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Modified Bar Recursion

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Abstract

We introduce a variant of Spector's bar recursion (called "modified bar recursion") in finite types to give a realizability interpretation of the classical axiom of countable choice allowing for the extraction of witnesses from proofs of Σ_1 formulas in classical analysis. As a second application of modified bar recursion we present a bar recursive definition of the fan functional. Moreover, we show that modified bar recursion exists in $\mathcal{M}$ (the model of strongly majorizable functionals) and is not S1-S9 computable in $\mathcal{C}$ (the model of total functionals). Finally, we show that modified bar recursion defines Spector's bar recursion primitive recursively.

1 Introduction

In [24] Spector extended Gödel's Dialectica Interpretation of Peano Arithmetic [11] to classical analysis using bar recursion in finite types. Although considered questionable from an intuitionistic point of view ([1], 6.6) there has been considerable interest in bar recursion, and several variants of this definition scheme and their interrelations have been studied by, e.g. Schwichtenberg [21], Bezem [8] and Kohlenbach [16]. In this paper we add another variant of bar recursion (so-called modified bar recursion) and use it to give a realizability interpretation of the classical, i.e. negatively translated, axiom of dependent choice that can be used to extract witnesses from proofs of Σ_1-formulas in full classical analysis. Our interpretation is inspired by a paper by Berardi, Bezem and Coquand [2] who use a similar kind of recursion in order to interpret dependent choice. The main difference to our paper is that in [2] a rather ad-hoc infinitary term calculus and a non-standard notion of realizability are used whereas we work with a straightforward combination of negative translation, A-translation, modified realizability, and Plotkin's adequacy result for the partial continuous functional semantics of PCF [20].

As a second application of bar recursion (in section 4) we show that the definition of the fan functional within PCF given in [3] and [19] can be derived from Kohlenbach's and our variant of bar recursion.

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The final part of the paper deals with the question of defining (primitive recursively) modified bar recursion MBR in other bar recursive definitions (namely, Spector's original bar recursion SBR and Kohlenbach's bar recursion KBR – see [24] and [16] respectively) and vice versa. In section 5 we show that modified bar recursion exists in $\mathcal{M}$ (the model of strongly majorizable functionals), [1] from which we can conclude that MBR cannot be used to define KBR primitive recursively. In section 6 we show that modified bar recursion is primitive recursively definable in SBR. Finally, (in section 7,) we prove that (as the fan functional) modified bar recursion is not S1-S9 computable in $\mathcal{C}$ (the model of total continuous functionals), which implies that MBR is not (primitive recursively) definable in SBR nor in KBR.

2 Bar recursion in finite types

We work in a suitable extension of Heyting Arithmetic in finite types, HA^ω. For convenience we enrich the type system by the formation of finite sequences. So, our *Types* are $\mathbb{N}$, function types $\rho \to \sigma$, product types $\rho \times \sigma$, and finite sequences ρ^*. We set $\rho^\omega :\equiv \mathbb{N} \to \rho$. The *level* of a type is defined by $\mathsf{level}(\mathbb{N}) = 0$, $\mathsf{level}(\rho \times \sigma) = \max(\mathsf{level}(\rho), \mathsf{level}(\sigma))$, $\mathsf{level}(\rho^*) = \mathsf{level}(\rho)$, $\mathsf{level}(\rho \to \sigma) = \max(\mathsf{level}(\rho)+1, \mathsf{level}(\sigma))$. By o we will denote an arbitrary but fixed type of level 0, and by ρ, τ, σ arbitrary. The terms of our version of HA^ω are a suitable extension of the terms of Gödel's system T [11] in lambda calculus notation. We use the variables $i, j, k, l, m, n \colon \mathbb{N}$; $s, t \colon \rho^*$; $\alpha, \beta \colon \rho^\omega$ unless the type of these variables is stated explicitly otherwise. Other letters will be used for different types in different contexts. By $\stackrel{\tau}{=}$ we denote equality of type τ for which we assume the usual equality axioms. However, equality between functions is *not* assumed to be extensional. We also do *not* assume decidability for $\stackrel{\tau}{=}$, when $\mathsf{level}(\tau) > 0$. Type information will be frequently omitted, when it is irrelevant or inferable from the context. We let k^ρ denote the canonical lifting of a number $k \in \mathbb{N}$ to type ρ, e.g. $k^{\rho \to \sigma} :\equiv \lambda x^\rho . k^\sigma$. By an $\exists$-*formula* respectively $\forall\exists$-*formula* we mean a formula of the form $\exists y^\tau \, B$ respectively $\forall z^\sigma \, \exists y^\tau \, B$, where B is provably equivalent to an atomic formulaquantifier free and contains only decidable predicates. We will also use the following operations:

$$
\begin{array}{rcl}
\langle x_0, \ldots, x_{n-1} \rangle & :\equiv & \text{the finite sequence with elements } x_0, \ldots, x_{n-1} \\
|s| & :\equiv & \text{the length of } s \text{, i.e. } |\langle x_0, \ldots, x_{n-1} \rangle| = n \\
s_k & :\equiv & \text{the } k\text{-th element of } s \text{ provided } k < |s| \\
s * t & :\equiv & \text{the concatenation of } s \text{ and } t \\
s * x & :\equiv & s * \langle x \rangle \\
s * \alpha & :\equiv & \text{appending } \alpha \text{ to } s \text{, i.e.} \\
 & & (s * \alpha)(k) = [\mathsf{if}\ k{<}|s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k{-}|s|)] \\
s @ \alpha & :\equiv & \text{overwriting } \alpha \text{ with } s \text{, i.e.} \\
 & & (s @ \alpha)(k) = [\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ \alpha(k)] \\
s @ x & :\equiv & s @ \lambda k.x, \text{ i.e.}
\end{array}
$$

[1] Nonetheless, the realizers for the countable and dependent choice presented by the authors does not necessarily exist in $\mathcal{M}$ since continuity is assumed for the proof of the soundness of the interpretation.

$$
\begin{aligned}
& (s \mathbin{@} x)(k) = [\mathsf{if}\ k < |s|\ \mathsf{then}\ s_k\ \mathsf{else}\ x] \\
\overline{\alpha}k \quad &:\equiv \quad \langle \alpha(0), \ldots, \alpha(k-1) \rangle \\
\overline{\alpha, k} \quad &:\equiv \quad \overline{\alpha}k \mathbin{@} \lambda x.0 \\
\beta \in \overline{\alpha}k \quad &:\equiv \quad \overline{\beta}k = \overline{\alpha}k.
\end{aligned}
$$

Definition 2.1 *Spector's definition of bar recursion [24] reads in our notation as follows:*

$$
\mathsf{SBR}_{\rho,\tau} \quad : \quad \Phi(s) \stackrel{\tau}{=} \begin{cases} G(s) & \mathsf{if}\ Y(s \mathbin{@} 0^\rho) \stackrel{o}{<} |s| \\ H(s, \lambda x^\rho.\Phi(s * x)) & \mathsf{otherwise}, \end{cases}
$$

where $\stackrel{\tau}{=}$ *denotes equality of type* τ *and* 0^ρ *denotes the constant functional zero of type* ρ*.*

In his thesis [16] Kohlenbach introduced the following kind of bar recursion which differs from Spector's only in the stopping condition:

$$
\mathsf{KBR}_{\rho,\tau} \quad : \quad \Phi(s) \stackrel{\tau}{=} \begin{cases} G(s) & \mathsf{if}\ Y(s \mathbin{@} 0^\rho) \stackrel{o}{=} Y(s \mathbin{@} 1^\rho) \\ H(s, \lambda x^\rho.\Phi(s * x)) & \mathsf{otherwise}. \end{cases}
$$

Finally, we define modified bar recursion at type ρ*:*

$$
\mathsf{MBR}_\rho \quad : \quad \Phi(s) \stackrel{o}{=} Y(s \mathbin{@} H(s, \lambda x^\rho.\Phi(s * x))).
$$

Instead of $\Phi(s)$ *we should have written more precisely* $\Phi(Y, G, H, s)$ *in* $\mathsf{SBR}_{\rho,\tau}$*,* $\mathsf{KBR}_{\rho,\tau}$*, and* $\Phi(Y, H, s)$ *in* MBR_ρ *in order to make clear that these equations specify functionals* Φ *of the respective types*

$$
(\rho^\omega \to o) \to (\rho^* \to \tau) \to (\rho^* \to (\rho \to \tau) \to \tau) \to \rho^* \to \tau
$$

$$
(\rho^\omega \to o) \to (\rho^* \to \tau) \to (\rho^* \to (\rho \to \tau) \to \tau) \to \rho^* \to \tau
$$

$$
(\rho^\omega \to o) \to (\rho^* \to (\rho \to o) \to \rho^\omega) \to \rho^* \to o
$$

By SBR *we mean* $\bigcup_{\tau,\rho} \mathsf{SBR}_{\rho,\tau}$ *and the same applies to* KBR *and* MBR*. We say a model* $\mathcal{S}$ *satisfies one of the respective variants of bar recursion if in* $\mathcal{S}$ *(for any given types* τ *and* ρ*) a functional exists satisfying* $\mathsf{SBR}_{\rho,\tau}$*,* $\mathsf{KBR}_{\rho,\tau}$*, or* MBR_ρ *for all possible values of* Y, G, H, s*.*

Recursive definitions similar to MBR occur in [2], and, in a slightly different form in [3] and [19] in connection with the fan functional (cf. section 4).

Remark 2.2 Note that in the definition of MBR it is inessential whether we use the operation $@$ (overwrite) or $*$ (concatenation). However it *is* essential that the type of $\Phi(s)$ is of level 0. If, for example, the type of $\Phi(s)$ were $\mathbb{N} \to \mathbb{N}$ we could set $Y(\alpha)(m) :\stackrel{o}{\equiv} \alpha(m) + 1$ and $H(s, F)(k) :\stackrel{o}{\equiv} F(0)(|s| + 1)$, and obtain the equation

$$
\Phi(s)(m) \stackrel{o}{=} (s \mathbin{@} \lambda k.\Phi(s * \langle 0 \rangle)(|s| + 1))(m) + 1
$$

implying

$$
\Phi(\langle\ \rangle)(0) = \Phi(\langle 0 \rangle)(1) + 1 = \Phi(\langle 0, 0 \rangle)(2) + 2 = \ldots
$$

which is inconsistent with HA^ω.

