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Abstract

In 1957 V. K. Saulyev proposed two so-called asymmetric methods for solving parabolic equations. We study these methods w.r.t. their stability and consistency, how to include first order derivative terms, how to apply boundary conditions with a derivative, and how to extend the methods to two space dimensions. We also prove that the various modifications proposed by Saulyev, Barakat and Clark, and Larkin also (as was to be expected) require $k = o(h)$ in order to be consistent. As a curiosity we show that the two original Saulyev methods in fact solve two different differential equations.

MSC 65M06, 65M12

1 Two Saulyev Methods

The first Saulyev method, called **LR**, for the equation

$$u_t = bu_{xx} \tag{1}$$

can be written

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^n - v_m^n - v_m^{n+1} + v_{m-1}^{n+1}}{h^2} \tag{2}$$

where h and k are the step sizes in the x- and t-direction, respectively, m and n are the corresponding step numbers, and v is an approximation to the true solution u. Here and in the following we shall use the notation of [7] (see pp. 7ff). Equation (2) can be rewritten as

$$(1 + b\mu)v_m^{n+1} = b\mu v_{m-1}^{n+1} + (1 - b\mu)v_m^n + b\mu v_{m+1}^n \tag{3}$$

where $\mu = k/h^2$. The **LR**-formula is implicit in nature but can be solved in an explicit fashion from left to right using the (Dirichlet) boundary condition on the left boundary to get started.

The second Saulyev method, called **RL**, for the same equation can be written

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m-1}^n - v_m^n - v_m^{n+1} + v_{m+1}^{n+1}}{h^2} \tag{4}$$

or

$$(1 + b\mu)v_m^{n+1} = b\mu v_{m+1}^{n+1} + (1 - b\mu)v_m^n + b\mu v_{m-1}^n \tag{5}$$

This formula can also be solved in an explicit fashion, now from right to left using the (Dirichlet) boundary condition on the right boundary for the first step.

2 Stability

To study the stability of the LR-method we use the von Neumann approach ([2], [8], p. 23) and compute the growth factor

$$g_{LR}(\varphi) - 1 = b\mu(e^{i\varphi} - 1 - g_{LR}(1 - e^{-i\varphi})) \tag{6}$$

or

$$g_{LR} = \frac{1 + b\mu(e^{i\varphi} - 1)}{1 + b\mu(1 - e^{-i\varphi})} = \frac{1 - b\mu(1 - \cos\varphi) + ib\mu\sin\varphi}{1 + b\mu(1 - \cos\varphi) + ib\mu\sin\varphi} \tag{7}$$

The condition $|g_{LR}| \leq 1$ is equivalent to

$$(1 - b\mu(1 - \cos\varphi))^2 + b^2\mu^2\sin^2\varphi \leq (1 + b\mu(1 - \cos\varphi))^2 + b^2\mu^2\sin^2\varphi$$

or

$$-2b\mu(1 - \cos\varphi) \leq 2b\mu(1 - \cos\varphi)$$

which is always satisfied for $b > 0$, and the Saulyev LR-method is therefore unconditionally stable.
A similar calculation reveals the same to be true for the RL-method.

3 Consistency

In order to check for consistency we apply the difference operator for the LR-method (cf. [8], p. 30) on a smooth function ψ:

$$\begin{aligned}
P^{LR}_{k,h}\psi &= \frac{\psi^{n+1}_m - \psi^n_m}{k} - b\frac{\psi^n_{m+1} - \psi^n_m - \psi^{n+1}_m + \psi^{n+1}_{m-1}}{h^2} \\
&= \psi_t + \frac{1}{2}k\psi_{tt} + \frac{1}{6}k^2\psi_{ttt} + \cdots \\
&\quad - \frac{b}{h}(\psi_x + \frac{1}{2}h\psi_{xx} + \frac{1}{6}h^2\psi_{xxx} + \cdots) \\
&\quad + \frac{b}{h}(\psi_x - \frac{1}{2}h\psi_{xx} + \frac{1}{6}h^2\psi_{xxx} + \\
&\quad + k\psi_{xt} - \frac{1}{2}hk\psi_{xxt} + \frac{1}{2}k^2\psi_{xtt} + \cdots) \qquad (8) \\
&= \psi_t - b\psi_{xx} + b\frac{k}{h}\psi_{xt} + \frac{1}{2}k(\psi_{tt} - b\psi_{xxt} + b\frac{k}{h}\psi_{xtt} + \frac{1}{3}k\psi_{ttt}) + \cdots
\end{aligned}$$

