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Greibach Normal Form in Algebraically Complete Semirings

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Abstract

We give inequational and equational axioms for semirings with a fixed-point operator and formally develop a fragment of the theory of context-free languages. In particular, we show that Greibach's normal form theorem depends only on a few equational properties of least pre-fixed-points in semirings, and elimination of chain- and deletion rules depend on their inequational properties (and the idempotency of addition). It follows that these normal form theorems also hold in non-continuous semirings having enough fixed-points.

Keywords: Greibach normal form, context-free languages, pre-fixed-point induction, equational theory, Conway semiring, Kleene algebra, algebraically complete semiring

1 Introduction

It is well-known that the equational theory of context-free languages is not recursively enumerable, i.e. the equivalence problem for context-free grammars is not semi-decidable. This may have been the reason why little work

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has been done to develop a formal theory for the rudiments of the theory of context-free languages.

In contrast, the equational theory of regular languages is decidable, and several axiomatizations of it appeared, using regular expressions as a notation system. In the 1970s, axiomatizations by schemata of equations between regular expressions were conjectured by Conway[9]. Salomaa[26] gave a finite first-order axiomatization based on a version of the unique fixed-point rule. Redko[24] showed that the theory does not have a finite equational basis. Twenty years later, Pratt[23] showed that a finite equational axiomatization is possible if one extends the regular operations $+, \cdot$ and * by the left- and right *residuals* $/$ and $\backslash$ of $\cdot$. The important new axiom was $(a/a)^* = (a/a)$, the axiom of 'pure induction'. (For a recent extension of Pratt's methods, see Santocanale[27].) Earlier, Krob[18] confirmed several conjectures of Conway including the completeness of Conway's group identities. He also gave several finite axiomatizations, including a system having, in addition to a finite number of equational axioms, a Horn formula expressing that a^*b is the *least solution of* $ax + b \leq x$. See also Boffa[7, 8], Bloom and Ésik[6], Bernátsky e.a.[4]. Independently, Kozen[16] defined a *Kleene algebra* as an idempotent semiring equipped with a * operation subject to the above Horn-formula and its dual asserting that b^*a is the least solution of $xa + b \leq x$. He gave a direct proof of the completeness of the Kleene algebra axioms with respect to the equational theory of the regular sets.

With a least-fixed-point operator μ, these axioms of KA can be expressed as $a^*b = \mu x(ax + b)$ and $ba^* = \mu x(xa + b)$. Hence it is natural to extend the regular expressions by a construction $\mu x.r$, which gives a notation system for context-free languages. Extensions of KA by μ have been suggested in [20] to axiomatize fragments of the theory of context-free languages.

In this paper we look at axioms for semirings with a least-fixed-point operator that are sufficient to prove some of the normal form theorems for context-free grammars. In particular, we derive the Greibach[12] normal form theorem using only equational properties of least fixed-points. Our proof gives the efficient algorithm of Rosenkrantz[25], but avoids the analytic method of power series of his proof. Our axioms also imply that context-free grammars have normal forms without chain rules or deletion rules. An important aspect is that we do not use the idempotency of $+$, except for the elimination of deletion rules, and so the classical theorems are extended to a wide class of semirings.

Recently, Parikh's theorem, another classical result on context-free languages, has been treated in a similar spirit. Hopkins and Kozen[15] generalized this theorem to an equation schema valid in all commutative idempotent semirings with enough solutions for recursion equations, also replacing analytic methods by properties of least fixed-points. A purely equational proof is given in [1].

2 Park μ-semirings and Conway μ-semirings

We will consider *terms*, or *μ-terms* defined by the following syntax, where x ranges over a fixed countable set X of variables:

$$T \quad ::= \quad x \mid 0 \mid 1 \mid (T+T) \mid (T \cdot T) \mid \mu x\, T$$

For example, $\mu x(x+1)$ is a term. To improve readability, we write $\mu x.t$ instead of $\mu x\, t$ when the term t is 0, 1, a variable or not concretely given. With $\mu x.t[s/y]$ we mean $\mu x(t[s/y])$, not $(\mu x.t)[s/y]$. The variable x is *bound* in $\mu x.t$. The set *free*(t) of *free variables* of a term t is defined as usual. We call a term *closed* if it has no free variables and *finite* if it contains no subterm of the form $\mu x.t$. Below we will write $t(x_1, \dots, x_n)$ or $t(\vec{x})$, where $\vec{x} = (x_1, \dots, x_n)$, to indicate that the free variables of term t belong to the set $\{x_1, \dots, x_n\}$. We identify any two terms that only differ in the names of the bound variables and write $t \equiv s$ for syntactic identity of s and t, up to renaming of bound variables. *Substitution* $t[t'/x]$ of t' for x in t and *simultaneous substitution* $t[(t_1, \dots, t_n)/(x_1, \dots, x_n)]$ are defined as usual.

We will be interested in interpretations where $\mu x.t$ provides solutions to the fixed-point equation $x = t$.

Definition 2.1 *A* μ-semiring *is a semiring* $(A, +, \cdot, 0, 1)$ *with an interpretation* $(\cdot)_A$ *of the terms* t *as functions* $t_A : A^X \to A$*, such that*

1. *for each environment* $\rho \in A^X$*, all variables* $x \in X$ *and all terms* t, t'*:*
   - *(a)* $0_A(\rho) = 0$*,* $1_A(\rho) = 1$*,* $x_A(\rho) = \rho(x)$*,* $(t+t')_A(\rho) = t_A(\rho) + t'_A(\rho)$*,* $(t \cdot t')_A(\rho) = t_A(\rho) \cdot t'_A(\rho)$*,*
   - *(b) the 'substitution lemma' holds, i.e.* $(t[t'/x])_A(\rho) = t_A(\rho[x \mapsto t'_A(\rho)])$*,*

2. *for all terms t, t' and variables $x \in X$, if $t_A = t'_A$, then $(\mu x.t)_A = (\mu x.t')_A$.*

A weak ordered μ-semiring *is a μ-semiring A equipped with a partial order $\leq$ such that all term functions t_A are monotone with respect to the pointwise order. An* ordered μ-semiring *is a weak ordered μ-semiring A such that for any two terms t, t' and variable x, if $t_A \leq t'_A$ in the pointwise order, then $(\mu x.t)_A \leq (\mu x.t)_A$.*

In a μ-semiring A, the value $t_A(\rho)$ does not depend on $\rho(x)$ if x does not have a free occurrence in t. As usual, $\rho[x \mapsto a]$ is the same as ρ except that it maps x to a. A term equation $t = t'$ *holds* or *is satisfied* in a μ-semiring A, if $t_A = t'_A$. A term inequation $t \leq t'$ *holds* in a μ-semiring A equipped with a partial order $\leq$, if $t_A \leq t'_A$ in the pointwise order on A^X. An implication $t = t' \to s = s'$ holds in A, if for all $\rho \in A^X$, whenever $t_A(\rho) = t'_A(\rho)$, then also $s_A(\rho) = s'_A(\rho)$. Likewise with inequations.

Definition 2.2 *A* strong μ-semiring *is a μ-semiring A where $\forall x(t = t') \to \mu x.t = \mu x.t'$ holds, for all terms t, t' and variables $x \in X$. A* strong ordered μ-semiring *is a weak ordered μ-semiring A where $\forall x(t \leq t') \to \mu x.t \leq \mu x.t'$ holds, for all terms t, t' and variables x.*

The validity of $\forall x(t = t') \to \mu x.t = \mu x.t'$ implies the second condition in Definition 2.1.

Definition 2.3 *A* Park μ-semiring *is a weak ordered μ-semiring satisfying the* fixed-point inequation (1) *and the pre-fixed-point* induction axiom (2), *also referred to as the* Park induction rule, *for all terms t and $x, y \in X$:*

$$t[\mu x.t/x] \quad \leq \quad \mu x.t, \tag{1}$$

$$t[y/x] \leq y \quad \to \quad \mu x.t \leq y. \tag{2}$$

Proposition 2.4 *Any Park μ-semiring A is a strong ordered μ-semiring satisfying the* composition identity (3) *and the* diagonal identity (4)

$$\mu x.t[t'/x] \quad = \quad t[\mu x.t'[t/x]/x] \tag{3}$$

$$\mu x.\mu y.t \quad = \quad \mu x.t[x/y], \tag{4}$$

for all terms t, t' and all variables x, y.

Note that taking t' to be x in (3) gives the *fixed point equation* for t,

$$\mu x.t \quad = \quad t[\mu x.t/x]. \tag{5}$$

Proof. To prove that A is a strong ordered μ-semiring, suppose for terms t, t' and $\rho \in A^X$ that $t_A(\rho[x \mapsto a]) \leq t'_A(\rho[x \mapsto a])$, for all $a \in A$. Since t_A is monotone, it follows that every pre-fixed-point of the map $a \mapsto t'_A(\rho[x \mapsto a])$ is a pre-fixed-point of the map $a \mapsto t_A(\rho[x \mapsto a])$. Hence, $(\mu x.t)_A(\rho) \leq (\mu x.t')_A(\rho)$. Equations (3) and (4) are established in Niwinski [22]. $\square$

DEFINITION 2.5 *A* Conway μ-semiring *is a μ-semiring satisfying the* Conway identities *(3) and (4), for all terms t, t' and variables x, y.*

In the following, when $t(\vec{x})$ is a term and $\vec{a}$ an appropriately sized tuple of elements of a μ-semiring A, we often write $t(\vec{a})$ instead of $t_A([\vec{x} \mapsto \vec{a}])$.

3 Algebraically complete semirings

An *ordered semiring* is a semiring $(S, +, \cdot, 0, 1)$ equipped with a partial order $\leq$ such that the $+$ and $\cdot$ operations are monotone in both arguments. If $+$ is idempotent and 0 is the least element, then $\leq$ is the semilattice order $x \leq y :\iff x + y = y$, because $x \leq y \Rightarrow x + y \leq y + y = y = 0 + y \leq x + y$ and $x + y = y \Rightarrow x = x + 0 \leq x + y = y$. Note that each weak ordered μ-semiring is an ordered semiring.

For any term t, we introduce the *left iteration* t^ℓ and the *right iteration* t^r *of* t by

$$t^\ell := \mu z(zt + 1) \qquad \text{and} \qquad t^r := \mu z(tz + 1),$$

where z is a variable not free in t.

DEFINITION 3.1 *An* algebraically complete semiring *is a Park μ-semiring which satisfies the inequations*

$$x^r y \quad \leq \quad \mu z(xz + y) \tag{6}$$
$$y x^\ell \quad \leq \quad \mu z(zx + y). \tag{7}$$

By Proposition 2.4, every algebraically complete semiring satisfies the composition (3) and diagonal identities (4), hence also the fixed-point identity (5).

PROPOSITION 3.2 *Any algebraically complete semiring S satisfies the following (in)equations:*

$$\begin{aligned} 0 &\leq x & (8)\\ x &\leq x+y & (9)\\ x^r y &= \mu z(xz+y) & (10)\\ yx^\ell &= \mu z(zx+y) & (11)\\ x^r &= x^\ell. & (12) \end{aligned}$$

Proof. As for (8), note that by (6)

$$0 \;=\; 1^r \cdot 0 \;\leq\; \mu x.x$$

holds in S. But by the Park induction rule, $(\mu x.x)_S$ is the least element of S.

Inequation (9) follows from (8) using the fact that each Park μ-semiring is an ordered semiring, hence $+$ is monotone.

To show (10), note that we have

$$xx^r + 1 \;\leq\; x^r$$

by the fixed-point inequation (1), hence

$$x(x^r y) + y \;\leq\; x^r y$$

by monotonicity. By the Park induction rule, this gives

$$\mu z(xz+y) \;\leq\; x^r y.$$

The reverse inequation is (6). Dually, we have (11).

As for (12), applying the composition identity

$$\mu z.t[s/z] \;=\; t[\mu z.s[t/z]/z]$$

to $t := xz + 1$ and $s := yz$, we obtain

$$\begin{aligned} (xy)^r &= x \cdot \mu z(y(xz+1)) + 1 \\ &= x \cdot \mu z((yx)z + y) + 1 \\ &= x(yx)^r y + 1, \end{aligned}$$

using equation (10) in the last step. In particular,

$$y^r = y^r y + 1.$$

Thus, by the Park induction rule we have

$$y^\ell = \mu z(zy + 1) \leq y^r.$$

Similarly, using (11) we get $y^r \leq y^\ell$, so that $y^\ell = y^r$. □

By (12), two possible definitions of iteration, left and right iteration, coincide in any algebraically complete semiring. With a variable z not free in t, we *define* the term

$$t^* := \mu z(tz + 1). \tag{13}$$

On algebraically complete semirings A, we obtain a *-operation with $a^* = a^r = a^\ell$ for all a.