The structures of primary interest to interpret bar recursion are the model $\mathcal{C}$ of *total continuous functionals* of Kleene [14] and Kreisel [17], the model $\widehat{\mathcal{C}}$ of *partial continuous functionals* of Scott [22] and Ershov [9], and the model $\mathcal{M}$ of (strongly) *majorizable functionals* introduced by Howard [12] and Bezem [7].

Theorem 2.3 *The models $\mathcal{C}$ and $\widehat{\mathcal{C}}$ satisfy all three variants of bar recursion.*

Proof. In the model $\widehat{\mathcal{C}}$ all three forms of bar recursion can simply be defined as the least fixed points of suitable continuous functionals. For $\mathcal{C}$ we use Ershov's result in [9] according to which the model $\mathcal{C}$ can be identified with the total elements of $\widehat{\mathcal{C}}$. Therefore it suffices to show that all three versions of bar recursion are total in $\widehat{\mathcal{C}}$. For Spector's version this has been shown by Ershov [9], and for the other versions similar argument apply. For example, in order to see that $\Phi(s)$ defined recursively by MBR is total for given total Y, H and s one uses bar induction on the bar

$$P(s) :\equiv Y(s \mathbin{@} \perp_\rho) \text{ is total}$$

where $\perp_\rho$ denotes the undefined element of type ρ. $P(s)$ is a bar because Y is continuous. $\Box$

Theorem 2.4 ([7], [16]) *$\mathcal{M}$ satisfies SBR but not KBR.*

We show in section 5 that $\mathcal{M}$ satisfies MBR.

3 Using MBR to realize countable choice

The aim of this section is to show how modified bar recursion can be used to extract witnesses from proofs of Σ^0_1-formulas in classical arithmetic plus the axiom (scheme) of countable choice

$$\textbf{AC} \quad \forall n\, \exists y\, A(n,y) \to \exists f\, \forall n\, A(n, f(n)).$$

Actually we will need only the following *weak modified bar recursion* which is a special case of MBR where H is constant:

$$\mathsf{wMBR}_\rho \quad : \quad \Phi(s) \stackrel{o}{=} Y(s \mathbin{@} \lambda k.H(s, \lambda x.\Phi(s * x))).$$

Note that in wMBR the returning type of H is ρ, i.e., the argument of Y consists of s followed by an infinite sequence with constant value of type ρ.

Remark 3.1 *In [4] the authors have shown how the same idea for the realizer of* **AC** *can be extended to give a realizer for the dependent choice [13]*

$$\textbf{DC} \quad \forall n\, \forall x\, \exists y\, A(n,x,y) \to \forall x \exists f\, (f(0) = x \wedge \forall n\, A(n, f(n), f(n+1))).$$

3.1 Witnesses from classical proofs

The method we use to extract witnesses from classical proofs is a combination of Gödel's negative translation (translation P^o in [18] page 42, see also [25]), the Dragalin/Friedman/Leivant trick, also called A-translation [27], and Kreisel's (formalized) modified realizability [26]. The method works in general for proofs in PA^ω, the classical variant of HA^ω. In order to extend it to PA^ω plus extra axioms Γ (e.g. $\Gamma \equiv \mathbf{DC}$) one has to find realizers for Γ^N, the negative translation of Γ[2], where $\perp$ is replaced by an $\exists$-formula (regarding negation, $\neg C$, is defined by $C \to \perp$). However, it is more direct and technically simpler to follow [5] and combine the Dragalin/Friedman/Leivant trick and modified realizability: instead of replacing $\perp$ by a $\exists$-formula we slightly change the definition of modified realizability by regarding $y \,\mathbf{mr}\, \perp$ as an (uninterpreted) atomic formula. More formally we define

$$y^\tau \,\mathbf{mr}_\tau\, \perp :\equiv P_\perp(y),$$

where $P_\perp$ is a new unary predicate symbol and τ is the type of the witness to be extracted. Therefore, we have a modified realizability for each type τ, according to the type of the existential quantifier in the $\forall\exists$-formula we are realizing. The other clauses of modified realizability are as usual, e.g.

$$f \,\mathbf{mr}_\tau\, (A \to B) :\equiv \forall x\, (x \,\mathbf{mr}_\tau\, A \to f x \,\mathbf{mr}_\tau\, B).$$

In the following proposition Δ is an axiom system possibly containing $P_\perp$ and further constants, which has the following closure property: If $D \in \Delta$ and B is a quantifier free formula with decidable predicates, then also the universal closure of $D[\lambda y^\tau.B/P_\perp]$ is in Δ, where $D[\lambda y^\tau.B/P_\perp]$ is obtained from D by replacing any occurrence of a formula $P_\perp(L)$ in D by $B[L/y]$.

Proposition 3.2 *Assume that Φ is a closed term such that*

$$\mathsf{HA}^\omega + \Delta \vdash \Phi \,\mathbf{mr}_\tau\, \Gamma^N$$

Then from any proof of

$$\mathsf{PA}^\omega + \Gamma \vdash \forall z^\sigma\, \exists y^\tau\, B(z, y)$$

where $\forall z^\sigma\, \exists y^\tau\, B(z, y)$ is a $\forall\exists$-formula in the language of HA^ω, one can extract a closed term $M^{\sigma\to\tau}$ such that

$$\mathsf{HA}^\omega + \Delta \vdash \forall z\, B(z, M z).$$

Proof. The proof is folklore. The main steps are as follows. Assuming w.l.o.g. that $B(z, y)$ is atomic ($P_\perp$ does not occur in $B(z, y)$) we obtain from $\mathsf{PA}^\omega + \Gamma \vdash \forall z^\sigma\, \exists y^\tau\, B(z, y)$ via negative translation

$$\mathsf{HA}^\omega + \Gamma^N \vdash_m \forall y\, (B(z, y) \to \perp) \to \perp$$

where $\vdash_m$ denotes derivability in minimal logic, i.e. ex-falso-quodlibet is not used. Now, soundness of modified realizability (which holds for our abstract version of modified realizability and minimal logic [5]), together with the assumption on Φ allows us to extract from this proof a closed term M such that

[2]The negative translation double-negates atomic formulas, replaces $\exists x$ by $\neg\forall x\neg$ and $A \vee B$ by $\neg(\neg A \wedge \neg B)$.

$$\mathsf{HA}^\omega + \Delta \vdash Mz\,\mathbf{mr}_\tau\,(\forall y\,(B(z,y) \to \bot) \to \bot)$$

i.e.

$$\mathsf{HA}^\omega + \Delta \vdash \forall f^{o \to o}\,(\forall y\,(B(z,y) \to P_\bot(fy)) \to P_\bot(Mzf)).$$

Replacing $P_\bot$ by $\lambda y.B(z,y)$ (remember the closure property of Δ) and instantiating f by the identity function we conclude

$$\mathsf{HA}^\omega + \Delta \vdash \forall z\,B(z, Mz(\lambda y.y)).\ \Box$$

We will apply this proposition with $\tau :\equiv o$ (and therefore we write just **mr** instead of $\mathbf{mr}_o$), $\Gamma :\equiv \mathbf{DC}$, or $\Gamma :\equiv \mathbf{AC}$ (countable choice, see below), and an axiom system Δ consisting of MBR (the defining equations), where the defined functional Φ is a new constant, together with the axiom

Continuity $\quad \forall F^{\rho^\omega \to o}, \alpha\,\exists n\,\forall \beta\,(\overline{\alpha}n = \overline{\beta}n \to F(\alpha) = F(\beta))$

(we call any n such that $\forall \beta\,(\overline{\alpha}n = \overline{\beta}n \to F(\alpha) = F(\beta))$ a point of continuity of F at α), and the following schema of

Relativized quantifier free pointwise bar induction

$$\forall \alpha \in S\,\exists n\,P(\overline{\alpha}n) \wedge \forall s \in S\,(\forall x\,[S(x,|s|) \to P(s * x)] \to P(s)) \to P(\langle\rangle)$$

where $S(x,n)$ is arbitrary, $P(s)$ is quantifier free, and $\alpha \in S$, $s \in S$ are shorthands for $\forall n\,S(\alpha(n),n)$ and $\forall i < |s|\,S(s_i,i)$, respectively. Clearly the condition on Δ in Proposition 3.2 is satisfied. This is similar to Luckhardt's higher bar induction over species, $(\mathrm{hBI})^\rho_\mathrm{D}$ ([18], page 144). [3]

In order to make sure that realizers can indeed be used to compute witnesses one needs to know that, 1. the axioms of $\mathsf{HA}^\omega + \Delta$ hold in a suitable model – we can choose the model $\mathcal{C}$ of continuous functionals – and, 2. every closed term of type level 0 (e.g. of type $\mathbb{N}$) can be reduced to a numeral in an effective and provably correct way. In [2] this is solved by building the notion of reducibility to normal form into the definition of realizability. In our case we solve this problem by applying Plotkin's adequacy result [20] as follows: each term in the language of HA^ω plus the bar recursive constants can be naturally viewed as a term in the language PCF [20], by defining the bar recursors by means of the general fixed point combinator. In this way our term calculus also inherits PCF's call-by-name reduction, i.e. if M is bar recursive and M reduces to M' then M' is bar recursive. Furthermore reduction is provably correct in our system, i.e. if M reduces to M' then $M = M'$ is provable. Now let M be a closed term of type $\mathbb{N}$. By Theorem 2.3 M has a total value, which is a natural number n, in the model of partial continuous functionals. Hence, by Plotkin's adequacy theorem M reduces to the numeral denoting n.