We recognize the differential operator for (1) in the first two terms, and the remaining terms (which constitute what we call the *local truncation error*) must tend to 0 as h and k tend to 0 for the LR-method to be consistent. We must

therefore require that k tends to 0 faster than h. For the method to be first order (in h) we must require k to be $O(h^2)$. This is a requirement much like the stability condition for the explicit method (cf. [8], p. 25), although we are no longer bound by the proportionality constant 0.5. On the other hand the LR-method is then only of order 1 in h.
A similar calculation for the RL-method gives

$$P_{k,h}^{RL}\psi = \psi_t - b\psi_{xx} - b\frac{k}{h}\psi_{xt} + \frac{1}{2}k(\psi_{tt} - b\psi_{xxt} - b\frac{k}{h}\psi_{xtt} + \frac{1}{3}k\psi_{ttt}) + \cdots \quad (9)$$

and similar comments on consistency and order apply for the RL-method. We note that the annoying $\frac{k}{h}$-term appears with opposite sign in the two expressions. Saulyev himself did not advise to use these methods by themselves ([7], p. 29) but instead suggested to use LR and RL alternately, e.g. LR in the odd steps and RL in the even steps ([6], [7], p.43), in order that the $\frac{k}{h}$-terms might partially compensate each other. Another suggestion ([1], [5]) with the same intention is to compute with both LR and RL in each step and take the average. This, however, means doubling the computational work.
We shall refer to these methods by the names ALT and AV, respectively.
It is obvious that either approach is unconditionally stable.
It is less obvious what the consistency requirements are.

4 The Local Error

Instead of the local truncation error we shift attention to the *local error* i.e. the difference between the true solution and the numerical solution obtained with true starting values and divided by k to compensate for the fact that when we use half the time step size we must take twice as many time steps. This division by k also makes a direct comparison with the local truncation error possible.
We rewrite (3) to

$$v_m^{n+1} = cv_{m-1}^{n+1} + dv_m^n + cv_{m+1}^n \quad (10)$$

with

$$c = \frac{b\mu}{(1+b\mu)} \quad \text{and} \quad d = \frac{(1-b\mu)}{(1+b\mu)}. \quad (11)$$

In order to study the local error we now assume that the values at time level n are values of the true solution, $v_m^n = u_m^n$, $0 \leq m \leq M$, and also for the boundary values, $v_0^{n+1} = u_0^{n+1}$ and $v_M^{n+1} = u_M^{n+1}$. The true solution at time level $n+1$ satisfies

$$u_m^{n+1} = u_m^n + ku_t + \frac{1}{2}k^2u_{tt} + \frac{1}{6}k^3u_{ttt} + \cdots \quad (12)$$

For the LR-method the numerical solution at $m = 1$ is

$$\begin{aligned}
(v_1^{n+1})_{LR} &= du_1^n + cu_2^n + cu_0^{n+1} \\
&= du_1^n + c(u_1^n + hu_x + \frac{1}{2}h^2u_{xx} + \frac{1}{6}h^3u_{xxx} + \cdots) \\
&\quad + c(u_1^n - hu_x + ku_t + \frac{1}{2}h^2u_{xx} - hku_{xt} + \frac{1}{2}k^2u_{tt} \\
&\qquad - \frac{1}{6}h^3u_{xxx} + \frac{1}{2}h^2ku_{xxt} - \frac{1}{2}hk^2u_{xtt} + \frac{1}{6}k^3u_{ttt} + \cdots) \\
&= u_1^n + c(ku_t + h^2u_{xx} - hku_{xt} + \frac{1}{2}k^2u_{tt} \\
&\qquad + \frac{1}{2}h^2ku_{xxt} - \frac{1}{2}hk^2u_{xtt} + \frac{1}{6}k^3u_{ttt} + \cdots) \qquad (13) \\
&= u_1^n + ku_t - chku_{xt} + \frac{1}{2}c(k^2u_{tt} + h^2ku_{xxt} - hk^2u_{xtt} + \frac{1}{3}k^3u_{ttt}) + \cdots
\end{aligned}$$

where we have used that $d + 2c = 1$ and $b\mu h^2u_{xx} = kbu_{xx} = ku_t$.
The m-th term of the computed solution can be expressed as