Remark 3.3 If we think of μ-terms as programs with $+$ as non-deterministic choice, $\cdot$ as sequential composition, and μ as recursion, then (10) and (11) reduce tail- and head recursion to iteration * and sequential composition (cf. [20]). Equation (10) is related to the parameter equation of [6].

We now give some examples of algebraically complete semirings.

Example 3.4 A *continuous semiring* is a semiring $S = (S, +, \cdot, 0, 1)$ equipped with a complete partial order $\leq$ such that 0 is its least element and the $+$ and $\cdot$ operations are continuous, i.e., they preserve in each argument the sup of any directed nonempty set. Any continuous semiring S gives rise to an algebraically complete semiring where $\mu x.t$ provides the least solution to the fixed-point equation $x = t$ (see [6]).

Let $\mathbb{N}$ denote the set of nonnegative integers and let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Equipped with the usual order and $+$ and $\cdot$ operations, $\mathbb{N}_\infty$ is a continuous semiring. Also, every finite ordered semiring having 0 as least element such as the boolean semiring $\mathbb{B} = \{0, 1\}$ is continuous. Thus, $\mathbb{N}_\infty$ and $\mathbb{B}$ are algebraically complete semirings.

Other prime examples of continuous semirings are the semiring $\mathbf{L}_A$ of all languages in A^*, where A is a set, $+$ is set union, $\cdot$ is concatenation and $\leq$ is set inclusion, and the semiring $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$ of power series over A with coefficients in $\mathbb{N}_\infty$, equipped with the pointwise order.

The set $\mathbf{R}_M$ of all binary relations on the set M, where $+$ is union, $\cdot$ the relation product, 0 the empty relation, 1 the diagonal on M and $\leq$ is inclusion, is a continuous semiring. In this example, r^* is the reflexive transitive closure of r.

Example 3.5 The context-free languages in $\mathbf{L}_A$ form an algebraically complete semiring as do the algebraic power series in $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$. Unless A is empty, neither of these semirings is continuous. Given a set A of binary relations over the set M, let $\mathbf{R}_M(A)$ be the values in $\mathbf{R}_M$ of all μ-terms with parameters from A. Then $\mathbf{R}_M(A)$ is also an algebraically complete semiring, which is generally not continuous.

Example 3.6 By the completeness of first-order logic, the first-order theory, and in particular the equational theory of (idempotent) algebraically complete semirings is recursively enumerable. Now, the context free languages are free for the class of idempotent semirings that can be embedded in continuous idempotent semirings (cf. [20]). Thus the equational theory of idempotent continuous semirings is not recursively enumerable. It follows that there exist algebraically complete idempotent semirings that cannot be embedded in a continuous (idempotent) semiring. In fact, when the alphabet A has two or more letters, the free idempotent algebraically complete semiring on A does not embed in a continuous semiring. The same holds for the free algebraically complete semirings.

Using the fixed-point inequation and monotonicity, one easily gets:

Proposition 3.7 *In any algebraically complete semiring, for all elements a and $n \in \mathbb{N}$*

$$(\sum_{i=0}^{n} a^i) \quad \leq \quad a^*. \tag{14}$$

We prove a few more basic equations. For any integer $n \geq 0$, we will denote by n also the term which is the n-fold sum of 1 with itself. When n is 0, this is just the term 0.

Proposition 3.8 *In any algebraically complete semiring, for any element a with $a^* + 1 \leq a^*$ we have*

$$a^* \;=\; a^* + 1 \;=\; a^* + a^* \;=\; a^* \cdot a^* \;=\; a^{**}. \tag{15}$$

In particular, (15) holds for any a such that $1 \leq a$.

Proof. First, note that if $1 \leq a$, then also

$$\begin{aligned} a^* + 1 &\leq aa^* + 1 \\ &\leq a^*, \end{aligned}$$

by the fixed-point inequation. Now suppose $a^* + 1 \leq a^*$. By the fixed-point inequation,

$$\begin{aligned} aa^* + 2 &= aa^* + 1 + 1 \\ &\leq a^* + 1 \\ &\leq a^*, \end{aligned}$$

so, by the induction rule, $\mu z(az + 2) \leq a^*$. It follows by (10) that

$$\begin{aligned} a^* + a^* &= 2a^* \\ &= \mu z(az + 2) \qquad (16) \\ &\leq a^*. \end{aligned}$$

¿From (16) and the fixed-point inequation,

$$\begin{aligned} aa^* + a^* &\leq aa^* + 1 + a^* \\ &\leq a^* + a^* \\ &\leq a^*, \end{aligned}$$

hence by the induction rule,

$$\begin{aligned} a^*a^* &= \mu z(az + a^*) \\ &\leq a^*. \qquad (17) \end{aligned}$$

¿From (17) and the assumption, we get

$$\begin{aligned} a^*a^* + 1 &\leq a^* + 1 \\ &\leq a^*, \end{aligned}$$

so again by induction,

$$\begin{aligned} a^{**} &= \mu z(a^*z + 1) \\ &\leq a^*. \end{aligned}$$

For the converse inequations, note that

$$\begin{aligned} 2 &= 1 + 1 \\ &\leq a^* + 1 \\ &\leq a^*, \end{aligned}$$

so by monotonicity,

$$\begin{aligned} a^* + a^* &= 2a^* \\ &\le a^*a^*. \end{aligned}$$

The remaining inequation $a^*a^* \le a^{**}$ follows from (14). □

REMARK 3.9 An element x of an ordered semiring is *reflexive* if $1 \le x$ and *transitive* if $xx \le x$. In an ordered semiring which is a Park μ-semiring, we call $x^{\odot} := \mu z(1 + zz + x)$ the *reflexive transitive closure* of x. We remark without proof that in an algebraically complete semiring, $x^* \le x^{\odot}$ and

$$x^{\odot} \le x^* \quad \Longleftrightarrow \quad x^* + x^* \le x^*.$$

In particular, when $+$ is idempotent as in $\mathbf{R}_M$ or $\mathbf{L}_A$, then iteration x^* coincides with reflexive transitive closure $x^{\odot}$, see also [21, 23, 4, 8]. By Propositions 3.7 and 3.8, we have $x^* = x^{\odot}$ for every reflexive element of an algebraically complete semiring.

COROLLARY 3.10 *In any algebraically complete semiring, if* $1 \le a$ *then* $(a+1)^* = a^*$.

Proof. By (14), $a+1 \le a^*$, hence by monotonicity and (15), $a^* \le (a+1)^* \le (a^*)^* = a^*$. □

PROPOSITION 3.11 *In any algebraically complete semiring, we have for* $n \in \mathbb{N}$

$$0^* = 1 \qquad \text{and} \qquad (n+1)^* = 1^*. \tag{18}$$

Proof. Clearly,

$$0^* = \mu x(0x + 1) = \mu x.1 = 1.$$

The second claim is clear for $n = 0$. If $1 \le n$, then $(n+1)^* = n^*$ by Corollary 3.10, and by induction hypothesis, $n^* = 1^*$. □

A *morphism* between μ-semirings is any function that commutes with the term functions. Thus, if $h : A \to B$ is a morphism between μ-semirings, its pointwise extension $h^X : A^X \to B^X$ satisfies $t_B \circ h^X = h \circ t_A$ for all terms t. A morphism of Park μ-semirings, Conway μ-semirings and algebraically complete semirings is just a μ-semiring morphism. A μ-semiring A is *initial* in a class of μ-semirings if for every B in the class there is a unique morphism $h : A \to B$.

Since $1^* = \infty$ in $\mathbb{N}_\infty$ by Proposition 3.8, any element of $\mathbb{N}_\infty$ can be considered as a closed term c. For algebraically complete semirings, the following converse also holds:

THEOREM 3.12 *If t is a closed term, then for some $c \in \mathbb{N}_\infty$, equation $t = c$ holds in all algebraically complete semirings.*

The proof is deferred to section 7. An immediate consequence is:

COROLLARY 3.13 $\mathbb{N}_\infty$ *is initial in the class of all algebraically complete semirings.*

We remark that the symmetric inductive *-semirings of [11] are right- and left-linear variants of algebraically complete semirings. An *inductive *-semiring* is an ordered semiring equipped with a *-operation, satisfying the inequation

$$xx^* + 1 \quad \leq \quad x^*$$

and the following instance of the Park induction rule:

$$xz + y \leq z \quad \to \quad x^*y \leq z. \tag{19}$$

A *symmetric* inductive *-semiring also satisfies

$$zx + y \leq z \quad \to \quad yx^* \leq z. \tag{20}$$

It follows that any inductive *-semiring satisfies $x^*x + 1 = x^*$ and has a monotone *-operation.

Note that (19) and (20) are instances of the pre-fixed-point axiom $t[z/v] \leq z \to \mu v.t \leq z$ for the terms $t = xv + y$ and $t = vx + y$ which are right- respectively left-linear in the recursion variable v, and $xx^* + 1 \leq x^*$ resp. $x^*x + 1 \leq x^*$ are instances of the corresponding pre-fixed-point inequations $t[\mu v.t/v] \leq \mu v.t$.

A *Kozen semiring*, called *Kleene algebra* in [16], is an idempotent symmetric inductive *-semiring. Note that Propositions 3.7, 3.8 and 3.11 hold in all inductive *-semirings. Via (13), any idempotent algebraically complete semiring is a Kozen semiring.

4 Algebraic Conway semirings

Next we turn to equational notions derived from algebraically complete semirings.

DEFINITION 4.1 *An* algebraic Conway semiring *is a Conway μ-semiring which satisfies (10), (11) and (12). A morphism of algebraic Conway semirings is a μ-semiring morphism.*

Thus, any algebraically complete semiring is an algebraic Conway semiring.

In [6], a *Conway semiring* is defined to be a semiring S equipped with an operation $^* : S \to S$ subject to the equations

$$\begin{aligned} (x+y)^* &= (x^*y)^*x^* \\ (xy)^* &= 1 + x(yx)^*y. \end{aligned}$$

It is known that also $(x + y)^* = x^*(yx^*)^*$ holds in any Conway semiring. By the following Proposition, any algebraic Conway semiring is a Conway semiring (using $x^r = x^\ell$ for x^*).

PROPOSITION 4.2 *For any terms t and s and variable z which does not occur in t and s, the following equations hold in any algebraic Conway semiring:*

$$\mu z(tz+s) = t^*s \tag{21}$$
$$\mu z(zt+s) = st^* \tag{22}$$
$$tt^*+1 = t^* \tag{23}$$
$$t^*t+1 = t^* \tag{24}$$
$$(t+s)^* = (t^*s)^*t^* \tag{25}$$
$$(t+s)^* = t^*(st^*)^* \tag{26}$$
$$(ts)^* = 1+t(st)^*s. \tag{27}$$

Proof. Equations (21) and (22) are (10) and (11), using (12) to identify $t^\ell = t^* = t^r$. Equations (23) and (24) are instances $xx^r + 1 = x^r$ and $x^\ell x + 1 = x^\ell$ of the fixed-point identity, which holds in any Conway μ-semiring. For equation (25), we apply the diagonal identity

$$\mu z.t[z/v] \quad = \quad \mu z.\mu v.t$$

to the term $t(x, y, z, v) := xv + yz + 1$ and use equations (10) as follows:

$$\begin{aligned}
(x+y)^r &= \mu z((x+y)z + 1) \\
&= \mu z.t[z/v] \\
&= \mu z.\mu v(xv + yz + 1) \\
&= \mu z(x^r(yz+1)) \\
&= \mu z((x^r y)z + x^r) \\
&= (x^r y)^r x^r.
\end{aligned}$$

Similarly, for equation (26) we obtain $(x + y)^\ell = x^\ell(yx^\ell)^\ell$ from (11). As for equation (27), note that in the proof of Proposition 3.2, we have already derived $(xy)^r = x(yx)^r y + 1$ from the composition identity and (10) only. □

Equations (21) and (22) can be generalized to the following ones, of which (29) will later be used to eliminate left- or head-recursion to obtain Greibach-Normal-Form.