3.2 Realizing $\mathbf{AC}^N$

We now construct a realizer of the classical (i.e. negatively translated) axiom of countable choice,

$$\mathbf{AC}^N \quad \forall n\,(\forall y\,(A(n,y)^N \to \bot) \to \bot) \to \forall f\,(\forall n\,A(n,f(n))^N \to \bot) \to \bot.$$

[3] The verification of the realizer for **DC** makes use of $(\mathrm{aBI})^\rho_\mathrm{D}$ ([18], page 144). (cf. [4])

Following Spector [24] we reduce $\mathbf{AC}^N$ to the double negation shift

$$\mathbf{DNS} \qquad \forall n\,((B(n) \to \bot) \to \bot) \to (\forall n\, B(n) \to \bot) \to \bot$$

observing that $\mathbf{AC} + \mathbf{DNS} \vdash_m \mathbf{AC}^N$, where $\mathbf{DNS}$ is used with the formula $B(n) :\equiv \exists y\, A(n,y)^N$. Therefore it suffices to show that this instance of $\mathbf{DNS}$ is realizable. The following lemma, whose proof is trivial, is necessary to see that the weak form of modified bar recursion wMBR suffices in the interpretation of $\mathbf{AC}$ and $\mathbf{DC}$.

Lemma 3.3 *Let B be a formula such that all of its atomic sub-formulas occur in negated form. Then there is a closed term H such that $\forall \vec{z}\, H\, \mathbf{mr}\, (\bot \to B)$ is provable (in minimal logic), where $\vec{z}$ are the free variables of B (it is important here that H is closed, i.e. does not depend on $\vec{z}$).*

Note that the formula $B(n) :\equiv \exists y\, A(n,y)^N$ to which we apply $\mathbf{DNS}$ is of the form specified in Lemma 3.3.

Theorem 3.4 *The double negation shift $\mathbf{DNS}$ for a formula $B(n)$ is realizable using* wMBR *provided $B(n)$ is of the form specified in Lemma 3.3.*

Proof. In order to realize the formula

$$\forall n((B(n) \to \bot) \to \bot) \to (\forall n B(n) \to \bot) \to \bot$$

we assume we are given realizers

$$Y^{\rho^\omega \to o}\, \mathbf{mr}\, (\forall n B(n) \to \bot)$$

$$G^{\mathbb{N} \to (\rho \to o) \to o}\, \mathbf{mr}\, \forall n((B(n) \to \bot) \to \bot)$$

and try to build a realizer for $\bot$. Using wMBR we define

$$\Psi(s) = Y(s \mathbin{@} \lambda n.H(G(|s|, \lambda x^\rho.\Psi(s * x))))$$

where $H^{o \to \rho}$ is a closed term such that $\forall n\, H\, \mathbf{mr}\, (\bot \to B(n))$ is provable, according to Lemma 3.3. We set

$$S(x,n) :\equiv x\, \mathbf{mr}\, B(n),$$

$$P(s) :\equiv \Psi(s)\, \mathbf{mr}\, \bot,$$

and, by quantifier free pointwise bar induction relativized to S, we show $P(\langle\,\rangle)$, i.e. $\Psi(\langle\,\rangle)\, \mathbf{mr}\, \bot$.

i) $\forall \alpha \in S\ \exists n\ P(\overline{\alpha}n)$. Let $\alpha \in S$ be fixed, and let n be the point of continuity of Y at α, according to the continuity axiom. By assumptions on α and Y we get $\forall \beta\ (Y(\overline{\alpha}n \mathbin{@} \beta)\, \mathbf{mr}\, \bot)$, which implies $\Psi(\overline{\alpha}n)\, \mathbf{mr}\, \bot$.

ii) $\forall s \in S(\forall x\ [S(x,|s|) \to P(s * x)] \to P(s))$. Let $s \in S$ be fixed. Suppose $\forall x\ [S(x,|s|) \to P(s*x)]$, i.e. $\forall x\ [x\, \mathbf{mr}\, B(|s|) \to \Psi(s*x)\, \mathbf{mr}\, \bot]$, or in another words

$$\lambda x^\rho.\Psi(s*x)\, \mathbf{mr}\, (B(|s|) \to \bot).$$

Using the assumption on G we obtain

$$G(|s|, \lambda x^\rho.\Psi(s*x))\, \mathbf{mr}\, \bot,$$

and from that, setting $w :\overset{\rho}{\equiv} H(G(|s|, \lambda x^\rho.\Psi(s * x)))$, we obtain $w\,\mathbf{mr}\,B(n)$, for all n. Because $s \in S$ it follows that $s \mathbin{@} \lambda n.w\,\mathbf{mr}\,\forall n\,B(n)$ and therefore

$$Y(s \mathbin{@} \lambda n.w)\,\mathbf{mr}\,\bot.$$

Since $\Psi(s) = Y(s \mathbin{@} \lambda n.w)$ we have $P(s)$. $\Box$

As explained above Theorem 3.4 yields

Corollary 3.5 *The negative translation of the countable axiom of choice,* $\mathbf{AC}^N$ *is realizable using* wMBR*.*

4 Bar recursion and the fan functional

A functional $\Psi^{(\mathbb{N}^\omega \to o) \to \mathbb{N}}$ is called *fan functional* if it computes a modulus of uniform continuity for every continuous functional $Y^{\mathbb{N}^\omega \to o}$ restricted to infinite $0,1$-sequences, i.e.

$$\mathsf{FAN}(\Psi) \quad :\equiv \quad \forall Y; \alpha, \beta \leq \lambda x.1(\overline{\alpha}(\Psi(Y)) = \overline{\beta}(\Psi(Y)) \to Y\alpha \overset{o}{=} Y\beta)$$

(note that $\rho = \mathbb{N}$ here). A recursive fan functional which is presented [3] and [19] uses two procedures,

$$\Phi(s, v) = s \mathbin{@} [\text{if } Y(\Phi(s * 0, v)) \neq v \text{ then } \Phi(s * 0, v) \text{ else } \Phi(s * 1, v)] \tag{1}$$

$$\Psi(Y, s) \overset{\mathbb{N}}{=} \begin{cases} 0 & \text{if } Y(\alpha) = Y(s \mathbin{@} 0), \\ & \text{ where } \alpha = \Phi(s, Y(s \mathbin{@} 0)) \\ 1 + \max\{\Psi(Y, s * 0), \Psi(Y, s * 1)\} & \text{otherwise.} \end{cases} \tag{2}$$

The first functional, $\Phi(s, v)$, returns an infinite path α having s as a prefix, such that $Y(s \mathbin{@} \alpha) \neq v$, if such a path exists, and returns s extended by $\lambda x.1$, otherwise, i.e. if Y is constant v on all paths extending s. The second functional, $\Psi(Y, s)$, returns the maximum point of continuity for Y on all extension of s. We show that the functional $\lambda Y.\Psi(Y, \langle\,\rangle)$ is a fan functional and is primitive recursive in MBR and KBR. The proof of the following two lemmas (which can be formalized in $\mathbf{HA}^\omega$+ rBI) can be found in [4].

Lemma 4.1 MBR *is equivalent to*

$$\Phi(s^{\rho^*}) \overset{\rho^\omega}{=} s \mathbin{@} H(s, \lambda t^{\rho^*} \lambda x^\rho. Y^{\rho^\omega \to o}(\Phi(s * t * x))). \tag{3}$$

Lemma 4.2 KBR *is equivalent to,*

$$\Phi(s) \overset{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \mathbin{@} 0^\rho) \overset{o}{=} Y(s \mathbin{@} J(s)) \\ H(s, \lambda x^\rho.\Phi(s * x)) & \text{otherwise,} \end{cases} \tag{4}$$

where the new parameter J *is of type* $\rho^* \to \rho^\omega$ *and, as usual,* $\Phi(s)$ *is shorthand for the more accurate* $\Phi(Y, G, H, J, s)$*.*

Theorem 4.3 *The functional* $\lambda Y.\Psi(Y, \langle\,\rangle)$ *is primitive recursively definable in* MBR *and* KBR *together.*

Proof. We show that procedures Φ and Ψ satisfying the equations (1) and (2) respectively can be defined using MBR and KBR.

For defining the functional $\Phi(s, v)$ we use equation (3) of Lemma 4.1.

$$\Phi(s, v) \overset{o^\omega}{=} s \mathbin{@} H(s, v, \lambda t \lambda x. Y(\Phi(s * t * x)))$$

where H is defined by course of value primitive recursion as

$$H(s, v, F)(n) \overset{o}{=} \begin{cases} s_n & \text{if } n < |s| \\ 0 & \text{if } n \geq |s| \wedge F(c, 0) \neq v \\ 1 & \text{if } n \geq |s| \wedge F(c, 0) = v, \end{cases}$$

with $c :\equiv \langle H(s, v, F)(|s|), \ldots, H(s, v, F)(n-1) \rangle$. Clearly Φ satisfies equation (1) at all $n < |s|$. For $n \geq |s|$ we first observe that

$$\Phi(s, v)(n) \overset{o}{=} \begin{cases} 0 & \text{if } Y(\Phi(s * c_{s,n} * 0, v)) \neq v \\ 1 & \text{if } Y(\Phi(s * c_{s,n} * 0, v)) = v, \end{cases}$$

where $c_{s,n} :\equiv \langle \Phi(s, v)(|s|), \ldots, \Phi(s, v)(n-1) \rangle$. Now if $Y(\Phi(s * 0, v)) \neq v$ then $\Phi(s, v)(|s|) = 0$ and therefore $s * c_{s,n} = s * 0 * c_{s*0,n}$. Hence $\Phi(s, v)(n) = \Phi(s * 0, v)(n)$ as required by (1). The case $Y(\Phi(s * 0, v)) = v$ is similar.

One immediately sees that a functional Ψ satisfying (2) can be defined from an instance of equation (4) using the functional Φ above. $\Box$

Theorem 4.4 *$\lambda Y. \Psi(Y, \langle\, \rangle)$ is a fan functional.*

Proof. See [3] and [19]. $\Box$

Remark 4.5 *Kohlenbach [16] has shown that* KBR *is primitive recursively definable in* SBR *and $\hat{\mu}$ (where $\hat{\mu}$ is the functional defined as,*

$$\hat{\mu}(Y, \alpha) :\equiv \min k \, [Y(\overline{\alpha}k * 0) = Y(\overline{\alpha}k * 1)]).$$

Since in section 6 we show that MBR *defines* SBR *primitive recursively, we can indeed say that* FAN *is primitive recursively definable in* $\mathsf{MBR} + \hat{\mu}$.