$$\begin{aligned}
(v_m^{n+1})_{LR} &= u_m^n + ku_t - (c + c^2 + \cdots + c^m)hku_{xt} + \frac{1}{2}c^mk^2u_{tt} \qquad (14) \\
&\quad + \frac{1}{2}(c + 3c^2 + 5c^3 + \cdots + (2m-1)c^m)h^2ku_{xxt} \\
&\quad - \frac{m}{2}c^mhk^2u_{xtt} + \frac{1}{6}c^mk^3u_{ttt} + \cdots
\end{aligned}$$

which we shall prove by induction. Above we have shown (14) for $m = 1$. Now assume (14) to hold for $m - 1$. Then

$$\begin{aligned}
(v_m^{n+1})_{LR} &= du_m^n + cu_{m+1}^n + cv_{m-1}^{n+1} \\
&= du_m^n + c(u_m^n + hu_x + \frac{1}{2}h^2u_{xx} + \frac{1}{6}h^3u_{xxx} + \cdots) \\
&\quad + c(u_m^n - hu_x + \frac{1}{2}h^2u_{xx} - \frac{1}{6}h^3u_{xxx} + ku_t - hku_{xt} + \frac{1}{2}h^2ku_{xxt} \\
&\qquad - (c + c^2 + \cdots + c^{m-1})(hku_{xt} - h^2ku_{xxt}) \\
&\qquad + \frac{1}{2}c^{m-1}(k^2u_{tt} - hk^2u_{xtt}) \\
&\qquad + \frac{1}{2}(c + 3c^2 + 5c^3 + \cdots + (2m-3)c^{m-1})h^2ku_{xxt} \\
&\qquad - \frac{m-1}{2}c^{m-1}hk^2u_{xtt} + \frac{1}{6}c^{m-1}k^3u_{ttt} + \cdots) \\
&= u_m^n + ku_t - (c + c^2 + \cdots + c^m)hku_{xt} + \frac{1}{2}c^mk^2u_{tt} \\
&\quad + \frac{1}{2}(c + 3c^2 + 5c^3 + \cdots + (2m-1)c^m)h^2ku_{xxt} \\
&\quad - \frac{m}{2}c^mhk^2u_{xtt} + \frac{1}{6}c^mk^3u_{ttt} + \cdots \qquad \square
\end{aligned}$$

Using that

$$c + c^2 + \cdots + c^m \;=\; c\frac{1-c^m}{1-c} \;=\; b\mu(1-c^m)$$

and $b\mu hk = b\frac{k^2}{h}$ we get the local error

$$\begin{aligned}\frac{u_m^{n+1} - (v_m^{n+1})_{LR}}{k} \;&=\; (1-c^m)(\frac{1}{2}ku_{tt} + b\frac{k}{h}u_{xt} + \frac{1}{6}k^2u_{ttt}) \\ &- \frac{1}{2}(c + 3c^2 + 5c^3 + \cdots + (2m-1)c^m)h^2u_{xxt} + \frac{m}{2}c^m hku_{xtt} + \cdots\end{aligned} \tag{15}$$

Note that $0 < c < 1$ and therefore $0 < c^m < 1$, and

$$0 \;<\; c + 3c^2 + 5c^3 + \cdots + (2m-1)c^m \;<\; 1 + 3 + 5 + \cdots + (2m-1) \;=\; m^2$$

so that the coefficient of u_{xxt} is bounded by $\frac{1}{2}(mh)^2$.
Another bound gives

$$\begin{aligned}(c + 3c^2 + 5c^3 + \cdots + (2m-1)c^m)h^2 \;&<\; (2m-1)(c + c^2 + \cdots + c^m)h^2 \\ =\; (2m-1)c\frac{1-c^m}{1-c}h^2 \;&=\; (2m-1)(1-c^m)bk.\end{aligned}$$

We note that the coefficients of the various terms of the local error depend on the step number, m, but apart from this there is a close similarity with the local truncation error, in particular we note the leading terms with $\frac{k}{h}u_{xt}$ and ku_{tt}. So the local error gives reliable information on the truncation error and therefore on the conditions for consistency.
For the RL-method we rewrite (5) to

$$v_m^{n+1} \;=\; dv_m^n + cv_{m-1}^n + cv_{m+1}^{n+1}. \tag{16}$$

The m-th term of the computed solution can be expressed as

$$\begin{aligned}(v_m^{n+1})_{RL} \;&=\; u_m^n + ku_t + (c + c^2 + \cdots + c^{M-m})hku_{xt} + \frac{1}{2}c^{M-m}k^2u_{tt} \\ &+ \frac{1}{2}(c + 3c^2 + 5c^3 + \cdots + (2M-2m-1)c^{M-m})h^2ku_{xxt} \\ &+ \frac{M-m}{2}c^{M-m}hk^2u_{xtt} + \frac{1}{6}c^{M-m}k^3u_{ttt} + \cdots\end{aligned} \tag{17}$$