PROPOSITION 4.3 *For any terms t and s which may have free occurrences of the variable z, the following equations hold in any algebraic Conway semiring:*

$$\mu z(tz + s) \quad = \quad \mu z(t^* s) \tag{28}$$

$$\mu z(zt + s) \quad = \quad \mu z(st^*). \tag{29}$$

Proof. By the diagonal identity, with a variable x not free in z, s, t we have

$$\begin{aligned}
\mu z(zt + s) &= \mu z(\mu x(xt + s)) \\
&= \mu z(st^\ell),
\end{aligned}$$

which is (29). Likewise, we get (28). □

Conway semirings may be seen as left- and right-linear versions of algebraic Conway semirings, in the sense that they satisfy the composition identity

$$\mu z.t[s/z] \quad = \quad t[\mu z.s[t/z]/z]$$

for the terms $t = xz + 1$ and $s = yz$ which are right-linear in z (resp. $t = zx + 1$ and $s = zy$ which are left-linear in z), and the diagonal identity

$$\mu z.t[z/v] \quad = \quad \mu z.\mu v.t$$

for the term $t = xv + yz + 1$ which is right-linear in v and z (resp. $t = vx + zy + 1$ which is left-linear in v and z). The basic fact which connects Conway semirings and inductive *-semirings is:

PROPOSITION 4.4 [11] *Every inductive *-semiring is a Conway semiring.*

5 Term vectors and term matrices

Suppose that t_i is a term for each $i \in \{1, \ldots, n\}$, $n \geq 1$. We write $\vec{t}$ for the term vector $(t_1, \ldots, t_n)$. When $\vec{x} = (x_1, \ldots, x_n)$ is a vector of *different* variables, we define the term vector $\mu\vec{x}.\vec{t}$ by induction on n:

- If $n = 1$, then $\mu\vec{x}.\vec{t} := (\mu x_1.t_1)$.
- If $n = m + 1$ with $m > 0$ and $\vec{y} = (x_1, \ldots, x_m)$, $\vec{z} = (x_n)$, $\vec{r} = (t_1, \ldots, t_m)$, $\vec{s} = (t_n)$ put

$$\mu\vec{x}.\vec{t} \quad := \quad (\mu\vec{y}.\vec{r}[\mu\vec{z}.\vec{s}/\vec{z}],\ \mu\vec{z}.\vec{s}[\mu\vec{y}.\vec{r}/\vec{y}]).$$

This definition is motivated by the Bekić–De Bakker–Scott rule [3, 2]. If $\vec{t} = (t_1, \ldots, t_n)$ and $\vec{t'} = (t'_1, \ldots, t'_n)$ are two term vectors of dimension $n \geq 1$, we say that the equation $\vec{t} = \vec{t'}$ *holds* in a μ-semiring A if each equation $t_i = t'_i$ does. Similarly for $\vec{t} \leq \vec{t'}$. We say that an implication $\vec{t} = \vec{t'} \to \vec{s} = \vec{s'}$ *holds* in A, if each implication $t_1 = t'_1 \wedge \ldots \wedge t_n = t'_n \to s_j = s'_j$ holds in A, for each j. We say $\vec{x} \notin \mathit{free}(\vec{t})$ if no x_i occurs free in any t_j.

PROPOSITION 5.1 *Let $\vec{t}$ be a term-vector and $\vec{x}$ a vector of variables of the same size, such that $\vec{x} \notin \mathit{free}(\vec{t})$. Then $\vec{t} = \mu\vec{x}.\vec{t}$ holds in any Conway μ-semiring.*

Proof. By induction on the length of $\vec{t}$, using the fixed-point equation for length 1. □

PROPOSITION 5.2 *Let $\vec{t}, \vec{s}$ and $\vec{x}, \vec{y}$ be vectors of terms and variables of the same size, such that $\vec{x}$ and $\vec{y}$ are distinct and $\vec{x} \notin free(\vec{s})$. Then*

$$(\mu\vec{x}.\vec{t})[\vec{s}/\vec{y}] \quad = \quad \mu\vec{x}.\vec{t}[\vec{s}/\vec{y}] \tag{30}$$

holds in any Conway μ-semiring.

Proof. By induction on the size of $\vec{t}$, using the definition of substitution in the base case. □

The following facts are proven in [6] in the more general context of Conway theories (Conway algebras). See Chapter 6, Section 2. Propositions 5.1 and 5.2 show the 'left zero identity' and the 'parameter identity' of [6], while the above definition of $\mu\vec{x}.\vec{t}$ is the 'scalar symmetric pairing identity' of [6].

By Theorem 6.2.20 of [6], a variation of this definition, the '(left) pairing identity', together with the parameter identity and the unary composition and diagonal identities, gives a Conway theory. By Corollary 6.2.4, in a Conway theory the 'symmetric paring identity' (31) holds, a generalized form of the Bekić-De Bakker-Scott equations. Therefore, it does not matter which of the definitions of $\mu\vec{x}.\vec{t}$ is used.

THEOREM 5.3 ([6], Corollary 6.2.4, 5.3.13) *Suppose that A is a Conway μ-semiring. Then for each term vector $\vec{t}$ and vector $\vec{x}$ of different variables as above, the equation*

$$\mu\vec{x}.\vec{t} \quad = \quad (\mu\vec{y}.\vec{r}[\mu\vec{z}.\vec{s}/\vec{z}],\ \mu\vec{z}.\vec{s}[\mu\vec{y}.\vec{r}/\vec{y}]) \tag{31}$$

holds in A for each *way of splitting $\vec{x}$ and $\vec{t}$ into two parts as $\vec{x} = (\vec{y}, \vec{z})$ and $\vec{t} = (\vec{r}, \vec{s})$ such that the dimension of $\vec{y}$ agrees with the dimension of $\vec{r}$.*

THEOREM 5.4 ([6], Corollary 6.2.4) *The vector version of the composition identity holds in any Conway μ-semiring: For all term vectors $\vec{t}(\vec{y}, \vec{z}), \vec{s}(\vec{x}, \vec{z})$ and variable vectors $\vec{x}, \vec{y}$ of appropriate sizes, any Conway μ-semiring satisfies*

$$\mu\vec{x}.\vec{t}[\vec{s}/\vec{y}] \quad = \quad \vec{t}[\mu\vec{y}.\vec{s}[\vec{t}/\vec{x}]/\vec{y}]. \tag{32}$$

The vector version of the diagonal identity holds in any Conway μ-semiring: for each term vector $\vec{t}(\vec{x}, \vec{y}, \vec{z})$ with distinct $\vec{x}, \vec{y}, \vec{z}$ such that the dimensions of $\vec{t}, \vec{x}, \vec{y}$ agree, any Conway μ-semiring satisfies

$$\mu\vec{x}.\mu\vec{y}.\vec{t} \quad = \quad \mu\vec{x}.\vec{t}[\vec{x}/\vec{y}]. \tag{33}$$

In particular, any Conway μ-semiring satisfies the vector version of the fixed-point equation,

$$\mu\vec{x}.\vec{t} \quad = \quad \vec{t}[\mu\vec{x}.\vec{t}/\vec{x}]. \tag{34}$$

Moreover, writing $\mu\vec{x}.\vec{t} = (r_1, \ldots, r_n)$, the permutation identity

$$\mu(x_{1\pi}, \ldots, x_{n\pi}).(t_{1\pi}, \ldots, t_{n\pi}) \quad = \quad (r_{1\pi}, \ldots, r_{n\pi})$$

holds in all Conway μ-semirings, for all permutations $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Proposition 5.5 *Suppose that $\vec{t}$ and $\vec{r}$ are term vectors of dimension m, n, respectively. Moreover, suppose that the components of the vectors of variables $\vec{x}$, $\vec{y}$ of dimension m and n, respectively, are pairwise distinct. Then*

$$\mu(\vec{x}, \vec{y}).(\vec{t}, \vec{r}) \quad = \quad \mu(\vec{x}, \vec{y}).(\vec{t}[\vec{r}/\vec{y}], \vec{r}) \tag{35}$$

holds in any Conway μ-semiring.

The following fact is essentially due to Bekić [3] and de Bakker and Scott [2]. See also [10].

Theorem 5.6 *The following vector version of the Park induction rule holds in all Park μ-semirings:*

$$\vec{t}[\vec{y}/\vec{x}] \leq \vec{y} \quad \to \quad \mu\vec{x}.\vec{t} \leq \vec{y},$$

for all vectors $\vec{t}$ of terms and $\vec{x}, \vec{y}$ of variables of the same size.

In addition to term vectors, we will also consider matrices whose entries are terms. A *term matrix* $T = (t_{i,j})$ of size $n \times m$, where $n, m \geq 1$, consists of a term vector $\vec{t}$ of length nm, listing the entries of T by rows, and the dimension (n, m).

We denote by 1_n the $n \times n$ matrix whose diagonal entries are 1 and whose other entries are 0, and by $0_{n,m}$ the $n \times m$ matrix whose entries are all 0. When S and T are term matrices of appropriate size, we define $S + T$ and ST in the obvious way. Suppose that T is a term matrix and X is a variable matrix of the same size $n \times m$, with pairwise distinct variables, and let $\vec{t}$

and $\vec{x}$ be obtained by listing their entries by rows. Then $\mu X.T$ is the $n \times m$ term matrix that corresponds to the term vector $\mu \vec{x}.\vec{t}$.

For square matrices T, we can define the left- and right iterations T^ℓ and T^r, using μ. Independently of μ, we now define a matrix T^* by induction on the dimension of T and then relate T^* to T^ℓ and T^r.

Definition 5.7 *For an $n \times n$ term matrix T, define a matrix T^* by induction on n:*

1. *If $n = 1$, then $T = (\, t\,)$ for some term t. We define $T^* := (\, t^*\,)$.*
2. *If $n = m + 1 > 1$ and*

$$T \;=\; \begin{pmatrix} R & S \\ U & V \end{pmatrix} \tag{36}$$

where R is $m \times m$ and V is 1×1, we define

$$T^* \;:=\; \begin{pmatrix} \overline{R} & \overline{S} \\ \overline{U} & \overline{V} \end{pmatrix} \tag{37}$$

where

$$\begin{array}{rclcrcl} \overline{R} & = & (R + SV^*U)^* & \qquad & \overline{S} & = & \overline{R}SV^* \\ \overline{U} & = & \overline{V}UR^* & & \overline{V} & = & (V + UR^*S)^*. \end{array}$$

Suppose that $T = (\, t_{ij}\,)$ and $S = (\, s_{ij}\,)$ are term matrices of the same size. We say that $T = S$ *holds* in a μ-semiring A if each equation $t_{ij} = s_{ij}$ holds in A.

The following result is proven in [6] using a different framework, cf. Chapter 9, Theorem 2.1.

Theorem 5.8 *Let A be an algebraic Conway semiring. Suppose that T is an $n \times n$ term matrix, S is an $n \times m$ (resp. $m \times n$) term matrix and let X denote an $n \times m$ (resp. $m \times n$) matrix of new variables. Then the equations*

$$\mu X(TX + S) \;=\; T^*S \tag{38}$$

$$\mu X(XT + S) \;=\; ST^* \tag{39}$$

hold in A. Moreover, (37) holds, if T splits like (36) for matrices R, U, V of appropriate dimensions.

In particular, the coincidence of left- and right iterations for square matrices holds in A,

$$T^\ell := \mu X(XT + 1_n) \quad = \quad T^* \quad = \quad \mu X(TX + 1_n) =: T^r. \tag{40}$$

The next results parallel with the fact that if A is a continuous semiring, then so is any matrix semiring $Mat_{n\times n}(A)$, for each $n \geq 1$.