5 Modified bar recursion and the model $\mathcal{M}$

The model $\mathcal{M}$ $(= \bigcup \mathcal{M}_\rho)$ of strongly majorizable functionals (introduced in [7] as a variation of Howard's majorizable functionals [12]) and the strongly majorizability relation $\text{s-maj}_\rho \subseteq \mathcal{M}_\rho \times \mathcal{M}_\rho$ are defined by induction on types as follows:

$$n \text{ s-maj}_o \, m :\equiv n \geq m, \qquad \mathcal{M}_o :\equiv \mathbb{N},$$

$$F^* \text{ s-maj}_{\rho \to \tau} \, F :\equiv F^*, F \in \mathcal{M}_\rho \to \mathcal{M}_\tau \wedge$$
$$\forall G^*, G \in \mathcal{M}_\rho \, [G^* \text{ s-maj}_\rho \, G \to F^* G^* \text{ s-maj}_\tau \, F^* G, FG],$$

$$\mathcal{M}_{\rho \to \tau} :\equiv \{F \in \mathcal{M}_\rho \to \mathcal{M}_\tau : \exists F^* \in \mathcal{M}_\rho \to \mathcal{M}_\tau \; F^* \text{ s-maj}_{\rho \to \tau} \, F\}.$$

In the following we abbreviate s-maj$_\rho$ by maj$_\rho$ and by "majorizable" always mean "strongly majorizable". We often omit the type in the relation maj$_\rho$.

In [16] it is shown that KBR is provably not primitive recursively definable from SBR, since SBR yields a well defined functional in the model of (strongly) majorizable functionals $\mathcal{M}$ (cf. [7]) and KBR does not (in the following we will by "majorizable" always mean "strongly majorizable"). SBR, however, can be primitive recursively defined from KBR (cf. [16]).[4] In this section we show that a functional satisfying MBR exists in $\mathcal{M}$. We first show that there exists a functional [5]

$$\Phi : \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \times (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$$

satisfying MBR, then we show that any such Φ has a majorant and therefore belongs to $\mathcal{M}$.

Most of our results in this section rely on Lemma 5.2 which can be viewed as a *weak continuity property* of functionals Y (of type $\rho^\omega \to o$) in $\mathcal{M}$. It says that a bound on the value of $Y(\alpha)$ can be determined from an initial segment of α. For the rest of this section all variables (unless stated otherwise) are assumed to range over the type structure $\mathcal{M}$.

Lemma 5.1 ([7], 1.4, 1.5) *Let* $\max^\rho$ *be inductively defined as,*

$$\max_{i \le n}{}^{o} m_i :\equiv \max\{m_0, \dots, m_n\},$$

$$\max_{i \le n}{}^{\tau \to \rho} X_i :\equiv \lambda Y^\tau . \max_{i \le n}{}^{\rho} X_i\, Y,$$

and for α^{ρ^ω}*, define* $\alpha^+(n) :\equiv \max_{i \le n}{}^{\rho} \alpha(i)$*. Then,*

$$\forall n (\alpha(n) \text{ maj } \beta(n)) \to \alpha^+ \text{ maj } \beta^+, \beta.$$

We also use addition in all types, which is done pointwise, e.g. if x, y are of type $\tau \to \rho$ then $x +_{\tau \to \rho} y :\equiv \lambda z^\tau \, (x(z) +_\rho y(z))$.

Lemma 5.2 (Weak continuity for $\mathcal{M}$) $\forall Y^{\rho^\omega \to \mathbb{N}}, \alpha\, \exists n^{\mathbb{N}}\, \forall \beta \in \overline{\alpha} n\, (Y(\beta) \le n)$.

Proof. Let Y and α be fixed, α^* maj α and Y^* maj Y. From the assumption

$$(a)\ \forall n\, \exists \beta \in \overline{\alpha} n (Y(\beta) > n)$$

we derive a contradiction. For any n, let β_n be the function whose existence we are assuming in (a). Let

$$\beta^*_n(i) :\equiv \begin{cases} 0^\rho & i < n \\ [\beta_n(i)]^* & i \ge n, \end{cases}$$

where $[\beta_n(i)]^*$ denotes some majorant of $\beta_n(i)$. Having defined the functional β^*_n we note two of its properties,

i) $\forall i < n (\beta^*_n(i) = 0^\rho)$,

ii) $(\alpha^* +_{\rho^\omega} \beta^*_n)^+$ maj β_n. (by Lemma 5.1)

[4]For the rest of the paper "s_1 is primitive recursively definable in s_2" should be understood as "there exists a closed term t such that $\mathbf{E\text{-}HA}^\omega \vdash s_1 = t(s_2)$"

[5]By $\mathcal{M}_\rho \to \mathcal{M}_\tau$ we mean an arbitrary function from $\mathcal{M}_\rho$ to $\mathcal{M}_\tau$. By $\mathcal{M}_{\rho \to \tau}$ we mean a functional from $\mathcal{M}_\rho$ to $\mathcal{M}_\tau$ which belongs to $\mathcal{M}$.

Consider the functional $\hat{\alpha}$ defined as, $\hat{\alpha}(n) :\equiv \alpha^*(n) +_\rho \sum_{i\in\mathbb{N}} \beta_i^*(n)$. Since at each point n only finitely many β_i^* are non-zero, α^* is well defined. Let $Y^*(\hat{\alpha}^+) = m$. Note that $\hat{\alpha}^+$ maj β_i, for all $i \in \mathbb{N}$, and from (a) we should have $m < Y(\beta_m) \leq m$, a contradiction. $\Box$

We extend, for convenience, the definition of majorizability for finite sequences as follows:

$$s^* \operatorname{maj}_{\rho^*} s :\equiv \langle s^* * \lambda k.s^*_{|s^*|-1}, |s^*|\rangle \operatorname{maj} \langle s * \lambda k.s_{|s|-1}, |s|\rangle,$$

(s^* and s not begin the empty sequence) where $\langle \alpha, m\rangle$ maj $\langle \beta, n\rangle$ stands for α maj $\beta \wedge m \geq n$. For the special case when one of the sequences is empty we set, s^* maj $\langle\ \rangle$ if s^* maj s^*. Moreover, $\langle\ \rangle$ majorizes only itself.

5.1 Finding $\Phi \in \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \times (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$ satisfying MBR

For any type ρ, the elements s of $\mathcal{M}_{\rho^*}$ (finite sequences of elements in ρ) can be viewed as nodes of an infinite tree which we call T. The infinite paths of T are the elements of $\mathcal{M}_{\rho^\omega}$ (which is just $\mathcal{M}_\rho^\omega$ as shown in [7]). For fixed Y and H, the functional Φ we are looking for should assign values to the nodes of T according to MBR. For each node s the set of nodes s' extending s is denoted by B_s.

Let $Y, H \in \mathcal{M}$ be fixed. We show that at each infinite path α there exists a point n such that a functional $\Phi_{\alpha,n} : \mathcal{M}_{\rho^*} \to \mathcal{M}_o$ can be defined satisfying MBR for all $s \in B_{\overline{\alpha}n}$. Then, by bar induction, a functional Φ can be defined for all nodes of T.

Let $\alpha \in \mathcal{M}_\rho^\omega$ be fixed, n the number whose existence is stated in Lemma 5.2, and $K :\equiv \{0, 1, \ldots, n\}$. We show how to define a functional $\Phi_{\alpha,n}(s)$ such that, for $s \in B_{\overline{\alpha}n}$, equation

$$\Phi_{\alpha,n}(s) = Y(s \mathbin{@} H(s, \lambda x.\Phi_{\alpha,n}(s * x)))$$

holds. Here we note that, for $s \in B_{\overline{\alpha}n}$, by Lemma 5.2, $\Phi_{\alpha,n}(s)$ must belong to the finite set K. Therefore, for those $s \in B_{\overline{\alpha}n}$, what we have is an instance of the more general equation,

$$(*)\quad \Psi(s) = G(s, \lambda x.\Psi(s * x)),$$

where $Img(G) \subseteq K$ (K as above). To see that modified bar recursion becomes an instance of $(*)$, let

$$G(s, F) :\equiv Y(\overline{\alpha}n * s \mathbin{@} H(\overline{\alpha}n * s, F)),$$

and, clearly, $Img(G) = Img(\lambda s \lambda F.Y(\overline{\alpha}n * s \mathbin{@} H(\overline{\alpha}n * s, F))) \subseteq K$. Hence, it suffices to show that equations of the form $(*)$ (with the mentioned restriction on G) always have a solution Ψ. That is what we will do now.

Consider the set $\mathcal{T} = \mathrm{T} \to 2^K \backslash \{\emptyset\}$. The set $\mathcal{T}$ can be viewed as the set of labelled trees whose labels range over non-empty subsets of K. We define a partial order $\sqsubseteq$ on $\mathcal{T}$ as follows (for $f, g \in \mathcal{T}$)

$$f \sqsubseteq g :\equiv \forall s\, (f(s) \subseteq g(s)).$$

Finally, we define an operation $\chi : \mathcal{T} \to \mathcal{T}$,

$$\chi(f)(s) :\equiv Img(\lambda F \in \mathsf{Cons}_s^f . G(s, F)),$$

where $\mathsf{Cons}_s^f :\equiv \{F : \forall x^\rho . F(x) \in f(s * x)\}$. We first observe the following.