a result which is shown in a similar way as above beginning with $m = M-1$ and then working your way down by induction to $m = 1$.
We thus find the local error for the RL-method:

$$\begin{aligned}\frac{u_m^{n+1} - (v_m^{n+1})_{RL}}{k} \;&=\; (1-c^{M-m})(\frac{1}{2}ku_{tt} - b\frac{k}{h}u_{xt} + \frac{1}{6}k^2u_{ttt}) \\ &- \frac{1}{2}(c + 3c^2 + 5c^3 + \cdots + (2M-2m-1)c^{M-m})h^2u_{xxt} \\ &- \frac{M-m}{2}c^{M-m}hku_{xtt} + \cdots\end{aligned} \tag{18}$$

Again we note a close similarity with the local truncation error, in particular that the $\frac{k}{h}$-term now has a negative coefficient.
We can now easily find the local error for the **AV**-method:

$$\frac{u_m^{n+1} - (v_m^{n+1})_{AV}}{k} = (1 - \frac{c^m + c^{M-m}}{2})\frac{1}{2}ku_{tt} + \frac{c^{M-m} - c^m}{2} b\frac{k}{h}u_{xt} + \cdots \quad (19)$$

We note that taking the average eliminates the $\frac{k}{h}$-term of the local error only for $m = M/2$, i.e. only at the center point of the interval. In general there is a $\frac{k}{h}$-contribution to the local error for **AV**, so the condition for consistency remains the same as for **LR** and **RL**.
For the **ALT**-method we begin with formula (14) and then take one step with the **RL**-method. As above we can prove by induction that

$$\begin{aligned}(v_m^{n+2})_{ALT} &= u_m^n + 2ku_t + 2c^m(c + c^2 + \cdots + c^{M-m})hku_{xt} \\ &\quad + (2c^{M-m} + (1-c)c^m(1 + c^2 + \cdots + c^{2(M-m)}))k^2u_{tt}.\end{aligned} \quad (20)$$

Therefore the local error is

$$\begin{aligned}\frac{u_m^{n+2} - (v_m^{n+2})_{ALT}}{k} &= (2(1 - c^{M-m}) - c^m\frac{1 - c^{2(M-m+1)}}{1+c})ku_{tt} \\ &\quad - 2c^m(1 - c^{M-m})b\frac{k}{h}u_{xt}\end{aligned} \quad (21)$$

and once again we note a $\frac{k}{h}$-term indicating that the alternating use of **LR** and **RL** only partially compensates for the asymmetry of the basic formulae.

5 A Word of Caution

The consistency requirement $\frac{k}{h} \to 0$ is concerned with the situation where the step sizes tend to 0 and we wish the numerical solution to converge towards the true solution. But in practice we compute with fixed, finite step sizes and wonder what the error might be.
Equation (2) can be rewritten as (cf. [7], pp. 30f)

$$\begin{aligned}\frac{v_m^{n+1} - v_m^n}{k} &= b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{2h^2} + b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{2h^2} - \\ &\quad b\frac{k}{h}\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} - v_{m+1}^n + v_{m-1}^n}{2hk}.\end{aligned} \quad (22)$$

We recognize the first two terms on the right-hand side as the (second order) Crank-Nicolson approximation to $u_{xx}((n + \frac{1}{2})k, mh)$ and the last term as an approximation to u_{xt} at the same point. So the **LR**-method is actually computing an approximate solution to

$$u_t = bu_{xx} - b\frac{k}{h}u_{xt} \quad (23)$$

a result which is actually apparent from formula (8).
Similarly it can be shown that the RL-method produces an approximate solution to

$$u_t = bu_{xx} + b\frac{k}{h}u_{xt} \tag{24}$$

When $\frac{k}{h} \to 0$ both these equations tend to the desired $u_t = bu_{xx}$, so everything works fine in the limit, but for finite step sizes there is a difference.

