of the lefthand side of this convergence converges to zero, which implies

\[
\frac{\lambda(n) \cdot \log \eta}{cn} \to \log \eta
\]

for \( n \to \infty \) in \( N_1 \),

and this proves the lemma. \( \qed \)

**Lemma 19.** Let \( N_2 \) denote the infinite subset of \( N_1 \) defined by

\[
N_2 = \{ n \in N_1 \mid i_0 = \gamma(n) = \gamma(n+r) \}.
\]

Then

\[
\varphi(n+r) - \varphi(n) \to 0 \quad \text{for} \quad n \to \infty \quad \text{in} \quad N_2.
\]
Proof. From (15) you get

\[
\frac{b}{a} \left( c + \frac{\varphi(n)}{n} \right) \downarrow n^{-k} \eta \varphi(n) \to 1 \quad \text{for } n \to \infty \text{ in } N_1,
\]

but since \( \varphi(n)/n \) converges to zero this implies

\[
n^{-k} \eta \varphi(n) \to \frac{a}{b} \frac{c}{1} \quad \text{for } n \to \infty \text{ in } N_1.
\]

Applying the logarithmic function to this convergence you get

\[
(l-k) \cdot \log(n) + \varphi(n) \cdot \log \eta \to \log \left( \frac{a}{b} \frac{c}{1} \right)
\]

\[
\quad \text{for } n \to \infty \text{ in } N_1.
\]

From this you get

\[
(l-k) \cdot (\log(n+r) - \log(n)) + (\varphi(n+r) - \varphi(n)) \cdot \log \eta \to 0
\]

\[
\quad \text{for } n \to \infty \text{ in } N_2.
\]

Clearly the first term of the above left-hand side expression converges to zero for \( n \) going to infinity in \( N_2 \), and hence so does the second, and this proves the lemma.

Consider the function \( \psi \) mapping \( N_2 \) to \( N \) defined by

\[
\psi(n) = \lambda(n+r) - \lambda(n) = cr + \varphi(n+r) - \varphi(n),
\]

for all \( n \in N_2 \).

From Lemma 19 you get that

\[
\psi(n) \to cr
\]

\[
\quad \text{for } n \to \infty \text{ in } N_2.
\]

But since \( \psi(n) \) is an integer function you get from this that \( cr \) is an integer, i.e., you get the following lemma corresponding to Lemma 13.

\textbf{Lemma 20.} \( c = \frac{s}{r} \) for some \( s \in N \).

\textbf{Lemma 21.} \( \exists n_0 \in N \forall n \in N_2, \ n > n_0 : \varphi(n+r) = \varphi(n) \).

Proof. Notice that \( \lambda(n) \) is an integer function, i.e., (from Lemma 20) \( \varphi(n) = \tau(n)/r \) for some integer function \( \tau \). The lemma follows now directly from Lemma 19.

Since \( \gamma \) satisfies the conditions of Lemma 10, Lemma 21 implies that there exists an \( n_1 \in N \), \( n_1 > n_0 \) such that

\[
\varphi(n_1) = \varphi(n_1+r) = \ldots = \varphi(n_1 + r(|\Sigma_H| + |\Sigma_G| - 1)).
\]
Applying Lemma 1 this means that $\text{SUB}(H, m_H, n_H, m_H, n_H, i_1 + n_1 n_H)$ is growth equivalent to $\text{SUB}(G, m_G, n_G, m_G, n_G, i_0 + (c n_1 + \varphi(n_1)) n_G)$.

From this point, the proof goes exactly like the proof in the previous section, and finally you end up with the following theorem.

**Theorem 3.** Let $H$ and $G$ be two DOL systems both of them with exponential growth functions. It is then decidable whether or not $\mathcal{R}(H) = \mathcal{R}(G)$.

**Proof.** The proof is identical to the proof of Theorem 2, with the exceptions that in the algorithm to decide whether $\mathcal{R}(H) \subseteq \mathcal{R}(G)$

$$m_H, n_H, m_G, n_G$$

are computed according to Lemma 17, and

$$y = c.$$ 

All other details are left to the reader. ■

**Theorem 4.** The growth range equivalence problem for DOL-systems is decidable.

**Proof.** Follows from the remarks just after Lemma 7, Theorem 2 and Theorem 3. ■

**Corollary 2.** For any two DOL systems $H$ and $G$ it is decidable whether or not there exist finite sets $F_H$ and $F_G$, $F_H, F_G \subseteq \mathbb{N}$, such that

$$\mathcal{R}(H) \backslash F_H = \mathcal{R}(G) \backslash F_G.$$ 

**Proof.** Follows from slight modifications of the proof of Theorem 4. The only interesting thing to notice is that the part of the algorithm corresponding to the algorithm of Theorem 2, consists simply of the first part of the algorithm of Theorem 2 (the part of finding growth equivalent SUB-systems) applied to $H$ and $G$. And similarly, for the part corresponding to the algorithm of Theorem 3. All details are left to the reader.

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