Lemma 5.3 $(\mathcal{T}, \sqsubseteq)$ *is a directed complete semi-lattice.*

Proof. Let S be a directed subset of $\mathcal{T}$. Since we assign non-empty finite sets to the nodes of T, it is easy to see that $\bigcap S$ belongs to $\mathcal{T}$ and it is smaller than any element in S. $\Box$

Lemma 5.4 $\chi : \mathcal{T} \to \mathcal{T}$ *is monotone.*

Proof. Let $f \sqsubseteq g$ and s be fixed. We get that $\mathsf{Cons}_s^f \subseteq \mathsf{Cons}_s^g$, which implies $\chi(f)(s) \subseteq \chi(g)(s)$. $\Box$

By the Knaster-Tarski fixed point theorem we obtain an $f \in \mathcal{T}$ such that $\chi(f) = f$, i.e. $f(s) = Img(\lambda F \in \mathsf{Cons}_s^f . G(s, F))$, for all s. Let $\chi_{f,s}^{-1}$ be a functional from the set $f(s)$ to Cons_s^f such that $c = G(s, \chi_{f,s}^{-1}(c))$, for all $c \in f(s)$. Define the functional $\Psi(s)$ recursively as follows,

$$\Psi(\langle\,\rangle) :\equiv \text{arbitrary element of } f(\langle\,\rangle);$$

$$\Psi(s * x) :\equiv \chi_{f,s}^{-1}(\Psi(s))(x).$$

Lemma 5.5 *The functional* Ψ *is total and satisfies equation* $(*)$*.*

Proof. We have just shown that Ψ is total. Moreover, note that, for all s, the values assigned to $\Psi(s * x)$ are such that $\Psi(s) = G(s, \lambda x . \Psi(s * x))$. $\Box$

Corollary 5.6 *There exists a functional*

$$\Phi : \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \times (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$$

satisfying modified bar recursion.

5.2 Finding a majorant for Φ

Now we show that Φ (from corollary above) has a majorant, and therefore belongs to $\mathcal{M}$.

Lemma 5.7 *Let* s^* *and* s *s.t.* $|s^*| = |s|$ *be fixed. If* s^* maj s *then*

$$\forall \beta \in s \exists \beta^* \in s^* \, (\beta^* \text{ maj } \beta).$$

Proof. Let s^*, s and $\beta \in s$ be fixed. Moreover, assume $|s^*| = |s| = n$ and s^* maj s. Define β^* as,

$$\beta^*(i) :\equiv \begin{cases} s_i^* & \text{if } i < n \\ \max^\rho \{\max\limits_{j<i}{}^\rho \beta^*(j), [\beta(i)]^*\} & \text{otherwise,} \end{cases}$$

where $[\beta(i)]^*$ is some majorant of $\beta(i)$. First note that, for all i, $\beta^*(i)$ maj $\beta(i)$. We show that β^* maj β. Let $k \geq i$.

If $k < n$ then $\beta^*(k) = s_k^*$ maj s_i^* maj $s_i = \beta(i)$.

If $k \geq n$ then $\beta^*(k) = \max^\rho\{\max\limits_{j<k}{}^\rho \beta^*(j), [\beta(k)]^*\}$ maj $\beta^*(i)$ maj $\beta(i)$. $\Box$

In the following we shall make use of two functionals Ω and Γ which we define now. The functional Ω was first introduced in [15], 3.40.

Lemma 5.8 ([15], 3.41) *Define functionals* $\min^\rho$ *(from non-empty sets* $X \subseteq \mathcal{M}_\rho$ *to elements of* $\mathcal{M}_\rho$*) and* $\Omega : \mathcal{M}_\rho \to \mathcal{M}_\rho$ *as*

$$\min{}^{\mathbb{N}} X :\equiv \min X, \text{ for } \emptyset \neq X \subseteq \mathbb{N},$$

$$\min{}^{\rho\to\tau} X :\equiv \lambda y^\rho . \min{}^\tau \{Fy : F \in X\}, \text{ for } \emptyset \neq X \subseteq \mathcal{M}_{\rho\to\tau},$$

$$\Omega(F) :\equiv \min{}^\rho \{F^* : F^* \text{ maj } F\}.$$

Then,

i) *For all* F*,* $\Omega(F)$ maj F*,*

ii) Ω maj Ω*. (Therefore,* $\Omega \in \mathcal{M}$*.)*

Lemma 5.9 *Define* $\Gamma : \mathcal{M}_{\rho^\omega \to \mathbb{N}} \to \mathcal{M}_{\rho^\omega} \to \mathcal{M}_{\mathbb{N}}$ *as,*

$$\Gamma(Y)(\alpha) :\equiv \min n \, [\forall \beta \in \overline{\alpha}n(\Omega(Y)(\beta) \leq n)].$$

Then,

i) $\Gamma(Y)$ maj Y*,*

ii) $\Gamma(Y)$ *is continuous and* $\Gamma(Y)(\alpha)$ *is a point of continuity for* $\Gamma(Y)$ *on* α*,*

iii) Γ maj Γ*. (therefore,* $\Gamma \in \mathcal{M}$*)*

Proof. First of all, we note that, by Lemma 5.2, the functional Γ is well defined. By Lemma 5.8 (i), $\Omega(Y)$ maj Y.

i) Let α^* maj α. We have to show $\Gamma(Y)(\alpha^*) \geq \Gamma(Y)(\alpha), Y(\alpha)$. By the definition of $\Gamma(Y)$, and Lemma 5.8 (i), we have $\Gamma(Y)(\alpha^*) \geq \Omega(Y)(\alpha^*) \geq Y(\alpha)$. It is only left to show that $\Gamma(Y)(\alpha^*) \geq \Gamma(Y)(\alpha)$. Suppose that $n = \Gamma(Y)(\alpha^*) < \Gamma(Y)(\alpha) = m$. Note that there exists a $\beta \in \overline{\alpha}(m-1)$ such that $\Omega(Y)(\beta) \geq m$ (otherwise we get a contradiction to the minimality in the definition of $\Gamma(Y)$). But since $m > n$, by Lemma 5.7, there exists a $\beta^* \in \overline{\alpha^*}n$ such that β^* maj β. Therefore, $\Omega(Y)(\beta^*) \leq n < m \leq \Omega(Y)(\beta)$. But by Lemma 5.8 ($i$) also $\Omega(Y)(\beta^*) \geq \Omega(Y)(\beta)$, a contradiction.

ii) Let α be fixed and take $n = \Gamma(Y)(\alpha)$. Suppose there exists a $\beta \in \overline{\alpha}n$ such that $\Gamma(Y)(\beta) \neq n$. If $\Gamma(Y)(\beta) < n$ we get, since $\alpha \in \overline{\beta}n$, that $\Gamma(Y)(\alpha) < n$, a contradiction. Suppose $\Gamma(Y)(\beta) > n$. Since $\beta \in \overline{\alpha}n$ we have, $\forall \gamma \in \overline{\beta}n(\Omega(Y)(\gamma) \leq n)$, also a contradiction.

iii) Assume Y^* maj Y and α^* maj α. We show $\Gamma(Y^*)(\alpha^*) \geq \Gamma(Y)(\alpha)$. By the self majorizability of $\Gamma(Y)$ we have $\Gamma(Y)(\alpha^*) \geq \Gamma(Y)(\alpha)$. We now show $\Gamma(Y^*)(\alpha^*) \geq \Gamma(Y)(\alpha^*)$. Let $n = \Gamma(Y^*)(\alpha^*)$ and suppose $m = \Gamma(Y)(\alpha^*) > n$. By the definition of $\Gamma(Y)$, there exists a $\beta \in \overline{\alpha^*}(m-1)$ s.t. $\Omega(Y)(\beta) \geq m$. But, since $m > n$, by Lemma 5.7, there exists a $\beta^* \in \overline{\alpha^*}n$ s.t. β^* maj β, and by Lemma 5.8 (ii), $\Omega(Y^*)(\beta^*) \geq m > n$, a contradiction. $\Box$

Lemma 5.10 *Let* Y^* maj Y *(of type* $\rho^\omega \to \mathbb{N}$*) and* α *be fixed, and* $n = \Gamma(Y^*)(\alpha)$*. If* $\overline{\alpha}n$ maj s *and* $|s| = n$ *then for all sequences* β *we have*

$$\Gamma(Y^*)(s \,@\, \beta), \Gamma(Y)(s \,@\, \beta), Y(s \,@\, \beta) \leq n.$$

Proof. We prove just that $\Gamma(Y^*)(s @ \beta) \leq n$. The other two cases follow similarly. Suppose there exists a β such that $n < \Gamma(Y^*)(s @ \beta)$. Since $\overline{\alpha}n$ maj s, by Lemma 5.7, there exists a β^* such that $\overline{\alpha}n * \beta^*$ maj $s @ \beta$. Therefore, by Lemma 5.9 (*iii*), we must have $n < \Gamma(Y^*)(\overline{\alpha}n * \beta^*)$. And by the fact that n is a point of continuity for $\Gamma(Y^*)$ on α we get $\Gamma(Y^*)(\overline{\alpha}n * \beta^*) = n$, a contradiction. $\Box$

In the following we extend the $(\cdot)^+$ operator of Lemma 5.1 for functionals F of type $\rho^* \to \mathbb{N}$ as

$$F^+ :\equiv \lambda s. \max_{s' \prec s} F(s'),$$

where $s' \prec s :\equiv |s'| \leq |s| \ \wedge \ \forall i < |s'| \ (s'_i = s_i)$.

Lemma 5.11 *Let F and G be of type $\rho^* \to \mathbb{N}$. If*

$$\forall s^*, s \ [s^* \text{ maj } s \wedge |s^*| = |s| \to F(s^*) \geq F(s), G(s)]$$

then F^+ maj G^+, G.