6 A First Order Term

A natural extension in the spirit of Saulyev of the LR-method to

$$u_t = bu_{xx} - au_x \tag{25}$$

is

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^n - v_m^n - v_m^{n+1} + v_{m-1}^{n+1}}{h^2} - a\frac{v_{m+1}^n - v_{m-1}^{n+1}}{2h} \tag{26}$$

or

$$(1 + b\mu)v_m^{n+1} = (b\mu + \frac{1}{2}a\lambda)v_{m-1}^{n+1} + (1 - b\mu)v_m^n + (b\mu - \frac{1}{2}a\lambda)v_{m+1}^n \tag{27}$$

with $\lambda = \frac{k}{h}$.
For the growth factor we now have

$$g_{LR}(\varphi) - 1 = b\mu(e^{i\varphi} - 1 - g_{LR}(1 - e^{-i\varphi})) - \frac{1}{2}a\lambda(e^{i\varphi} - g_{LR}e^{-i\varphi}) \tag{28}$$

or

$$\begin{aligned} g_{LR} &= \frac{1 + b\mu(e^{i\varphi} - 1) - \frac{1}{2}a\lambda e^{i\varphi}}{1 + b\mu(1 - e^{-i\varphi}) - \frac{1}{2}a\lambda e^{-i\varphi}} \\ &= \frac{1 - b\mu(1 - \cos\varphi) - \frac{1}{2}a\lambda\cos\varphi + i(b\mu - \frac{1}{2}a\lambda)\sin\varphi}{1 + b\mu(1 - \cos\varphi) - \frac{1}{2}a\lambda\cos\varphi + i(b\mu + \frac{1}{2}a\lambda)\sin\varphi} \end{aligned} \tag{29}$$

A short calculation shows that $|g_{LR}| \leq 1$ is equivalent to $a\lambda \geq -2$.
Similarly the RL-method is absolutely stable when $a\lambda \leq 2$. It thus looks like LR is stable for positive a and RL for negative a. But consistency requires that $\lambda \to 0$ when $h \to 0$ so both methods will be absolutely stable for small step sizes even for convection-dominated problems.
The asymmetric approximation to the convection term gives rise to yet another $\frac{k}{h}$-term in $P_{k,h}$. One can avoid this by using an explicit approximation:

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^n - v_m^n - v_m^{n+1} + v_{m-1}^{n+1}}{h^2} - a\frac{v_{m+1}^n - v_{m-1}^n}{2h} \tag{30}$$

or

$$(1+b\mu)v_m^{n+1} = b\mu v_{m-1}^{n+1} + \frac{1}{2}a\lambda v_{m-1}^n + (1-b\mu)v_m^n + (b\mu - \frac{1}{2}a\lambda)v_{m+1}^n \quad (31)$$

For the growth factor we now have

$$g(\varphi) - 1 = b\mu(e^{i\varphi} - 1 - g(1 - e^{-i\varphi})) - \frac{1}{2}a\lambda(e^{i\varphi} - e^{-i\varphi}) \quad (32)$$

or

$$g = \frac{1 - b\mu(1-\cos\varphi) + i(b\mu - a\lambda)\sin\varphi}{1 + b\mu(1-\cos\varphi) + ib\mu\sin\varphi}. \quad (33)$$

$|g| \leq 1$ is equivalent to

$$(1 - b\mu(1-\cos\varphi))^2 + (b\mu - a\lambda)^2\sin^2\varphi \leq (1 + b\mu(1-\cos\varphi))^2 + b^2\mu^2\sin^2\varphi$$

or

$$a\lambda(a\lambda - 2b\mu)\sin^2\varphi \leq 4b\mu(1-\cos\varphi)$$

or

$$a\lambda(a\lambda - 2b\mu)\cos^2\frac{\varphi}{2} \leq 2b\mu.$$

For this to be true for all φ we must have

$$a\lambda(a\lambda - 2b\mu) \leq 2b\mu.$$

or

$$a\lambda(ah - 2b) \leq 2b.$$

If $ah \leq 2b$ then this inequality is satisfied. If $ah > 2b$ then we have an upper limit on λ, but λ is supposed to be small already for reasons of consistency. The reader is referred to [3] for a thorough discussion of advection-diffusion problems.

7 Derivative boundary conditions

We have assumed a Dirichlet boundary condition on the left boundary in order to get the LR-method started on the next time level. If the boundary condition involves a derivative the first x-step becomes slightly more complicated. Assume the boundary condition to be (cf. [8], pp. 36ff.)

$$\alpha u(t, X_1) - \beta u_x(t, X_1) = \gamma \quad (34)$$

where α, β, and γ are known and non-negative (cf. [8], p. 4).