Lemma 5.9 *Let A be a μ-semiring. For all vectors $\vec{t}, \vec{t}'$ of terms and $\vec{x}$ of variables of the same dimension, if $\vec{t}_A = \vec{t}'_A$, then $(\mu\vec{x}.\vec{t})_A = (\mu\vec{x}.t')_A$.*

Proof. By induction on the dimension n of $\vec{x}$. For $n = 1$, the claim is true since A is a μ-semiring. For $n > 1$, consider a vector $(\vec{x}, x)$ of distinct variables and term-vectors $(\vec{t}, t)$ and $(\vec{s}, s)$ of the same size. By the definition of $\mu(\vec{x}, x).(\vec{t}, t)$ and $\mu(\vec{x}, x).(\vec{s}, s)$, we have to show that $(\mu\vec{x}.\vec{t}[\mu x.t/x])_A = (\mu\vec{x}.\vec{s}[\mu x.s/x])_A$ and $(\mu x.t[\mu\vec{x}.\vec{t}/\vec{x}])_A = (\mu x.s[\mu\vec{x}.\vec{s}/\vec{x}])_A$. But by induction, $(\mu x.t)_A = (\mu x.s)_A$ and $(\mu\vec{x}.\vec{t})_A = (\mu\vec{x}.\vec{s})_A$, so for each $\rho \in A^X$

$$\begin{aligned}
(\mu\vec{x}.\vec{t}[\mu x.t/x])_A(\rho) &= (\mu\vec{x}.\vec{t})_A(\rho[x \mapsto (\mu x.t)_A(\rho)]) \\
&= (\mu\vec{x}.\vec{s})_A(\rho[x \mapsto (\mu x.s)_A(\rho)]) \\
&= (\mu\vec{x}.\vec{s}[\mu x.s/x])_A(\rho).
\end{aligned}$$

Hence $(\mu\vec{x}.\vec{t}[\mu x.t/x])_A = (\mu\vec{x}.\vec{s}[\mu x.s/x])_A$. The proof for $(\mu x.t[\mu\vec{x}.\vec{t}/\vec{x}])_A = (\mu x.s[\mu\vec{x}.\vec{s}/\vec{x}])_A$ is symmetric. □

Corollary 5.10 *If A is a μ-semiring, so is $Mat_{n\times n}(A)$, for each $n \geq 1$.*

Proof. For each term t we define a term matrix t' of size $n \times n$ inductively, using the addition and multiplication of matrices and different new variables $x_{i,j}$ in the first case:

$$\begin{aligned}
x' &:= (x_{i,j}), & (t_1 + t_2)' &:= t_1' + t_2', \\
0' &:= 0_{n,n}, & (t_1 \cdot t_2)' &:= t_1' \cdot t_2', \\
1' &:= 1_n, & (\mu x.t)' &:= \mu x'.t'.
\end{aligned}$$

Let $M := Mat_{n\times n}(A)$. Note that each $\rho : X \to M$ is obtained from some $\hat{\rho} : X \to A$ such that $\rho(x) = (\hat{\rho}(x_{i,j}))$ when $x' = (x_{i,j})$. To define the interpretation $t_M : M^X \to M$, we put

$$t_M(\rho) \quad := \quad t'_A(\hat{\rho}). \tag{41}$$

To see that M is a μ-semiring, we check the conditions 1. – 3. of Definition 2.1. Let s, t be terms, $x \in X$ and $\rho : X \to M$.

1. By definition, $x_M(\rho) = x'_A(\hat{\rho}) = (\hat{\rho}(x_{i,j})) = \rho(x)$. Likewise, $0_M(\rho) = 0'_A(\hat{\rho}) = 0_{n,n}$ and $(s+t)_M(\rho) = (s'+t')_A(\hat{\rho}) = s'_A(\hat{\rho})+t'_A(\hat{\rho}) = s_M(\rho)+t_M(\rho)$, and similarly for 1 and $(s \cdot t)$.

2. We need to show

$$(t[s/x])_M(\rho) = (t[s/x])'_A(\hat{\rho}) = t'_A(\widehat{\rho[x \mapsto s_M(\rho)]}) = t_M(\rho[x \mapsto s_M(\rho)]).$$

By induction on t, one first shows that $(t[s/x])' = t'[s'/x']$, where substitution for matrices of variables is done componentwise. This is clear when t is atomic, and immediate by induction when t is $(t_1 + t_2)$ or $(t_1 \cdot t_2)$. If t is $\mu x.r$, then

$$(t[s/x])' = (\mu x.r)' = \mu x'.r' = (\mu x'.r')[s'/x'] = t'[s'/x'].$$

If t is $\mu z.r$ where $z \not\equiv x$ then, assuming z is not free in s, $(\mu z.r)[s/x] = \mu z.r[s/x]$ and hence

$$(t[s/x])' = (\mu z.r[s/x])' = \mu z'(r[s/x])' = \mu z'.r'[s'/x'] = (\mu z'.r')[s'/x'] = t'[s'/x']$$

by induction and using (30). Moreover,

$$t'_A(\widehat{\rho[x \mapsto s_M(\rho)]}) = t'_A(\hat{\rho}[x' \mapsto s'_A(\hat{\rho})]) = (t'[s'/x'])_A(\hat{\rho}).$$

3. Assume $t_M = s_M$. Then $t'_A = s'_A$, thus $(\mu x'.t')_A = (\mu x'.s')_A$. So for all ρ,

$$(\mu x.t)_M(\rho) = (\mu x'.t')_A(\hat{\rho}) = (\mu x'.s')_A(\hat{\rho}) = (\mu x.s)_M(\rho).$$

Hence $(\mu x.t)_M = (\mu x.s)_M$. □

THEOREM 5.11 *If A is an algebraic Conway semiring, then so is $Mat_{n\times n}(A)$, for each $n \geq 1$.*

Proof. Let $M := Mat_{n\times n}(A)$. By Corollary 5.10, M is a μ-semiring. By Theorem 5.4 it follows that M satisfies the Conway identities (3) and (4). By (38) – (40), M satisfies (10), (11) and (12). □

Corollary 5.12 *Let A be an algebraic Conway semiring. For all term matrices S, T of appropriate size, the following equations hold in A:*

$$TT^* + 1_n = T^* \quad (42)$$
$$T^*T + 1_n = T^* \quad (43)$$
$$(T+S)^* = (T^*S)^*T^* \quad (44)$$
$$(T+S)^* = T^*(ST^*)^* \quad (45)$$
$$(TS)^* = 1_n + T(ST)^*S. \quad (46)$$

For an $n \times n$ matrix X of distinct variables and any $n \times n$ term matrices T and S, which may contain variables of X, the following equations hold in A:

$$\mu X(TX+S) = \mu X(T^*S) \quad (47)$$
$$\mu X(XT+S) = \mu X(ST^*). \quad (48)$$

Proof. By Theorem 5.11, Proposition 4.2 and Proposition 4.3. □

The equations (47) and (48) also hold when X is of size $m \times n$ and T and S have appropriate sizes. Indeed, with an $m \times n$ matrix Y of new variables, by (33) and (38) we get

$$\mu X(TX+S) = \mu X \mu Y(TY+S) = \mu X(T^*S),$$

and similarly for (48).

Theorem 5.13 *If A is a Park μ-semiring, or an algebraically complete semiring, so is $Mat_{n\times n}(A)$, for each $n \geq 1$.*

Proof. By Theorem 5.6, the vector version of the Park induction rule holds in A. □

6 Normal forms

In this section we present a Greibach normal form theorem applicable to all algebraically complete semirings. We also show that analogs of elimination of chain rules and deletion rules in context-free grammars hold in algebraically complete semirings, although we can prove the latter only when $+$ is idempotent.

Let K be the set of terms $\{0, 1, \ldots\} \cup \{1^*\}$, which, by Theorem 3.12, amount to *all* closed terms over algebraically complete semirings.

PROPOSITION 6.1 *For $k \in K$, the equation $kx = xk$ holds in all algebraic Conway semirings.*

Proof. This is obvious when $k \neq 1^*$. Moreover, we have

$$\begin{aligned} 1^r \cdot x &= \mu z(1 \cdot z + x) \\ &= \mu z(z \cdot 1 + x) \\ &= x \cdot 1^\ell \end{aligned}$$

in any algebraic Conway semiring, and $1^* = 1^r = 1^\ell$. □

A *monomial* is a term of the form ku, where $k \in K$ and u is a product of variables. When u is the empty product, the monomial ku is called *constant*. The *leading factor* of a monomial ku, where $u = x_1 \cdots x_n$ is a nonempty product of variables, is the variable x_1. A *polynomial* is any finite sum of monomials. A *finite polynomial* is a polynomial which is also a finite term, i.e. a polynomial whose constants and leading factors belong to $\mathbb{N}$. In particular, 0 is a finite polynomial.

With respect to the semiring equations, any finite term is equivalent to a finite polynomial. The following normal form theorem is quite standard.

THEOREM 6.2 *(See, e.g., [6]) In algebraic Conway semirings, any μ-term is equivalent to the first component of a term vector of the form*

$$\mu(x_1, \ldots, x_n).(p_1, \ldots, p_n),$$

where each p_i is a finite polynomial.

DEFINITION 6.3 *A term vector $\mu\vec{x}.\vec{t}$, where $\vec{t} = (t_1(\vec{x}, \vec{y}), \ldots, t_n(\vec{x}, \vec{y}))$, is a* context-free grammar *if each t_i is a polynomial. The context-free grammar $\mu\vec{x}.\vec{t}(\vec{x}, \vec{y})$* has no chain rules*, if no t_i has a monomial of the form kx where $k \in K \setminus \{0\}$ and x is among the variables $\vec{x}$; it* has no ϵ-rules *if no t_j has a monomial of the form k where $k \in K \setminus \{0\}$.*

A context-free grammar $\mu\vec{x}.\vec{t}$ is in Greibach normal form *if each t_i is a polynomial which is a sum of non-constant monomials whose leading factors are among the* parameters $y_1, \ldots, y_m$.

The next theorem is a first version of Greibach's normal form theorem. The algorithm in the proof is due to Rosenkrantz [25] (cf. [13], Algorithm 4.9.1). We use properties of least pre-fixed-points rather than power series to prove its correctness, and thus show that it is applicable to any algebraic Conway semiring.

If $\mu\vec{x}.\vec{t}$ has dimension n and $m \leq n$, we denote by $(\mu\vec{x}.\vec{t})_{[m]}$ the vector whose components are the first m components of $\mu\vec{x}.\vec{t}$. We simply write $(\mu\vec{x}.\vec{t})_1$ for $(\mu\vec{x}.\vec{t})_{[1]}$.

THEOREM 6.4 *Let $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{z} = (z_1, \ldots, z_p)$ be different variables, and suppose that $\mu\vec{x}.\vec{t}(\vec{x}, \vec{z})$ is a context-free grammar that has no chain-rules and no ϵ-rules. Then there is a context-free grammar*

$$\mu(x_1, \ldots, x_n).(s_1, \ldots, s_n)(x_1, \ldots, x_n, z_1, \ldots, z_p)$$

in Greibach normal form, such that $m \leq n \leq m + m^2$ and the equation

$$\mu\vec{x}.\vec{t} \quad = \quad (\mu(x_1, \ldots, x_n).(s_1, \ldots, s_n))_{[m]}$$

holds in any algebraic Conway semiring.

Proof. By distributivity, we can write

$$t_j(\vec{x}, \vec{z}) \quad = \quad \sum_{k=1}^{m} (x_k \cdot t_{kj}(\vec{x}, \vec{z})) + r_j(\vec{x}, \vec{z}),$$

where r_j is 0 or a sum of non-constant monomials whose leading factors are parameters; constant monomials $\neq 0$ do not occur since $\mu\vec{x}.\vec{t}$ has no ϵ-rules. So we can write $\mu\vec{x}.\vec{t}$ as

$$\mu\vec{x}(\vec{x} \cdot T(\vec{x}, \vec{z}) + \vec{r}(\vec{x}, \vec{z})),$$

using

$$T = \begin{pmatrix} t_{11} & \ldots & t_{1m} \\ \vdots & \ddots & \vdots \\ t_{m1} & \ldots & t_{mm} \end{pmatrix} \quad \text{and} \quad \vec{r} = (r_1, \ldots, r_m).$$

With an $m \times m$ matrix $Y = (\, y_{ij}\,)$ of new variables, consider the term

$$\mu(\vec{x}, Y).(\vec{r}Y + \vec{r}, TY + T). \tag{49}$$

Claim The equation

$$\mu\vec{x}.\vec{t} \;=\; (\mu(\vec{x}, Y).(\vec{r}Y + \vec{r}, TY + T))_{[m]}$$

holds in all algebraic Conway semirings.