Proof. Let s^* maj s be fixed. For all prefixes t^* (of s^*) and t (of s) of the same length, by the assumption of the lemma, we have $F(t^*) \geq F(t), G(t)$. Therefore,

$$\max_{s' \prec s^*} F(s') \geq \max_{s' \prec s} F(s'), \max_{s' \prec s} G(s').$$

Therefore, F^+ maj G^+, G. $\Box$

Theorem 5.12 *If Φ is a functional of type*

$$\mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}},$$

which for any given $Y, H, s \in \mathcal{M}$ (of appropriate types) satisfies MBR*, then $\Phi \in \mathcal{M}$.*

Proof. Our proof is based on the proof of the main result of [7]. The idea is that, if Φ satisfies MBR then the functional

$$\Phi^* :\equiv \lambda Y, H. [\lambda s. \Phi(\hat{Y}, \hat{H}, s)]^+ \text{ maj } \Phi,$$

where

$$\hat{Y}(\alpha) :\equiv \Gamma(Y)(\alpha^+) \text{ and}$$

$$\hat{H}(s, F) :\equiv H(s, \lambda x. F(\{x\}_s)),$$

and $\{x\}_s$ abbreviates $\max^\rho\{\max_{i<|s|}{}^\rho s_i, x\}$. Let Y^* maj Y and H^* maj H be fixed. The fact that Φ^* maj Φ follows from,

$$[\lambda s. \Phi(\hat{Y^*}, \hat{H^*}, s)]^+ \text{ maj } [\lambda s. \Phi(\hat{Y}, \hat{H}, s)]^+, \lambda s. \Phi(Y, H, s),$$

which follows, by Lemma 5.11, from $\forall s^* \ P(s^*)$ where (For the rest of the proof s^* maj s is a shorthand for s^* maj $s \ \wedge \ |s^*| = |s|$, i.e. majorizability is only considered for sequences of equal length.)

$$P(s^*) :\equiv \forall s \ [s^* \text{ maj } s \to \Phi(\hat{Y^*}, \hat{H^*}, s^*) \geq \Phi(\hat{Y^*}, \hat{H^*}, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)].$$

We prove $\forall s^* \ P(s^*)$ by bar induction:

i) $\forall \alpha \exists n\ P(\overline{\alpha}n)$. Let α be fixed and $n :\equiv \hat{Y^*}(\alpha) = \Gamma(Y^*)(\alpha^+)$. If $\overline{\alpha}n$ does not majorize any sequence s we are done. Let s be such that $\overline{\alpha}n$ maj s. Note that $\overline{\alpha^+}n = \overline{(\overline{\alpha}n @ \beta)^+}n$, for all β. Therefore, by Lemma 5.9 (*ii*) and our assumption that Φ satisfies MBR we get $\Phi(\hat{Y^*}, \hat{H}^*, \overline{\alpha}n) = n$. Since $\overline{\alpha^+}n$ maj $\overline{(s @ \beta)^+}n$ (for all β), by Lemma 5.10, we have $n \geq \Phi(\hat{Y^*}, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)$.

ii) $\forall s^* (\forall x\ P(s^* * x) \to P(s^*))$. Let s^* be fixed. Assume that $\forall x\ P(s^* * x)$, i.e.

$$\forall x, s\ [s^* * x \text{ maj } s \to \Phi(\hat{Y^*}, \hat{H}^*, s^* * x) \geq \Phi(\hat{Y^*}, \hat{H}^*, s), \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s)].$$

Note that if s^* does not majorize any sequence we are again done. Assume s is such that s^* maj s. If x^* maj x then (by $\forall x\ P(s^* * x)$),

$$\underbrace{\Phi(\hat{Y^*}, \hat{H}^*, s^* * \{x^*\}_{s^*})}_{\equiv:\ \Phi_1(\{x^*\}_{s^*})} \geq \underbrace{\Phi(\hat{Y^*}, \hat{H}^*, s * \{x\}_s)}_{\equiv:\ \Phi_2(\{x\}_s)}, \underbrace{\Phi(\hat{Y}, \hat{H}, s * \{x\}_s)}_{\equiv:\ \Phi_3(\{x\}_s)}, \underbrace{\Phi(Y, H, s * x)}_{\equiv:\ \Phi_4(x)}.$$

and also $\Phi_1(\{x^*\}_{s^*}) \geq \Phi_1(\{x\}_{s^*})$, which implies

$$\lambda x.\Phi_1(\{x\}_{s^*}) \text{ maj } \lambda x.\Phi_2(\{x\}_s), \lambda x.\Phi_3(\{x\}_s), \lambda x.\Phi_4(x),$$

and by the definition of majorizability

$$\underbrace{H^*(s^*, \lambda x.\Phi_1(\{x\}_{s^*}))}_{\hat{H}^*(s^*, \lambda x.\Phi_1(x))} \text{ maj } \underbrace{H^*(s, \lambda x.\Phi_2(\{x\}_s))}_{\hat{H}^*(s, \lambda x.\Phi_2(x))}, \underbrace{H(s, \lambda x.\Phi_3(\{x\}_s))}_{\hat{H}(s, \lambda x.\Phi_3(x))}, H(s, \lambda x.\Phi_4(x)),$$

which implies

$$\begin{array}{lll} (s^* @ \hat{H}^*(s^*, \lambda x.\Phi_1(x)))^+ & \text{maj} & (s @ \hat{H}^*(s, \lambda x.\Phi_2(x)))^+, \\ & & (s @ \hat{H}(s, \lambda x.\Phi_3(x)))^+, \\ & & s @ H(s, \lambda x.\Phi_4(x)). \end{array}$$

And finally, by Lemma 5.9 (*i*) and (*iii*),

$$\begin{array}{lll} (\Phi(\hat{Y^*}, \hat{H}^*, s^*) =) & & \\ \hat{Y^*}(s^* @ \hat{H}^*(s^*, \lambda x.\Phi_1(x))) & \geq & \hat{Y^*}(s @ \hat{H}^*(s, \lambda x.\Phi_2(x))), \ \ (= \Phi(\hat{Y^*}, \hat{H}^*, s)) \\ & & \hat{Y}(s @ \hat{H}(s, \lambda x.\Phi_3(x))), \ \ (= \Phi(\hat{Y}, \hat{H}, s)) \\ & & Y(s @ H(s, \lambda x.\Phi_4(x))). \ \ (= \Phi(Y, H, s)) \end{array}$$

□

Corollary 5.13 *There exists a $\Phi \in \mathcal{M}$ (not unique) satisfying* MBR*.*

Proof. In section 5.1 we have constructed a

$$\Phi \in \mathcal{M}_{\rho^\omega \to o} \times \mathcal{M}_{\rho^* \to (\rho \to o) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_o$$

satisfying MBR. By Theorem 5.12, $\Phi \in \mathcal{M}$. The fact that Φ is not unique follows by taking, e.g.,

$$F(\alpha) = \begin{cases} 1 & \text{if } \alpha = \lambda x.1 \\ 0 & \text{otherwise,} \end{cases}$$

which means that the sets Set_s^σ (cf. section 5.1) will not be singletons. □

Corollary 5.14 KBR *is not primitive recursively definable in* MBR*.*

6 Defining SBR primitive recursively in MBR

Assuming we have a term t satisfying MBR we build a term t' (primitive recursively in t) which satisfies SBR.

Definition 6.1 $\tilde{\mu}(Y, \alpha^{\rho^\omega}, k) :\equiv \min n \geq k\, [Y(\overline{\alpha, n}) < n]$.

Kohlenbach [16] has shown that $\tilde{\mu}$ is primitive recursively definable in SBR.

Theorem 6.2 *$\tilde{\mu}$ is primitive recursively definable in* MBR.

Proof. Let n be the value of $\tilde{\mu}(Y, \alpha, k)$. The case when $n = k$ is simple and will be treated at the end of the proof. We will assume that $n > k$. In this case we note that, by the minimality condition, $Y(\overline{\alpha, n-1}) \geq n-1$. Hence, $Y(\overline{\alpha, n-1})+1$ can be used (for bounded search) as an upper bound for the value of n. Using MBR, in order for Y to return the required upper bound we have to give as input the sequence $\overline{\alpha, n-1}$. We show how this sequence can be computed by an appropriate H (in the definition of MBR). By MBR we can define a Φ_α satisfying $\Phi_\alpha(s) = Y(s \,@\, (\overline{\alpha, m-1}))$, where,

$$(*)\quad m \stackrel{o}{=} \begin{cases} |s|+1 & \text{if } Y(\overline{\alpha, |s|+1}) < |s|+1 \\ \tilde{\mu}^b(Y, \alpha, k, \Phi_\alpha(s * \alpha(|s|)) + 1) & \text{otherwise,} \end{cases}$$

and $\tilde{\mu}^b$ is the bounded version of $\tilde{\mu}$ (which is primitive recursive). We then define,

$$\tilde{\mu}(Y, \alpha, k) :\equiv \begin{cases} k & \text{if } Y(\overline{\alpha, k}) < k \\ \tilde{\mu}^b(Y, \alpha, k, \Phi_\alpha(\overline{\alpha}k) + 1) & \text{otherwise.} \end{cases}$$

We show that this is a good definition of $\tilde{\mu}$ by showing that $\Phi_\alpha(\overline{\alpha}k)+1$ is a good upper bound on the value of $\tilde{\mu}(Y, \alpha, k)$ (assume this value is $n > k$). In fact, we show by induction on j that, for $k \leq j < n$, n is bounded by $\Phi_\alpha(\overline{\alpha}j)+1$.

i) $j = n-1$. We see that the first case of $(*)$ will be satisfied, m is equal n and $\Phi_\alpha(\overline{\alpha}j)+1 = Y(\overline{\alpha}j \,@\, (\overline{\alpha, m-1}))+1 = Y(\overline{\alpha, n-1})+1 \geq n$.

ii) $j < n-1$. By induction hypothesis $\Phi_\alpha(\overline{\alpha}j * \alpha(j))+1$ is a bound for n. Therefore, m (see second case of $(*)$) has value n, and as above we get $\Phi_\alpha(\overline{\alpha}j)+1 \geq n$. $\Box$

Theorem 6.3 $\mathsf{SBR}_{\rho,o}$ *is primitive recursively definable in* MBR_ρ.