7.1 First order approximation

The first order approximation to u_x in (34) gives

$$\alpha v_0^{n+1} - \beta \frac{v_1^{n+1} - v_0^{n+1}}{h} = \gamma$$

or

$$(h\alpha + \beta) v_0^{n+1} - \beta v_1^{n+1} = h\gamma \tag{35}$$

which together with the first LR-equation

$$-b\mu v_0^{n+1} + (1 + b\mu) v_1^{n+1} = (1 - b\mu) v_1^n + b\mu v_2^n \tag{36}$$

provide two equations in the two unknowns v_0^{n+1} and v_1^{n+1} which we can solve to get the process started.
In the special case $u_x = 0$ we have $v_0^{n+1} = v_1^{n+1}$ and (36) reduces to

$$v_1^{n+1} = (1 - b\mu) v_1^n + b\mu v_2^n.$$

7.2 Asymmetric second order approximation

A better approximation to u_x is

$$\frac{-v_2^{n+1} + 4v_1^{n+1} - 3v_0^{n+1}}{2h} \tag{37}$$

which inserted in (34) gives

$$\alpha v_0^{n+1} - \beta \frac{-v_2^{n+1} + 4v_1^{n+1} - 3v_0^{n+1}}{2h} = \gamma$$

or

$$(2h\alpha + 3\beta) v_0^{n+1} - 4\beta v_1^{n+1} + \beta v_2^{n+1} = 2h\gamma \tag{38}$$

which together with the two first LR-equations produce 3 equations in the three unknowns v_0^{n+1}, v_1^{n+1} and v_2^{n+1}.

7.3 Symmetric second order approximation

If the differential equation can be extended to hold at the left boundary (and a little bit beyond) then we can introduce a fictitious point v_{-1}^{n+1} and use a symmetric approximation to u_x. The boundary condition then reads

$$\alpha v_0^{n+1} - \beta \frac{v_1^{n+1} - v_{-1}^{n+1}}{2h} = \gamma$$

or

$$\beta v_{-1}^{n+1} + 2h\alpha v_0^{n+1} - \beta v_1^{n+1} = 2h\gamma. \qquad (39)$$

Two LR-equations centered at 0 and 1 provide the remaining two equations.
In the special case $u_x = 0$ (39) reduces to $v_{-1}^{n+1} = v_1^{n+1}$ and the LR-equation at 0 becomes

$$(1 + b\mu)v_0^{n+1} - b\mu v_1^{n+1} = (1 - b\mu)v_0^n + b\mu v_1^n.$$

7.4 The RL-formula

The RL-formula can calculate all values down to v_1^{n+1}, and v_0^{n+1} can be calculated from the first order or the asymmetric second order boundary approximation. For symmetric second order we use LR down to $m = 0$ and calculate v_{-1}^{n+1} from (39). In order to get started at time 0 we calculate v_{-1}^0 from (39) (with $n = -1$).

8 Two space dimensions

The Saulyev methods can be extended in a straightforward manner to

$$u_t = b_1 u_{xx} + b_2 u_{yy} \qquad (40)$$

The first formula is

$$\frac{v_{l,m}^{n+1} - v_{l,m}^n}{k} = b_1 \frac{v_{l+1,m}^n - v_{l,m}^n - v_{l,m}^{n+1} + v_{l-1,m}^{n+1}}{h_1^2} + b_2 \frac{v_{l,m+1}^n - v_{l,m}^n - v_{l,m}^{n+1} + v_{l,m-1}^{n+1}}{h_2^2} \qquad (41)$$

where h_1 and h_2 are the step sizes in the x- and y-direction, respectively, and l and m are the corresponding step numbers.
Equation (41) can be rewritten as

$$(1 + b_1\mu_1 + b_2\mu_2)v_m^{n+1} = b_1\mu_1 v_{l-1,m}^{n+1} + b_2\mu_2 v_{l,m-1}^{n+1} + (1 - b_1\mu_1 - b_2\mu_2)v_{l,m}^n + b_1\mu_1 v_{l+1,m}^n + b_2\mu_2 v_{l,m+1}^n \qquad (42)$$

where $\mu_1 = k/h_1^2$ and $\mu_2 = k/h_2^2$. This formula looks implicit but can be solved in an explicit fashion using the (Dirichlet) boundary conditions at $y = 0$ and $x = 0$. If we solve from left to right on each row ($y =$ constant) and take the rows from $y = 0$ and up we get what we shall call the LRDU-method (left-right, down-up). Analogously we can consider the RLUD-method.
If we have derivative boundary conditions for $x = 0$ then the considerations of section 7 apply. If we have derivative boundary conditions at $y = 0$ then it might be a good idea to reverse the order of x and y.

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