Indeed, by Corollary 5.12, equation (48),

$$\begin{aligned}\mu\vec{x}.\vec{t} &= \mu\vec{x}(\vec{x}T + \vec{r})\\ &= \mu\vec{x}(\vec{r}T^*).\end{aligned}$$

On the other hand, by Theorem 5.3, equation (31), and Theorem 5.8, equation (38) and Corollary 5.12, equation (42), we have that

$$\begin{aligned}(\mu(\vec{x}, Y).(\vec{r}Y + \vec{r}, TY + T))_{[m]} &= \mu\vec{x}(\vec{r}T^*T + \vec{r})\\ &= \mu\vec{x}(\vec{r}(T^*T + 1_m))\\ &= \mu\vec{x}(\vec{r}T^*).\end{aligned}$$

It remains to be shown that (49) contains no essential left recursion. First, each component of $\vec{r}Y + \vec{r}$ is of the form

$$(\vec{r}Y)_j + r_j \;=\; \sum_{k=1}^{m}(r_k \cdot y_{kj}) + r_j,$$

which is 0 or can be written as a sum of non-constant monomials whose leading factors are parameters. Second, each component of the term $TY + T$ is of the form

$$\sum_{k=1}^{m} t_{ik} \cdot y_{kj} + t_{ij}. \tag{50}$$

By Proposition 5.5 leading factors x_u in summands of t_{ik} and t_{ij} can be replaced by $(\vec{r}Y)_u + r_u$. Since $\mu\vec{x}.\vec{t}$ has no chain-rules, none of the t_{ik} or t_{ij} is a constant $k \in K \setminus \{0\}$, so y_{kj} is not a leading factor of $t_{ik} \cdot y_{kj}$ and no monomial in the new polynomials is a constant $\neq 0$. $\square$

Example 6.5 Let G be the context-free grammar

$$\begin{array}{ccccc} A & = & BC & + & a \\ B & = & Ab & + & CA \\ C & = & AB & + & CC \end{array}$$

over the alphabet $\{a, b\}$. In matrix notation, this is

$$(A, B, C) \;=\; (A, B, C) \cdot T \;+\; (a, 0, 0) \qquad \text{where} \quad T = \begin{pmatrix} 0 & b & B \\ C & 0 & 0 \\ 0 & A & C \end{pmatrix}. \tag{51}$$

By the proof, the least solution of (51) is the same as the least solution of the (essentially) right-recursive system

$$\begin{array}{ccc} (A, B, C) & = & (a, 0, 0) \cdot Y + (a, 0, 0) \\ Y & = & T \cdot Y + T \end{array} \qquad \text{where} \quad Y = \begin{pmatrix} Y_{1,1} & Y_{1,2} & Y_{1,3} \\ Y_{2,1} & Y_{2,2} & Y_{2,3} \\ Y_{3,1} & Y_{3,2} & Y_{3,3} \end{pmatrix}. \tag{52}$$

Multiplying out gives

$$\begin{array}{lcllcl} A & = & aY_{1,1} + a & Y_{1,1} & = & bY_{2,1} + BY_{3,1} \\ B & = & aY_{1,2} & Y_{1,2} & = & bY_{2,2} + BY_{3,2} + b \\ C & = & aY_{1,3} & Y_{1,3} & = & bY_{2,3} + BY_{3,3} + B \\ Y_{2,1} & = & CY_{1,1} + C & Y_{3,1} & = & AY_{2,1} + CY_{3,1} \\ Y_{2,2} & = & CY_{1,2} & Y_{3,2} & = & AY_{2,2} + CY_{3,2} + A \\ Y_{2,3} & = & CY_{1,3} & Y_{3,3} & = & AY_{2,3} + CY_{3,3} + C \end{array}$$

Plugging in the right hand sides for A, B, C in the Y-equations gives 28 rules in GNF. Standard textbooks like Hopcroft and Ullman [14] give an exponential algorithm, producing 119 rules for this example.

For algebraically complete semirings, we obtain a slightly more general version of the Greibach normal form theorem, based on the following

Lemma 6.6 (Elimination of chain rules) *For every context-free grammar $\mu\vec{x}.\vec{t}(\vec{x}, \vec{z})$ there is a context-free grammar $\mu\vec{x}.\vec{s}$ that has no chain rules, such that*

$$\mu\vec{x}.\vec{t} = \mu\vec{x}.\vec{s}$$

holds in all algebraically complete semirings. If $\mu\vec{x}.\vec{t}$ has no ϵ-rules, then $\mu\vec{x}.\vec{s}$ has no ϵ-rules.

Proof. By distributivity, we can write

$$t_j(\vec{x}, \vec{z}) \;=\; \sum_{i=1}^{m} x_i k_{ij} + r_j(\vec{x}, \vec{z})$$

where $k_{ij} \in K$ and the polynomials r_j have no monomials of the form kx, with $k \in K \setminus \{0\}$ and x is one of the variables $\vec{x}$. Let $E = (\, k_{ij} \,)$. By (48),

$$\begin{aligned} \mu\vec{x}.\vec{t} &= \mu\vec{x}(\vec{x}E + \vec{r}) \\ &= \mu\vec{x}(\vec{r}E^*). \end{aligned}$$

By Proposition 3.8 and the definition of E^*, we obtain that E^* has entries in K only, so the terms $\vec{s} := \vec{r}E^*$ are polynomials. By the choice of $\vec{r}$, the polynomials $\vec{s}$ have no monomials of the form kx with $k \in K \setminus \{0\}$ and x a variable from $\vec{x}$. If the polynomials $\vec{t}$ have no monomials k with $k \in K \setminus \{0\}$, then those of $\vec{r}$ and hence those of $\vec{r}E^*$ have no such monomials. □

Corollary 6.7 *For each context-free grammar $\mu\vec{x}.\vec{t}(\vec{x},\vec{z})$ that has no ϵ-rules, there is a context-free grammar $\mu(\vec{x},\vec{y}).\vec{s}(\vec{x},\vec{y},\vec{z})$ in Greibach normal form such that*

$$\mu\vec{x}.\vec{t} = (\mu(\vec{x},\vec{y}).\vec{s})_{[m]}$$

holds in all algebraically complete semirings, where $\vec{x} = (x_1,\ldots,x_m)$.

Proof. By Lemma 6.6, we may assume that $\mu\vec{x}.\vec{t}$ has no chain-rules and no ϵ-rules. Hence, we can apply Theorem 6.4. □

Finally, we give a version of the Greibach normal form theorem that applies to all context-free grammars. Additional effort is needed to get rid of ϵ-rules:

Lemma 6.8 (Elimination of ϵ-rules) *Let $\vec{t}(\vec{x},\vec{z})$ be a vector of m polynomials in $\vec{x} = (x_1,\ldots,x_m)$ with parameters $\vec{z}$. There are constants $\vec{k} \in K^m$ and polynomials $\vec{s}(\vec{x},\vec{z})$ without non-zero constant monomials such that*

1. *in all algebraically complete semirings S,*

$$\mu\vec{x}.\vec{t} \;\leq\; \vec{k} + \mu\vec{x}.\vec{s}, \tag{53}$$

2. *in all continuous semirings and all idempotent algebraically complete semirings S,*

$$\mu\vec{x}.\vec{t} \;\geq\; \vec{k} + \mu\vec{x}.\vec{s}. \tag{54}$$

Proof. We will simplify the notation by suppressing the parameters $\vec{z}$.

Claim 1: There is $\vec{k} \in K^m$ and a vector $\vec{s}(\vec{x})$ of m polynomials such that

(i) $\vec{k} \leq \vec{\mu}\vec{x}.\vec{t}$ holds in all algebraically complete semirings S,

(ii) the polynomials $\vec{s}(\vec{x})$ have no constant monomials $\neq 0$, and

(iii) $\vec{t}[\vec{x} + \vec{k}/\vec{x}] = \vec{s}(\vec{x}) + \vec{k}$ with respect to the semiring equations.

Proof. First, we determine $\vec{k}$ from $\vec{t}$. Call a monomial $m(\vec{x})$ *pure* if it has no parameters and is not a constant. Using the semiring equations, we can write $\vec{t}$ in the form

$$\vec{t}(\vec{x}) \quad = \quad \vec{q}(\vec{x}) + \vec{p}(\vec{x}) + \vec{c} \tag{55}$$

where, in the i-th component, $p_i(\vec{x})$ is 0 or the sum of the pure monomials of $t_i(\vec{x})$ and c_i is 0 or the sum of the constant monomials of $t_i(\vec{x})$. Put

$$\vec{k} \quad := \quad \mu\vec{x}(\vec{p}(\vec{x}) + \vec{c}),$$

and note that $\vec{k} \in K^m = \mathbb{N}_\infty^m$ by Theorem 3.12. ¿From $\vec{0} \leq \vec{q}(\vec{x})$ and monotonicity, we obtain that $\vec{k} \leq \mu\vec{x}.\vec{t}$ holds in all algebraically complete semirings S, showing (i).

By the semiring equations, the polynomials $\vec{t}[\vec{x} + \vec{k}/\vec{x}]$ can be written as

$$\vec{t}[\vec{x} + \vec{k}/\vec{x}] \quad = \quad \vec{s}(\vec{x}) + \vec{d},$$

where, by Proposition 3.8, $\vec{d} \in K^m$ and the polynomials $\vec{s}(\vec{x})$ contain no non-zero constant monomials. Part (ii) is clear by the choice of $\vec{s}(\vec{x})$. For (iii), note that $\vec{d}$ is the sum of the constant monomials of

$$\vec{t}[\vec{x} + \vec{k}/\vec{x}] \quad = \quad \vec{q}[\vec{x} + \vec{k}/\vec{x}] + \vec{p}[\vec{x} + \vec{k}/\vec{x}] + \vec{c},$$

so $\vec{d} = \vec{p}[\vec{k}/\vec{x}] + \vec{c}$. But in algebraically complete semirings, the vector version of the fixed-point equation holds, so $\vec{k} = \vec{p}[\vec{k}/\vec{x}] + \vec{c} = \vec{d}$.

Claim 2: For all algebraically complete semirings S and $\vec{b} \in S$, if $\vec{s}(\vec{b}) \leq \vec{b}$, then $\mu\vec{x}.\vec{t} \leq \vec{b} + \vec{k}$.

Proof. ¿From Claim 1, (iii), and the assumption on $\vec{b}$ we have

$$\vec{t}(\vec{b} + \vec{k}) \quad = \quad \vec{s}(\vec{b}) + \vec{k} \quad \leq \quad \vec{b} + \vec{k},$$

which implies $\mu\vec{x}.\vec{t} \ \leq \ \vec{b} + \vec{k}$ by fixed-point induction.

Claim 3: If S is a continuous semiring or an algebraically complete idempotent semiring and $\vec{a} \in S$, then if $\vec{t}(\vec{a}) \leq \vec{a}$, then $\vec{k} + \mu\vec{x}.\vec{s} \leq \vec{a}$.

Proof. If S is continuous, for $\vec{b}_0 := \vec{0}$ and $\vec{b}_{n+1} := \vec{s}(\vec{b}_n)$ we have

$$\mu\vec{x}.\vec{s} \;=\; \bigsqcup_{n\in\mathbb{N}} \vec{b}_n. \tag{56}$$

Note that by the choice of $\vec{k}$ and Claim 1, (i), $\vec{b}_0 + \vec{k} = \vec{k} \leq \vec{a}$. Assuming $\vec{b}_n + \vec{k} \;\leq\; \vec{a}$, we obtain from the monotonicity of $\vec{t}$ and the choice of $\vec{s}$ that

$$\begin{aligned}
\vec{b}_{n+1} + \vec{k} &= \vec{s}(\vec{b}_n) + \vec{k} \\
&= \vec{t}(\vec{b}_n + \vec{k}) \\
&\leq \vec{t}(\vec{a}) \\
&\leq \vec{a}.
\end{aligned}$$

Hence, by induction, (56) and continuity of $+$,

$$\mu\vec{x}.\vec{s} + \vec{k} \;=\; \bigsqcup_{n\in\mathbb{N}} \vec{b}_n \;+ \vec{k} \;\leq\; \vec{a}.$$

If S is algebraically complete and idempotent, then from $\vec{0} \leq \vec{k} \leq \mu\vec{x}.\vec{t} \leq \vec{a}$ we obtain

$$\begin{aligned}
\vec{s}(\vec{a}) &\leq \vec{k} + \vec{s}(\vec{a}) \\
&= \vec{t}(\vec{a} + \vec{k}) \\
&\leq \vec{t}(\vec{a} + \vec{a}) \\
&= \vec{t}(\vec{a}) \\
&\leq \vec{a},
\end{aligned}$$

hence $\mu\vec{x}.\vec{s} \leq \vec{a}$. Using idempotency once more, we have $\vec{k} + \mu\vec{x}.\vec{s} \leq \vec{a} + \vec{a} \;=\; \vec{a}$.