Proof. We show how to define (primitive recursively in MBR) a Ψ satisfying the equation ($\mathsf{SBR}_{\rho,o}$),

$$(i)\quad \Psi(Y, G, H, s) \stackrel{o}{=} \begin{cases} G(s) & \text{if } Y(s * 0) < |s| \\ H(s, \lambda x.\Psi(Y, G, H, s * x)) & \text{otherwise.} \end{cases}$$

Let Φ be a functional satisfying MBR. In the following π_0 and π_1 will denote the projection functional, i.e. $\pi_i(\langle x_0^\rho, x_1^\rho \rangle) = x_i$, $i \in \{0, 1\}$. If $s^{\langle \rho, \rho \rangle^*} = \langle s_0, \ldots, s_n \rangle$, $\pi_i(s)$ also denotes $\langle \pi_i(s_0), \ldots, \pi_i(s_n) \rangle$. In the same way we define $\pi_i(\alpha^{\langle \rho, \rho \rangle^\omega})$. Note that s, for the rest of the proof, has type $\langle \rho, \rho \rangle^\omega$. We first define two tilde operations,

$$(ii)\quad \tilde{H}(s, F) :\equiv \lambda n. \langle 1, H(\pi_1(s), \lambda x.F(\langle 0, x \rangle)) \rangle$$

and [6]

[6]Since, by our construction, the first element of the pair will either be 0^ρ or 1^ρ, the test $\pi_0(s_i) = 0$ in the definition (iii) is primitive recursive.

$$(iii)\ \ \tilde{Y}_{G,k}(\alpha) :\equiv \begin{cases} G(\pi_1(s)) & \text{if } \bigwedge_{i=0}^{n} (\pi_0(s_i) = 0) \\ \pi_1(s_n) & \text{otherwise,} \end{cases}$$

where (in the definition of $\tilde{Y}_{G,k}$) $s = \langle s_0, \ldots, s_n \rangle = \overline{\alpha}\ \tilde{\mu}(Y, \pi_1(\alpha), k)$. Note that the first operation is primitive recursive in H, s and F; and the second is primitive recursive in Y, G, k, α and MBR (since it uses $\tilde{\mu}$). Moreover, $(*)$ if $Y(s * 0) \geq |s|$ then $\tilde{Y}_{G,|s|} = \tilde{Y}_{G,|s|+1}$.

We abbreviate $\langle\langle 0, s_0\rangle, \ldots, \langle 0, s_{|s|-1}\rangle\rangle$ by $\langle 0, s\rangle$. Define

$$(iv)\ \ \Psi(Y, G, H, s) :\equiv \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle).$$

We show that Ψ satisfies equation (i), i.e.

$$(v)\ \ \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) = \begin{cases} G(s) & \text{if } Y(s * 0) < |s| \\ H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|+1}, \tilde{H}, \langle 0, s * x\rangle)) & \text{otherwise.} \end{cases}$$

We first note that, by the definition of MBR (and (ii)),

$$(vi)\ \ \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) = \tilde{Y}_{G,|s|}(\langle 0, s\rangle \ @\ \lambda n.\langle 1, H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle))\rangle).$$

We will show that (v) holds. Assume $Y(s * 0) < |s|$, we have,

$$\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) \overset{(vi)}{=} \tilde{Y}_{G,|s|}(\langle 0, s\rangle \ @\ \ldots) \overset{(iii)}{=} G(s).$$

On the other hand, if $Y(s * 0) \geq |s|$ then,

$$\begin{aligned} \Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s\rangle) &\overset{(vi)}{=} \tilde{Y}_{G,|s|}(\langle 0, s\rangle \ @\ \lambda n.\langle 1, H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle))\rangle) \\ &\overset{(iii)}{=} H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|}, \tilde{H}, \langle 0, s * x\rangle)) \\ &\overset{(*)}{=} H(s, \lambda x.\Phi(\tilde{Y}_{G,|s|+1}, \tilde{H}, \langle 0, s * x\rangle)), \end{aligned}$$

and the proof is concluded. □

Theorem 6.4 $\mathsf{SBR}_{\rho,\tau}$ *is primitive recursively definable in* $\mathsf{SBR}_{\rho',o}$*, where if* $\tau = \tau_1 \to \ldots \to \tau_n \to o$ *then* $\rho' = \rho \times \tau_1 \times \ldots \times \tau_n$.

Proof. Let $\tau = \tau_1 \to \ldots \to \tau_n \to o$. We will show that $\mathsf{SBR}_{\rho,\tau}$ can be defined from $\mathsf{SBR}_{\rho\times\tau_1\times\ldots\times\tau_n,o}$. Let G, H and Y be given, we have to define a functional Φ such that,

$$(i)\ \ \Phi(Y, G, H, s) \overset{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \ @\ 0^{\rho}) \overset{\mathbb{N}}{<} |s| \\ H(s, \lambda x^{\rho}.\Phi(s * x)) & \text{otherwise.} \end{cases}$$

From Y, G and H we define,

(ii) $\tilde{Y}(\alpha) :\equiv Y(\pi_0^{n+1}(\alpha))$;

(iii) $\tilde{G}(t) :\equiv G(\pi_0^{n+1}(t))(\mathbf{y})$;

(iv) $\tilde{H}(t, F) :\equiv H(\pi_0^{n+1}(t), \lambda x^{\rho}, z_1^{\tau_1}, \ldots, z_n^{\tau_n}.F(\langle x, z_1, \ldots, z_n\rangle))(\mathbf{y})$;

where $\mathbf{y}$ denotes $\pi_1^{n+1}(t_{|t|-1}), \ldots, \pi_n^{n+1}(t_{|t|-1})$ and the types are,

$$\begin{array}{rcl} \alpha & : & (\rho \times \tau_1 \times \ldots \times \tau_n)^\omega \\ \mathbf{y} & : & \tau_1 \times \ldots \times \tau_n \\ t & : & (\rho \times \tau_1 \times \ldots \times \tau_n)^* \\ F & : & (\rho \times \tau_1 \times \ldots \times \tau_n) \to o. \end{array}$$

We define (using $\mathsf{SBR}_{\rho \times \tau_1 \times \ldots \times \tau_n, o}$),

$$(v)\ \ \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, t) \stackrel{o}{=} \begin{cases} \tilde{G}(t) & \text{if } \tilde{Y}(t \mathbin{@} 0) \stackrel{\mathbb{N}}{<} |t| \\ \tilde{H}(t, \lambda x^{\rho \times \tau_1 \times \ldots \times \tau_n}.\Psi(t * x)) & \text{otherwise.} \end{cases}$$

Finally we set, ($\langle s, \mathbf{y} \rangle$ abbreviates $\langle \langle s_0, \mathbf{y} \rangle, \ldots, \langle s_{|s|-1}, \mathbf{y} \rangle \rangle$)

$$(vi)\ \ \Phi(Y, G, H, s) :\stackrel{\tau}{=} \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle).$$

We show that equation (i) is satisfied by Φ. One easily verifies that

$$(vii)\ \ \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle) = \Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle \langle s_0, \mathbf{z} \rangle, \ldots, \langle s_{|s|-2}, \mathbf{z} \rangle, \langle s_{|s|-1}, \mathbf{y} \rangle \rangle),$$

for arbitrary $\mathbf{z}$. Let Y, G, H and s be fixed and t abbreviate $\langle s, \mathbf{y} \rangle$. By (ii), $Y(s \mathbin{@} 0) < |s|$ iff $\tilde{Y}(t \mathbin{@} 0) < |t|$. Therefore, if $Y(s \mathbin{@} 0) < |s|$ then

$$\begin{array}{rcl} \Phi(Y, G, H, s) & \stackrel{(vi)}{=} & \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle) \\ & \stackrel{(v)}{=} & \lambda \mathbf{y}.\tilde{G}(\langle s, \mathbf{y} \rangle) \stackrel{(iii)}{=} \lambda \mathbf{y}.G(s)(\mathbf{y}) = G(s). \end{array}$$

On the other hand, if $Y(s \mathbin{@} 0) \geq |s|$ then

$$\begin{array}{rcl} \Phi(Y, G, H, s) & \stackrel{(vi)}{=} & \lambda \mathbf{y}.\Psi(\tilde{Y}, \tilde{G}, \tilde{H}, \langle s, \mathbf{y} \rangle) \stackrel{(v)}{=} \lambda \mathbf{y}.\tilde{H}(t, \lambda x.\Psi(t * x)) \\ & \stackrel{(iv)}{=} & \lambda \mathbf{y}.H(s, \lambda x, \mathbf{z}.\Psi(t * \langle x, \mathbf{z} \rangle))(\mathbf{y}) \\ & \stackrel{(vii)}{=} & \lambda \mathbf{y}.H(s, \lambda x, \mathbf{z}.\Psi(\langle s, \mathbf{z} \rangle))(\mathbf{y}) \\ & \stackrel{(vi)}{=} & \lambda \mathbf{y}.H(s, \lambda x.\Phi(s * x))(\mathbf{y}) = H(s, \lambda x.\Phi(s * x)) \end{array}$$

□

Corollary 6.5 SBR *is primitive recursively definable in* MBR.

7 S1-S9 Computability and MBR

Definition 7.1 (Axioms S1-S9) *In any applicative type structure $\mathcal{S}$ (containing $\mathbb{N}$) we define a set of relations Γ (parametrized by their arity and type of arguments) on $\mathcal{S}$ inductively as follows,* [7]

S1 $\{e\}^{\mathcal{S}}(m, \overrightarrow{y}) = m + 1$, *where* $e = \langle 1, \sigma \rangle$.

[7] We abbreviate $y_1, \ldots, y_n$ (of arbitrary type) by $\overrightarrow{y}$. The variables $e_1, e_2, m, n, i, k, k_1, k_2$ range over natural numbers, σ ranges over codes for finite types and $f, x, \overrightarrow{y}$ over functionals of appropriate types. We write $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$ instead of $\mathcal{S} \models \Gamma(e, \overrightarrow{y}, k)$.

S2 $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$, *where* $e = \langle 2, \sigma, k\rangle$.

S3 $\{e\}^{\mathcal{S}}(m, \overrightarrow{y}) = m$, *where* $e = \langle 3, \sigma\rangle$.

S4 *If* $\{e_1\}^{\mathcal{S}}(\overrightarrow{y}) = k_1$ *and* $\{e_2\}^{\mathcal{S}}(k_1, \overrightarrow{y}) = k_2$ *then* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k_2$, *where* $e = \langle 4, e_1, e_2, \sigma\rangle$.

S5 *Can be omitted in the presence of* S9,

S6 *If* $\{e_1\}^{\mathcal{S}}(\tau(\overrightarrow{y})) = k$ *then* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$, *where* $e = \langle 6, e_1, \tau, \sigma\rangle$.

S7 $\{e\}^{\mathcal{S}}(f, x, \overrightarrow{y}) = f(x)$, *where* $e = \langle 7, \sigma\rangle$.

S8 *If* $\{e_1\}^{\mathcal{S}}(x, \overrightarrow{y}) = f(x)$, *for all* x, *then* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = y_1(f)$, *where* $e = \langle 8, e_1, \sigma\rangle$.

S9 *If* $\{e_1\}^{\mathcal{S}}(y_1, \ldots, y_i) = k$ *then* $\{e\}^{\mathcal{S}}(e_1, \overrightarrow{y}) = k$, *where* $i \leq n$ *and* $e = \langle 9, i, \sigma\rangle$.

One can prove by induction on S1-S9 *that for each* e *and* $\overrightarrow{y}$ *there exists at most one* k *such that* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$. *Therefore, each index* e *gives rise to a partial functional (denoted by* $\{e\}^{\mathcal{S}}$*) which on input* $\overrightarrow{y}$ *takes value* k *if* $\{e\}^{\mathcal{S}}(\overrightarrow{y}) = k$ *and is undefined otherwise. It is important to note that the functional* $\{e\}^{\mathcal{S}}$ *yielded by an index* e *need not belong to* $\mathcal{S}$. *The set of all indices* e *such that* $\{e\}^{\mathcal{S}} \in \mathcal{S}$ *is denoted by* $\mathcal{R}ec^{\mathcal{S}}$. *If* $\{e\}^{\mathcal{S}}$ *is a functional of the form* $\lambda\Psi, \overrightarrow{y}.\{e\}^{\mathcal{S}}(\Psi, \overrightarrow{y})$ *then* $\{e\}^{\mathcal{S}}_{\Psi}$ *denotes the functional* $\lambda\overrightarrow{y}.\{e\}(\Psi, \overrightarrow{y})$.

Definition 7.2 *A formula* P *in the language of* $\mathbf{HA}^{\omega}$ *having a unique free variable is called an* specification of a functional *or just* functional. *(e.g.* SBR *having variables* Y, G, H *and* s *universally quantified is an specification for Spector's bar recursor.)*

Definition 7.3 (S1-S9 computability) *Let* P, Q *be specifications and* $\mathcal{S}$ *any applicative type structure (containing* $\mathbb{N}$*). Then,*

P *is* S1-S9 computable in $\mathcal{S}$ *if* $\mathcal{S} \models \exists e \in \mathcal{R}ec^{\mathcal{S}}.\mathsf{P}(\{e\}^{\mathcal{S}})$.

P *is* S1-S9 + Q computable in $\mathcal{S}$ *if* $\mathcal{S} \models \exists\Psi\ \big(\mathsf{Q}(\Psi) \wedge \exists e \in \mathcal{R}ec^{\mathcal{S}}.\mathsf{P}(\{e\}^{\mathcal{S}}_{\Psi})\big)$.

Lemma 7.4 KBR *and* SBR *are* S1-S9 *computable in* $\mathcal{C}$.

Proof. One shows $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{KBR}(\{e\}^{\mathcal{C}})$ and $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{SBR}(\{e\}^{\mathcal{C}})$ using the recursion theorem. □

The total elements of $\mathcal{C}$ can be viewed as equivalence classes of elements of $\hat{\mathcal{C}}$. We denote these equivalence classes by $[F]$, i.e. if $F \in \hat{\mathcal{C}}$ is total then $[F] \in \mathcal{C}$. We have a *transfer principle* which says that if $\{e\}^{\mathcal{C}}([F]) = k$ then $\{e\}^{\hat{\mathcal{C}}}(F) = k$. Moreover, (+) if $\{e\}^{\hat{\mathcal{C}}}(F) = k$ then there exists a compact element $G \in \hat{\mathcal{C}}$ such that $G \sqsubseteq F$ and $\{e\}^{\hat{\mathcal{C}}}(G) = k$.

Lemma 7.5 *If*

(*i*) *e is a* S1-S9 *code of type 3,*

(*ii*) $\overrightarrow{x}, \overrightarrow{y} \in \hat{\mathcal{C}}$ *(of type 2) coincide in all total recursive arguments,* (*iii*) $\overrightarrow{x}$ *total* S1-S9 *computable in* $\hat{\mathcal{C}}$,

(*iv*) $\{e\}^{\mathcal{C}}([\overrightarrow{x}]) = k$,

then $\{e\}^{\hat{\mathcal{C}}}(\overrightarrow{y}) = k$.

Proof. By induction on S1-S9 codes, the critical point being S8. Assume e is of the form $\langle 8, e_1, \sigma\rangle$ and that $(i) - (iv)$ hold. From (iv), by the definition of S1-S9, there must exist a function $f \in \mathcal{C}$ such that

(*v*) $f(n) = \{e_1\}^{\mathcal{C}}(n, [\overrightarrow{x}])$, for all $n \in \mathbb{N}$, and

(*vi*) $[x_1](f) = k$,

By (iii) and (v) we get that f is recursive. Let n be fixed and assume that $\{e_1\}^{\mathcal{C}}(n, [\overrightarrow{x}]) = l$. By induction hypothesis we have that

$$\{e_1\}^{\hat{\mathcal{C}}}(n, \overrightarrow{y}) = l,$$

i.e. $\lambda n^{\mathbb{N}}.\{e_1\}^{\hat{\mathcal{C}}}(n, \overrightarrow{y})$ $(= [\lambda p^{\mathbb{N}_\perp}.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})])$ is identical to f. By (vi),

$$[x_1]([\lambda p^{\mathbb{N}_\perp}.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})]) = k.$$

Hence,

$$x_1(\lambda p.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})) = k.$$

Note that $\lambda p.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})$ is total and recursive. Therefore, by assumption (ii),

$$y_1(\lambda p.\{e_1\}^{\hat{\mathcal{C}}}(p, \overrightarrow{y})) = k,$$

and by the definition of S1-S9, $\{e\}^{\hat{\mathcal{C}}}(\overrightarrow{y}) = k$. □

Theorem 7.6 ([10]) FAN *is not* S1-S9 *computable in* $\mathcal{C}$.

Proof. Assume $e \in \mathcal{R}ec^{\mathcal{C}}$ is such that $\mathcal{C} \models \mathsf{FAN}(\{e\}^{\mathcal{C}})$. Let O be a total (S1-S9 computable) element of $\hat{\mathcal{C}}$ which is constant zero. Assume $\{e\}^{\mathcal{C}}([O]) = k$. Let F be another type two functional (in $\hat{\mathcal{C}}$) such that $F(f) = 0$ whenever f is total and recursive, but which is $\perp$ for other f. By Lemma 7.5 $\{e\}^{\hat{\mathcal{C}}}(F) = k$. By (+) there must be a compact $G \sqsubseteq F$ (still in $\hat{\mathcal{C}}$) such that $\{e\}^{\hat{\mathcal{C}}}(G) = k$, G defined on a closed-open set that does not cover all of $\hat{\mathcal{C}}$. We can then extend G to a total G' that is not constant and that k is not a modulus of uniform continuity for G'. Assume $\{e\}^{\mathcal{C}}([G']) = l$. By the *transfer principle* $\{e\}^{\hat{\mathcal{C}}}(G') = l$ and l must equal k, i.e. $\{e\}^{\mathcal{C}}([G']) = k$, a contradiction. □

Lemma 7.7 FAN *is* S1-S9 + MBR *computable in* $\mathcal{C}$.

Proof. By Theorem 2.3 there exists a $\Psi \in \mathcal{C}$ such that $\mathcal{C} \models \mathsf{MBR}(\Psi)$. In Theorem 4.4 we have shown that $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{FAN}(\{e\}^{\mathcal{C}}_{\Psi})$. □

Corollary 7.8 MBR *is not* S1-S9 *computable in* $\mathcal{C}$.

Proof. Assume $\mathcal{C} \models \exists e \in \mathcal{R}ec^{\mathcal{C}}.\mathsf{MBR}(\{e\}^{\mathcal{C}})$. By Lemma 7.7 we have that FAN is S1-S9 computable in $\mathcal{C}$, contradicting Theorem 7.6. □

Corollary 7.9 MBR *is not primitive recursively definable in* KBR *nor* SBR.

Proof. Follows from the corollary above, Lemma 7.4 and the fact that the set of functionals S1-S9 computable in $\mathcal{C}$ is closed under primitive recursion. □

8 Conclusion

In this paper we discussed modified bar recursion a variant of Spector's bar recursion that seems to be of some significance in proof theory and the theory and higher type recursion theory. Our main result was an abstract modified realizability interpretation (where realizability for falsity is uninterpreted) of the axioms of countable and dependent choice that can be used to extract programs from non-constructive proofs using this axiom. A similar result is in [2], but we claim that our solution is more accessible and, in a sense, more 'civilised'. It can be noted here that the weak form of modified bar recursion used for the realization of dependent choice can be implemented quite efficiently by equipping the functional with an internal memory that records the value of $H(s, \lambda x.\Phi(s * x))$ (which is of type o) and thus avoids its repeated computation. Such an optimisation does not seem to be possible for the (allegedly more efficient) solution given in [2]. In order to make the realizability interpretation of dependent choice useful for program synthesis it seems necessary to combine it with optimisations of the A-translation as development e.g. in [5] and [6]. To find out whether this is possible will be a subject of further research.

Another important result was a definition of the fan functional using modified bar recursion and a version of bar recursion due to Kohlenbach, improving [3] and [19] where a PCF definition of the fan functional was given. In [23] this definition of the fan functional has been applied to give a purely functional algorithm for exact integration of real functions.

Finally, we have also established the relation between modified bar recursion, Spector's and Kohlenbach's bar recursions. It turns out that MBR and KBR are primitive recursively incomparable (none is primitive recursively definable in the other). Spector's bar recursion, however, is primitive recursively definable in MBR but not the other way around. All these results hold in $\mathbf{HA}^\omega$. A by-product of this investigation is that modified bar recursion exists in $\mathcal{M}$, the model of strongly majorizable functionals. We also proved a *weak continuity property* for the model $\mathcal{M}$.

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