¿From Claims 1–3 we obtain $\mu\vec{x}.\vec{t} = \vec{k} + \mu\vec{x}.\vec{s}$ holds, completing the proof of Lemma 6.8. □

EXAMPLE 6.9 Let $t(x) = xx + c$ with $c > 1$. Splitting t into its non-pure, pure and constant monomials, we obtain

$$t(x,z) \;\equiv\; q(x,z) + p(x) + c$$

where $q \equiv 0$ and $p \equiv xx$. Therefore,

$$k := \mu x(p(x) + c) \;=\; 1^*$$

and hence

$$t(x+k) \;\equiv\; (x+1^*)^2 \;\equiv\; x^2+2x1^*+(1^*)^2+c \;\equiv\; x^2+x1^*+1^*+c \;\equiv\; s(x)+k$$

for $s(x) := x^2 + 1^*x$. Hence

$$\begin{aligned} \mu x.t &= \mu x(xx+c) \;=\; 1^*, \\ \mu x.s &= \mu x(xx+x1^*) \;=\; 0, \end{aligned}$$

and indeed $k + \mu x.s = 1^* + 0 = 1^* = \mu x.t$.

We don't know if Lemma 6.8 holds for algebraically complete semirings in general, although some further cases are given in the appendix. Hence, of the version of Greibach's normal form theorem involving elimination of ϵ-rules we only have:

THEOREM 6.10 *For each context-free grammar $\mu\vec{x}.\vec{t}$ of length m there is $\vec{k} \in K^m$ and a context-free grammar $\mu\vec{x}.\vec{r}$ in Greibach normal form such that $\mu\vec{x}.\vec{t} = \vec{k} + (\mu\vec{x}.\vec{r})_{[m]}$ holds in all continuous semirings and in all idempotent algebraically complete semirings.*

Proof. By Lemma 6.8, there are $\vec{k} \in K^m$ and a context-free grammar $\mu\vec{x}.\vec{s}$ without ϵ-rules such that

$$\mu\vec{x}.\vec{t} \;\;=\;\; \vec{k} + \mu\vec{x}.\vec{s}$$

holds in all continuous semirings and in all idempotent algebraically complete semirings. By Lemma 6.6, we may assume that $\mu\vec{x}.\vec{s}$ does not have chain-rules. Hence, by Theorem 6.4, there is a context-free grammar $\mu(\vec{x},\vec{y}).\vec{r}$ such that

$$\mu\vec{x}.\vec{s} = (\mu(\vec{x},\vec{y}).\vec{r})_{[m]}$$

holds in all algebraic Conway semirings, hence in all algebraically complete semirings. □

For continuous semirings, the Greibach normal form theorem including ϵ-elimination also can be shown using formal power series, cf. [19].

Since the set of context-free languages over the alphabet A form an idempotent algebraically complete semiring, Theorem 6.10 implies the classical Greibach normal form theorem.

By Theorem 6.2, for every term t there are finite terms $\vec{t}$ such that $t = (\mu\vec{x}.\vec{t})_1$ holds in all algebraic Conway semirings. Therefore, we obtain:

Corollary 6.11 *For each term t either t is closed and there is some $k \in K$ such that $t = k$ holds in all algebraically complete semirings, or t is not closed and there is a $k \in K$ and a term $\mu\vec{x}.\vec{s}$ in Greibach normal form such that equation $t = k + (\mu\vec{x}.\vec{s})_1$ holds in all continuous semirings and in all idempotent algebraically complete semirings.*

7 The initial algebraically complete semiring

Recall from section 3 that a morphism between μ-semirings or algebraically complete semirings is any function that commutes with the term functions. A morphism of continuous semirings is a semiring morphism which is a continuous function.

It is not difficult to prove that for any set A, the power series semiring $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$, equipped with the pointwise order is the free continuous semiring generated by A: For any continuous semiring S and function $h : A \to S$, there is a unique morphism of continuous semirings $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle \to S$ extending h. (As usual, we identify each letter in A with the corresponding series.) In particular, $\mathbb{N}_\infty$ from Example 3.4 is the initial continuous semiring.

Recall that since $\mathbb{N}_\infty$ is a continuous semiring, it is also an algebraically complete semiring and a symmetric inductive *-semiring. In [11], it has been shown:

Theorem 7.1 *$\mathbb{N}_\infty$ is initial in the category of (symmetric) inductive *-semirings.*

In this section we prove Theorem 3.12 and its Corollary 3.13, which we recall as

Theorem 7.2 *If t is a closed term, then for some $c \in \mathbb{N}_\infty$, equation $t = c$ holds in all algebraically complete semirings.*

Corollary 7.3 *$\mathbb{N}_\infty$ is initial in the class of all algebraically complete semirings.*

Recall from Propositions 3.8 and 3.11 that in all algebraically complete semirings,

$$1^* = 2^* = \ldots = n^* = \ldots = 1^* + 1^* = 1^*1^* = 1^{**}.$$

By Theorem 6.2, any closed μ-term is equivalent to a term of the form $\mu(x_1,\ldots,x_n).(p_1,\ldots,p_n)$ where each p_i is a finite polynomial in $x_1,\ldots,x_n$. We may assume that the words appearing in the monomials of a polynomial are pairwise different and each monomial has a nonzero coefficient.

Let $\vec{p} = (p_1,\ldots,p_n)$ be a vector of finite polynomials, containing at most the variables $x_1,\ldots,x_n$. We call an integer $i \in [n] := \{1,\ldots,n\}$ *eventually nonzero* in $\vec{p}$ if p_i has a nonzero constant monomial, or a monomial of the form cu with $c \neq 0$ such that for each variable x_j appearing in u it holds that j is eventually nonzero in $\vec{p}$. If i is not eventually nonzero in $\vec{p}$, we call i *eventually zero* in $\vec{p}$.

The above definition is recursive. We may give an alternative inductive definition. Let C_0 consist of those integers $i \in [n]$ such that p_i contains a nonzero constant monomial. Given C_m, define C_{m+1} to be the union of C_m with the set of all $i \in [n]$ such that p_i contains a monomial cu with $c \neq 0$ such that $j \in C_m$ holds for each variable x_j occurring in u. Then let C denote the union of the C_m. An integer $i \in [n]$ is eventually nonzero in $\vec{p}$ iff $i \in C$.

PROPOSITION 7.4 *For the vector $\vec{p}$ as above and each $i \in [n]$, equation*

$$(\mu\vec{x}.\vec{p})_i \quad = \quad 0 \tag{57}$$

holds in all algebraically complete semirings iff i is eventually zero in $\vec{p}$.

Proof. Let i be eventually nonzero in $\vec{p}$, so that $i \in C_m$, for some m. We argue by induction on m to show that $(\mu\vec{x}.\vec{p})_i = 1$ in the Boolean semiring $\mathbb{B} = \{0,1\}$, which is algebraically complete. When $m = 0$, p_i contains a nonzero constant monomial c, so that by the fixed point equation,

$$(\mu\vec{x}.\vec{p})_i \quad \geq \quad c$$

holds in all algebraically complete semirings. In particular, $(\mu\vec{x}.\vec{p})_i = 1$ in $\mathbb{B}$. Suppose now that $m > 0$ and that our claim holds for all integers in C_{m-1}. Since $i \in C_m$, either $i \in C_{m-1}$ in which case the result is immediate from the induction hypothesis, or p_i contains a monomial cu with $c \neq 0$ such that $j \in C_{m-1}$ holds for all variables x_j appearing in u. By induction, $(\mu\vec{x}.\vec{p})_j = 1$ in $\mathbb{B}$ for all such j, so that by the fixed point equation, $(\mu\vec{x}.\vec{p})_i$ evaluates in $\mathbb{B}$ to an element greater than or equal to c times the product of the $(\mu\vec{x}.\vec{p})_j$, for all occurrences of the variables x_j in u, i.e., $(\mu\vec{x}.\vec{p})_i = 1$ holds in $\mathbb{B}$ as claimed.

For the converse, suppose that i is eventually zero in $\vec{p}$. Then for each monomial cu of p_i with $c \neq 0$ it holds that u contains at least one occurrence of a variable x_j with j eventually zero in $\vec{p}$. Thus, if in p_i we substitute 0 for all the variables that are eventually zero in $\vec{p}$, then we obtain 0 using just the semiring equations. In the rest of the argument, assume without loss of generality (by the permutation identity) that the eventually zero integers in $[n]$ are the last $n-m$ ones. Let A be an algebraically complete semiring and denote $\vec{a} = (a_1, \ldots, a_n) = (\mu\vec{x}.\vec{p})_A$. We want to prove that the last $n-m$ components of $\vec{a}$ are all 0. For each $i \in [n]$, let q_i denote the polynomial which results from p_i by substituting 0 for all the x_j with $j > m$, and let $\vec{b} = (b_1, \ldots, b_n) = (\mu\vec{x}.\vec{q})_A$. By the preceding observation,

$$\begin{aligned}\vec{b} &= (b_1, \ldots, b_m, 0, \ldots, 0)\\ &= \vec{q}_A(b_1, \ldots, b_m, 0, \ldots, 0)\\ &= \vec{p}_A(b_1, \ldots, b_m, 0, \ldots, 0),\end{aligned}$$

since all the q_j with $j > m$ are 0. But since A is algebraically complete,

$$\vec{a} \leq (b_1, \ldots, b_m, 0, \ldots, 0),$$

yielding that $a_j = 0$ for all $j > m$. □

Corollary 7.5 *Equation (57) holds in all algebraic Conway semirings iff it holds in all continuous semirings iff it holds in $\mathbb{N}_\infty$ iff it holds in the boolean semiring $\mathbb{B}$.*

Let $\vec{p} = (p_1, \ldots, p_n)$ be as above. For $i, j \in [n]$, we say that i *directly depends on* j in $\vec{p}$ if the variable x_j occurs in some monomial cu of p_i, where $c \neq 0$. We say that i *depends on* j if there is a chain of integers $i_0, \ldots, i_k$ in $[n]$ such that $i_0 = i$, $i_k = j$ and each i_m directly depends on i_{m+1}.

We say that an integer $i \in [n]$ is *eventually finite* in $\vec{p}$ if there is no infinite chain $i = i_0, i_1, \ldots$ of integers in $[n]$ such that for each m, integer i_m directly depends on i_{m+1}. Alternatively, i is eventually finite in $\vec{p}$ if it belongs to one of the following sets F_m. The set F_0 consists of all those $j \in [n]$ such that p_j is constant, and F_{m+1} is the union of F_m with the set of all $j \in [n]$ such that $k \in F_m$ for all $k \in [n]$ on which j directly depends.

If $i \in [n]$ is eventually finite in $\vec{p}$, we define *the value* v_i *of* i as follows: if $i \in F_0$, the value v_i is the constant p_i. If $i \in F_{m+1} - F_m$, then p_i is a sum of monomials $cx_{j_1}\cdots x_{j_k}$ such that $c \neq 0$ and $j_1,\ldots,j_k \in F_m$; the value v_i is the corresponding sum of the $cv_{j_1}\cdots v_{j_k}$.

PROPOSITION 7.6 *If i is eventually finite in $\vec{p}$, then $v_i \in \mathbb{N}$ and*

$$(\mu\vec{x}.\vec{p})_i = v_i$$

holds in all algebraic Conway semirings.

Proof. This is based on the vector form of the fixed point identity that holds in all algebraic Conway semirings. The details are routine. □

LEMMA 7.7 *For every $k \geq 0$,*

$$\mu x.(x^k)^* = 1^* \tag{58}$$

holds in any algebraically complete semiring.

Proof. This is clear when $k = 0$, since $\mu x.1^* = 1^*$ by the fixed point identity. Suppose that $k \geq 1$. Since $(1^*)^k = 1^*$ and $1^{**} = 1^*$ hold, it follows that $((1^*)^k)^* = 1^*$. Thus, by the least pre-fixed point rule, $\mu x.(x^k)^* \leq 1^*$. In order to prove the reverse inequation, suppose that S is an algebraically complete semiring. Let f denote the function $S \to S$ which maps any $x \in S$ to $(x^k)^*$. Then $f(0) = 1^*$, hence the least pre-fixed point of f is $\geq 1^*$. Thus, in all algebraically complete semirings, it holds that $1^* \leq \mu x.(x^k)^*$. □

LEMMA 7.8 *For every $k \geq 1$, the equation*

$$\mu x(x^k + 1) = 1^* \tag{59}$$

holds in any algebraically complete semiring.

Proof. Using (58) and the diagonal identity,

$$\begin{aligned}\mu x(x^k + 1) &= \mu x.\mu y(x^{k-1}y + 1)\\ &= \mu x.(x^{k-1})^*\\ &= 1^*. \quad \square\end{aligned}$$

Lemma 7.9 *For every non-constant polynomial $p = p(x)$ in the variable x, if the constant term of p is not zero (i.e., $p(x) = q(x) + c$ for some $q(x)$ and $c \neq 0$), then*

$$\mu x.p \quad = \quad 1^* \tag{60}$$

holds in all algebraically complete semirings.

Proof. First, note that $p(1^*) = 1^*$ holds in all algebraically complete semirings, so that $\mu x.p \leq 1^*$. We can write $p(x) = x^k + 1 + q(x)$ for some polynomial $q(x)$ and some $k \geq 1$. Thus, $x^k + 1 \leq p(x)$ holds, and thus also $\mu x(x^k + 1) \leq \mu x.p$. But $\mu x(x^k + 1) = 1^*$. □

Lemma 7.10 *For each μ-term $t(x_1, \ldots, x_n)$, either*

$$t(0, \ldots, 0) \quad = \quad 0$$

holds in all algebraically complete semirings, or

$$t(0, \ldots, 0) \quad \geq \quad 1$$

holds in all algebraically complete semirings.

Proof. We argue by induction on the structure of t. Our claim is clear when t is a variable or one of the constants $0, 1$, as is the induction step when t is the sum or product of two terms. Suppose finally that $t = \mu x.t'$, where $t' = t'(x_1, \ldots, x_n, x)$. By the induction hypothesis, we have that either $t'(0, \ldots, 0) = 0$ in all algebraically complete semirings, or else $t'(0, \ldots, 0) \geq 1$ in all algebraically complete semirings. In the first case, clearly 0 is the least pre-fixed point of the map $a \mapsto t'_A(0, \ldots, 0, a)$, over all algebraically complete semirings A, so that $t(0, \ldots, 0) = 0$ holds in all such semirings. In the second case, the least pre-fixed point of the above map is at least 1 in each A, since $t'_A(0, \ldots, 0) \geq 1$. Thus, $t(0, \ldots, 0) \geq 1$ holds. □

Proposition 7.11 *Suppose that $\vec{p} = (p_1, \ldots, p_n)$ is a vector of polynomials in the variables $\vec{x} = (x_1, \ldots, x_n)$ such that no $i \in [n]$ is eventually zero or finite in $\vec{p}$. Then*

$$\mu\vec{x}.\vec{p} \quad = \quad (1^*, \ldots, 1^*)$$

holds in all algebraically complete semirings.

Proof. It is clear that $(1^*, \ldots, 1^*)$ is a solution to the fixed point equation $\vec{x} = \vec{p}$, in all algebraically complete semirings, since no $i \in [n]$ is eventually zero or finite in $\vec{p}$. Hence, $\mu\vec{x}.\vec{p} \leq (1^*, \ldots, 1^*)$ holds. Below we show that $\mu\vec{x}.\vec{p} \geq (1^*, \ldots, 1^*)$ holds.

Consider the direct dependency graph on the set of integers $[n]$ determined by $\vec{p}$, which has a directed edge from i to j iff i directly depends on j (in $\vec{p}$.)

Let $H_1, \ldots, H_k$ denote all the maximal strongly connected subgraphs of the dependency graph with the property that whenever they contain a vertex i, they contain each vertex j on which i depends. Clearly, from every vertex there is a directed path to at least one of the subgraphs $H_1, \ldots, H_k$, i.e., each vertex depends on at least one vertex in $H_1 \cup \ldots \cup H_k$.

First we show that $(\mu\vec{x}.\vec{p})_i \geq 1^*$ holds in all algebraically complete semirings for each $i \in H_1 \cup \ldots \cup H_k$, say in H_1. Without loss of generality we may assume that $H_1 = [m]$, for some $m \in [n]$. Since H_1 contains each vertex accessible from the vertices of H_1 by a directed path, each p_j with $j \in [m]$ contains only variables in the set $\{x_1, \ldots, x_m\}$. Thus, by Theorem 5.3,

$$\mu(x_1, \ldots, x_m).(p_1, \ldots, p_m) \quad = \quad (\mu\vec{x}.\vec{p})_{[m]}$$

holds in all algebraically complete semirings, in fact in all algebraic Conway semirings. It cannot be the case that each p_j with $j \in [m]$ has a zero constant term, since otherwise the integers in $[m]$ would all be eventually zero in $\vec{p}$. Let i denote an integer in $[m]$ such that p_i has a nonzero constant term. Now, since H_1 is strongly connected, by repeated substitutions of components p_j for the variables x_j starting from p_i, we obtain a polynomial q_i which contains x_i (in a monomial with nonzero coefficient) and has a nonzero constant term. Proposition 5.5 and the permutation identity guarantee that

$$\mu(x_1, \ldots, x_m).(p_1, \ldots, p_i, \ldots, p_m) \quad = \quad \mu(x_1, \ldots, x_m).(p_1, \ldots, q_i, \ldots, p_m).$$

Also, by Proposition 7.4, no component of $\mu(x_1, \ldots, x_m).(p_1, \ldots, q_i, \ldots, p_m)$ is 0 in all algebraically complete semirings, so that by Lemma 7.10 each is at least 1 in each algebraically complete semiring. Thus, by the fixed point equation and monotonicity,

$$(\mu(x_1, \ldots, x_m).(p_1, \ldots, q_i, \ldots, p_m))_i \quad \geq \quad \mu x_i.q_i(1, \ldots, 1, x_i, 1, \ldots, 1)$$

which is 1^* in all algebraically complete semirings, by Lemma 7.9. Now, using the fact that each integer in $[m]$ depends on i, it follows by the fixed point identity that all other components of $\mu(x_1, \ldots, x_m).(p_1, \ldots, q_i, \ldots, p_m)$ and hence of $\mu(x_1,\ldots,x_m).(p_1,\ldots,p_m)$ are also $\geq 1^*$, in all algebraically complete semirings. Thus, for all $i \in H_1\cup\ldots\cup H_k$ we have $(\mu(x_1,\ldots,x_n).(p_1,\ldots,p_n))_i \geq 1^*$ in all algebraically complete semirings.

Finally, since any other component depends on some component in the union of the H_j, the same applies to any component of $\mu(x_1,\ldots,x_n).(p_1,\ldots,p_n)$. □

Proof of Theorem 7.2. We know that in algebraically complete semirings, t is equivalent to the first component of $\mu\vec{x}.\vec{p}$, for some vector $\vec{p}$ of finite polynomials in the variables $\vec{x}$. If 1 is eventually 0 or finite with value v, then $t = 0$ or $t = v$ holds in all algebraically complete semirings, respectively. Otherwise $t = 1^*$ holds. To see this, substitute 0 for all variables x_j in $\vec{p}$ such that j is eventually 0 and the constant v for all variables x_j eventually finite with value v, and apply the previous proposition. □

8 Open problems

By general arguments, free algebraically complete (idempotent) semirings exist.

Problem 8.1 Find concrete representations of the free algebraically complete (idempotent) semirings.

We conjecture that the one-generated free algebraically complete (idempotent) semiring consists of the algebraic series in $\mathbb{N}_\infty\langle\langle a^*\rangle\rangle$ (regular = context-free languages in $\{a\}^*$, respectively), where a is a single letter. When $|A| \geq 2$, it is *not* true that the free algebraically complete semiring on A is the semiring of algebraic series in $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$. Also, when $|A| \geq 2$, the free algebraically complete idempotent semiring on A is not the semiring of context-free languages in A^*.

We have established ϵ-elimination in algebraically complete idempotent semirings.

Problem 8.2 Does ϵ-elimination hold in all algebraically complete semirings? Does it hold in all algebraic Conway semirings satisfying $1^* = 1^{**}$?

It is known that $\mathbb{N}_\infty$ is initial in the class of all Conway semirings satisfying $1^* = 1^{**}$.

Problem 8.3 Is $\mathbb{N}_\infty$ initial in the class of all algebraic Conway semirings satisfying $1^* = 1^{**}$?

Problem 8.4 To what extent do the normal form theorems hold when, as in process algebra, we only have one-sided distributivity of multiplication over sum?

Problem 8.5 Is every Kleene algebra embeddable in an idempotent algebraically complete semiring? Is every symmetric inductive *-semiring embeddable in an algebraically complete semiring?

If so, then the Horn theory of Kleene algebras, which is undecidable (by a result of E. Cohen, cf. [17]), is the same as the rational Horn theory of idempotent algebraically closed semirings.

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9 Appendix

We mention some partial results concerning ϵ-elimination in algebraically complete semirings that extend Lemma 6.8. For a grammar $\mu\vec{x}.\vec{t}(\vec{x},\vec{z})$, let $\vec{k}$ be the constant terms and $\vec{s}(\vec{x},\vec{z})$ the polynomials defined in the proof of Lemma 6.8. We want to prove

$$\vec{k} + \mu\vec{x}.\vec{s} \quad \leq \quad \mu\vec{x}.\vec{t} \tag{61}$$

without continuity or idempotency. This is possible if there are no parameters $\vec{z}$:

Proposition 9.1 *Suppose the decomposition $\vec{t}(\vec{x},\vec{z}) = \vec{q}(\vec{x},\vec{z}) + \vec{p}(\vec{x}) + \vec{c}$ of Lemma 6.8 satisfies $\vec{q}(\vec{x},\vec{z}) \equiv 0$. Then (61) and hence ϵ-elimination holds in all algebraically complete semirings.*

Proof. By the assumption, $\vec{k} = \mu\vec{x}(\vec{p}(\vec{x}) + \vec{c}) = \mu\vec{x}.\vec{t}$. Hence it is sufficient to show that $\mu\vec{x}.\vec{s} = \vec{0}$. Recall that $\vec{s}(\vec{x})$ consists of the non-constant monomials of $\vec{t}[\vec{x}+\vec{k}/\vec{x}] = \vec{p}[\vec{x}+\vec{k}/\vec{x}] + \vec{c}$. Using the distribution laws of the semiring, with fresh variables $\vec{y}$ we can write

$$\vec{p}(\vec{x}+\vec{y}) \quad = \quad \vec{r}(\vec{u},\vec{x},\vec{y})[\vec{x}+\vec{y}/\vec{u}] + p(\vec{y}),$$

where $\vec{p}(\vec{y})$ collects all monomials of $p(\vec{x}+\vec{y})$ in $\vec{y}$ alone, and $\vec{r}(\vec{u},\vec{x},\vec{y})$ are polynomials in $\vec{u},\vec{x},\vec{y}$. By the construction, no monomial of $\vec{r}$ contains only variables from $\vec{y}$. Also, no monomial of $\vec{r}$ contains only variables from $\vec{u}$, since otherwise $\vec{r}[\vec{x}+\vec{y}/\vec{u}]$ contained a monomial in the variables $\vec{y}$ alone. Hence for all elements $\vec{u},\vec{y}$ we have $\mu\vec{x}.\vec{r}(\vec{u},\vec{x},\vec{y}) = \vec{0}$. But since

$$t(\vec{x}+\vec{k}) = \vec{p}(\vec{x}+\vec{k}) + \vec{c} = \vec{r}(\vec{x}+\vec{k},\vec{x},\vec{k}) + \vec{p}(\vec{k}) + \vec{c},$$

we have $\vec{s}(\vec{x}) = \vec{r}(\vec{x}+\vec{k},\vec{x},\vec{k})$ and hence $\mu\vec{x}.\vec{s} = \mu\vec{z}\mu\vec{x}.r(\vec{z}+\vec{k},\vec{x},\vec{k}) = \mu\vec{z}.\vec{0} = \vec{0}$.
□

A sufficient condition for (61) is the following:

Proposition 9.2 *Let $\vec{s}(\vec{x},\vec{z})$ and $\vec{k}$ be obtained from $\vec{t}(\vec{x},\vec{z})$ as in Lemma 6.8. Then*

$$\vec{k} + \mu\vec{x}.\vec{t} \leq \mu\vec{x}.\vec{t} \quad \Rightarrow \quad \vec{k} + \mu\vec{x}.\vec{s} \leq \mu\vec{x}.\vec{t} \tag{62}$$

holds in any algebraically complete semiring.

Proof. We assume $\vec{k} + \mu\vec{x}.\vec{t} \leq \mu\vec{x}.\vec{t}$ and suppress the parameters $\vec{z}$ in the following. By the choice of $\vec{s}$, we have $\forall\vec{x}(\vec{s}(\vec{x}) \leq \vec{t}(\vec{x}+\vec{k}))$, hence

$$\begin{aligned} \mu\vec{x}.\vec{s} &\leq \mu\vec{x}.\vec{t}(\vec{x}+\vec{k}) \\ &\leq \vec{k} + \mu\vec{x}.\vec{t}(\vec{x}+\vec{k}) \\ &= \mu\vec{x}(\vec{k}+\vec{t}(\vec{x})), \end{aligned}$$

using the composition identity in the last step. But $\mu\vec{x}(\vec{k}+\vec{t}(\vec{x})) \leq \mu\vec{x}.\vec{t}$ by the induction rule, because from the fixed-point inequation and the assumption, we get

$$\begin{aligned} \vec{k} + \vec{t}(\mu\vec{x}.\vec{t}) &\leq \vec{k} + \mu\vec{x}.\vec{t} \\ &\leq \mu\vec{x}.\vec{t}. \end{aligned}$$

Thus, $\mu\vec{x}.\vec{s} \leq \mu\vec{x}.\vec{t}$, and using the assumption again, the claim $\vec{k} + \mu\vec{x}.\vec{s} \leq \vec{k} + \mu\vec{x}.\vec{t} \leq \mu\vec{x}.\vec{t}$ follows from the monotonicity of $+$. □

Recall that $\vec{k} \le \mu\vec{x}.\vec{t}$ always holds, which in the idempotent case is equivalent to $\vec{k} + \mu\vec{x}.\vec{t} = \mu\vec{x}.\vec{t}$. Hence Propostion 9.2 provides another proof of ϵ-elimination for idempotent algebraically closed semirings. It also yields a further case of ϵ-elimination:

COROLLARY 9.3 *Suppose A is an algebraically complete semiring satisfying $x \le y \iff \exists z(x + z = y)$, and that $\vec{t}(\vec{x}, \vec{z})$, $\vec{s}(\vec{x}, \vec{z})$ and $\vec{k}$ are as in Lemma 6.8. If each component of $\vec{k}$ is 0 or 1^*, then $\vec{k} + \mu\vec{x}.\vec{s} \le \mu\vec{x}.\vec{t}$ holds in A.*

Proof. By (62), it is sufficient to show $\vec{k} + \mu\vec{x}.\vec{t} \le \mu\vec{x}.\vec{t}$. From the definition of $\vec{k}$, we know that $\vec{k} \le \mu\vec{x}.\vec{t}$ holds, so for given elements $\vec{z}$ there exists some $\vec{b}$ such that $\vec{k} + \vec{b} = \mu\vec{x}.\vec{t}$. By the assumption on $\vec{k}$, we have $\vec{k} + \vec{k} = \vec{k}$, and hence $\vec{k} + \mu\vec{x}.\vec{t} = \mu\vec{x}.\vec{t}$. □

LEMMA 9.4 *For every polynomial $t(x, \vec{z})$ in a single variable x and parameters $\vec{z}$ there is a constant $k \in K$ and a polynomial $s(x, \vec{z})$ without non-zero constant monomials such that*

$$\mu x.t \quad = \quad k + \mu x.s$$

holds in all algebraically complete semirings.

Proof. We can write $t(x, \vec{z}) = q(x, \vec{z}) + p(x) + c$ where c is the sum of constant, $p(x)$ the sum of pure and $q(x, \vec{z})$ that of the remaining monomials of t. If $c = 0$ or $p = 0$, then $k := \mu x(p(x) + c) = c$, and the claim follows by Proposition 9.7 below. Otherwise, $1 \le c$ and p has a monomial x^m for some $m \ge 1$. Then $k = \mu x(p(x) + c) = 1^*$ by Lemma 7.9. Next, we show that

$$k + \mu x.t \quad \le \quad \mu x.t. \tag{63}$$

We distinguish two cases. If $m > 1$, then from $1 \le 1^* = k \le \mu x.t$ we get

$$\begin{aligned} k + \mu x.t &\le \mu x.t + \mu x.t \\ &= 2(\mu x.t) \\ &\le (\mu x.t)(\mu x.t) \\ &\le p(\mu x.t) \\ &\le t(\mu x.t) \\ &= \mu x.t. \end{aligned}$$

Otherwise, if $m = 1$ for all monomials x^m of p, then $p(x) = dx$ for some constant $d \in K - \{0\}$. Using equation (28) and $d^*c = k = k + k$, we obtain

$$\begin{aligned} \mu x.t &= \mu x(q(x, \vec{z}) + dx + c) \\ &= \mu x(d^* q(x, \vec{z}) + d^* c) \\ &= \mu x(\tilde{q} + k) \\ &= \tilde{q}[\mu x(\tilde{q} + k)/x] + k \\ &= \tilde{q}[\mu x(\tilde{q} + k)/x] + k + k \\ &= \mu x.t + k, \end{aligned}$$

where $\tilde{q}(x, \vec{z}) = d^* q(x, \vec{z})$. Having (63), the claim follows by (62) and Lemma 6.8. □

EXAMPLE 9.5 Let $t(x, z) \equiv xxz + xx + 2x + 1^*$. Separating the pure and constant monomials, we get $t(x, y) \equiv q(x, z) + p(x) + c$ for $q(x, z) := xxz$, $p(x) := xx + 2x$, and $c := 1^*$. From $p(c) + c = 1^* \cdot 1^* + 2 \cdot 1^* + 1^* = 1^* = c$ we obtain $k := \mu x(p(x) + c) = 1^*$ and hence

$$\begin{aligned} t(x + k, z) &= (x + k)(x + k)z + (x + k)(x + k) + 2(x + k) + 1 \\ &= (x + k)(x + k)z + 2x(x + k) + kk + 2x + 2k + 1 \end{aligned}$$

So $t(x + k, z) = s(x, z) + k$ for

$$\begin{aligned} s(x, z) &:= (x + k)^2 z + 2x(x + k + 1) \\ &= x^2 z + 1^* xz + 1^* z + 2x^2 + 1^* x. \end{aligned}$$

Therefore, elimination of ϵ-rules leads to

$$\mu x.t(x, z) = 1^* + \mu x.s(x, z)$$

The next example demonstrates that (63) does not hold for term vectors $\mu\vec{x}.\vec{t}$ of length > 1, and hence is not a necessary condition for (61).

EXAMPLE 9.6 Let $\vec{x} = (x, y, z)$ and $\vec{t}(\vec{x}) = (x + 1, 1, xy)$. We obtain

$$\begin{aligned} \vec{c} &= (1, 1, 0), \\ \vec{p}(\vec{x}) &= (x, 0, xy), \\ \vec{k} &= \mu\vec{x}(\vec{p}(\vec{x}) + \vec{c}) \\ &= \mu\vec{x}.\vec{t} \\ &= (1^*, 1, 1^*). \end{aligned}$$

In particular, note that $\vec{k} + \vec{c} = (1^*, 2, 1^*) \not\leq (1^*, 1, 1^*) = \vec{k}$, hence also $\vec{k} + \vec{k} \not\leq \vec{k}$ and so we don't have

$$\vec{k} + \mu\vec{x}.\vec{t} \ \leq \ \mu x.\vec{t}.$$

We also don't have $\mu\vec{x}.\vec{t}(\vec{x} + \vec{c}) \leq \mu\vec{x}.\vec{t}$. But by Proposition 9.1, $\vec{k} + \mu\vec{x}.\vec{s} \leq \mu\vec{x}.\vec{t}$ still holds.

PROPOSITION 9.7 *Suppose the decomposition $\vec{t}(\vec{x},\vec{z}) = \vec{q}(\vec{x},\vec{z}) + \vec{p}(\vec{x}) + \vec{c}$ of Lemma 6.8 satisfies $\vec{c} = \vec{k}$ where $\vec{k} := \mu\vec{x}(\vec{p}(\vec{x}) + \vec{c})$. Then (61) and hence ϵ-elimination holds.*

In particular, we have $\vec{c} = \vec{k}$ whenever $\vec{p}(\vec{x}) = \vec{0}$ or $\vec{c} = \vec{1}^*$ or $\vec{c} = \vec{0}$.

Proof. The polynomials $\vec{s}(\vec{x},\vec{z})$ as defined in Lemma 6.8 satisfy

$$\vec{s}(\vec{x},\vec{z}) \quad \leq \quad \vec{q}(\vec{x}+\vec{k},\vec{z}) + \vec{p}(\vec{x}+\vec{k}).$$

Using monotonicity, the composition identity and $\vec{k} \leq \vec{c}$ we get

$$\begin{aligned}
\vec{k} + \mu\vec{x}.\vec{s} &\leq \vec{k} + \mu\vec{x}(\vec{q}(\vec{x}+\vec{k}) + \vec{p}(\vec{x}+\vec{k})) \\
&= \mu\vec{x}(\vec{q}(\vec{x}) + \vec{p}(\vec{x}) + \vec{k}) \\
&\leq \mu\vec{x}(\vec{q}(\vec{x}) + \vec{p}(\vec{x}) + \vec{c}) \\
&= \mu\vec{x}.\vec{t}.
\end{aligned}$$

Note that

$$\mu\vec{x}(\vec{q}(\vec{x},\vec{z}) + \vec{p}(\vec{x}) + \vec{k}) \quad \leq \quad \mu\vec{x}(\vec{q}(\vec{x},\vec{z}) + \vec{p}(\vec{x}) + \vec{c}). \tag{64}$$

is sufficient for the proof. □

EXAMPLE 9.8 Let $\vec{t}(x,y,a,b) = (ax+1, bx^2y+1)$, where $\vec{x} = (x,y)$ and $\vec{z} = (a,b)$. We have $\vec{p} = \vec{0}$ and so $\vec{k} = \vec{c} = (1,1)$. By the definition of $\mu\vec{x}.\vec{t}$ we obtain

$$\begin{aligned}
\mu\vec{x}.\vec{t} &= \mu(x,y)(ax+1, bx^2y+1) \\
&= (\mu x.(ax+1)[\mu y(bxy+)/y], \mu y.(bx^2y+1)[\mu x(ax+1)/x]) \\
&= (a^*, \mu y(ba^*a^*y+1)) \\
&= (a^*, (ba^*a^*)^*).
\end{aligned}$$

Since $\vec{p} = \vec{0}$, the non-constant part of $\vec{t}(\vec{x}+\vec{k})$ is $\vec{s}(\vec{x}) = (a(x+1), b(x+1)^2(y+1))$. Hence, writing a^+ for a^*a,

$$\begin{aligned}
\mu\vec{x}.\vec{s} &= \mu(x,y)(ax+a, b(x+1)^2(y+1)) \\
&= (\mu x.(ax+a), \mu y.(b(x+1)^2(y+1))[\mu x(ax+a)/x]) \\
&= (a^+, \mu y.(b(a^++1)^2(y+1))) \\
&= (a^+, (ba^*a^*)^+),
\end{aligned}$$

and indeed, $\vec{k} + \mu\vec{x}.\vec{s} = (1,1) + (a^+, (ba^*a^*)^+) = (a^*, (ba^*a^*)^*) = \mu\vec{x}.\vec{t}$.

In Example 9.6, we do have (64) with $\vec{k} \neq \vec{c}$, because $\vec{q} = \vec{0}$ and

$$\mu\vec{x}(\vec{p}+\vec{k}) = \mu\vec{x}.(x+1^*, 1, xy+1^*) = (1^*1^*, 1, 1^*1+1^*) = \vec{k} = \mu\vec{x}(\vec{p}+\vec{c}).$$

The next Example 9.9 shows that (64) is also not a neccessary condition for (61).

EXAMPLE 9.9 Let $\vec{t}(\vec{x}) := (0,0,xy) + (1,1,0) = \vec{p}(\vec{x}) + \vec{c}$. Then $\vec{k} = \mu\vec{x}(\vec{p} + \vec{c}) = \mu(x,y,z).(1,1,xy) = (1,1,1)$, but

$$\begin{aligned} \mu\vec{x}(\vec{p} + \vec{k}) &= \mu(x,y,z)(0+1,0+1,xy+1) \\ &= (1,1,2) \\ &\not\leq (1,1,1) \\ &= \mu\vec{x}(\vec{p} + \vec{c}). \end{aligned}$$

By Proposition 9.1, $\vec{k} + \mu\vec{x}.\vec{s} = \mu\vec{x}.\vec{t}$ still holds.

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