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Inclusion–Exclusion(3) Implies Inclusion–Exclusion(n)

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1 Introduction

The *inclusion–exclusion formula* is one of the most basic techniques in combinatorics [1]. In its most familiar form, it expresses the cardinality of the union of an arbitrary family of sets in terms of the cardinalities of the family of sets given by all possible intersections of the sets in the original family. Precisely, if $A_1, \ldots, A_n$ are arbitrary finite sets, with n a positive integer, then

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|, \quad (1)$$

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or, more compactly,

$$|\bigcup_{i\in[n]} A_i| = \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|-1}|\bigcap_{i\in I} A_i|. \qquad (2)$$

One can consider a more general version of this formula in the setting of a *ranked lattice*, see [1], for example. This is a pair (L, r), where $\mathsf{L} := (L, \vee, \wedge, \bot)$ is a lattice [1] and the *rank function* $r : L \to \mathsf{N}$ satisfies the property that $r(y) = r(x) + 1$ whenever y *covers* x in L, that is, $x \leq y$ and if $x \leq z \leq y$ then $z = x$ or $z = y$. One can ask: For which ranked lattices does the following generalised inclusion-exclusion formula hold?

$$r(\bigvee_{i\in[n]} A_i) = \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|-1} r(\bigwedge_{i\in I} A_i). \qquad (3)$$

The case $n = 2$ of this formula,

$$r(A) + r(B) = r(A \vee B) + r(A \wedge B), \qquad (4)$$

is obviously a necessay condition for the general formula to hold. When (4) holds, the rank function r is said to be *modular*, and we shall refer to (4) as the *modular law.*

In § 2, we give some natural and important examples of ranked lattices and see if the generalisation holds in these cases. In § 3, we give a complete answer to the question.

2 Examples

In this section, we give some natural examples of ranked lattices that are of importance in combinatorics and examine if the generalised inclusion-exclusion formula (3) holds in them.

Example 1 (Sets) The first example is our starting point: the lattice of finite subsets of a universal set U. The lattice is $(2^U, \cup, \cap, \emptyset)$, and the rank function is just the cardinality. The fact that the inclusion–exclusion formula holds is one of the first theorems in any textbook on combinatorics. ∎

[1]We are being a bit non-standard here since a lattice usually also comes with a top element ⊤. However, in some of the examples below, it is rather artificial to impose a top element. Moreover, for our results, ⊤ is completely superfluous (but ⊥ is not!).

Example 2 (Divisors) Let N be a positive integer and let $\mathcal{D}_N$ denote the set of all positive divisors of N, ordered by divisibility i.e. for divisors a, b of N, $a \leq b$ iff a divides b. Let $N := p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$ be the (unique) prime factorisation of N. The lattice operations can be described by regarding an element a as a r vector $(a_1, \dots, a_r)$ where a_i is aan integer with $0 \leq a_i \leq \alpha_i$ for each $i \in [r]$. The elements $a \vee b$ and $a \wedge b$ are given by the following equations on the components $i \in [r]$:

$$(a \vee b)_i := \max(a_i, b_i) \quad (a \wedge b)_i := \min(a_i, b_i).$$

The bottom element is 1.

The rank of an element $a := (a_1, \dots, a_r)$ is $\sum_i a_i$. The inclusion–exclusion formula holds in this ranked lattice.

One can consider also, the lattice of *all* positive integers ordered by divisibility. Representing the elements of this lattice as *infinte* vectors of exponents of prime powers, the lattice operations are given exactly as before. ∎

Example 3 (Multisets) This example generalises the previous one. Let $\mathbf{e} := (e_1, e_2, \dots)$ be an (infinite) vector whose components are non–negative integers or ∞. Let $\mathcal{M}_{\mathbf{e}}$ denote the collection of "all finite multisets of integers with multiplicities restricted to $\mathbf{e}$". By this we mean the family of all finite unordered collections of positive integers, where repetitions are allowed but each i can appear at most e_i times. Each multiset in $\mathcal{M}_{\mathbf{e}}$ can be represented by a vector $\sigma := (\sigma_1, \sigma_2, \dots)$ of non–negative integers, where $0 \leq \sigma_i \leq e_i$ for each $i \geq 1$, and also $\sum_i \sigma_i < \infty$. This collection forms a partial order under component–wise ordering. In fact it forms a lattice, where the lattice operations are given by the max and min operations respectively, as in the previous example. The bottom element is $(0, 0, , \dots)$.

The rank of an element σ in this lattice is $\sum_i \sigma_i$. The inclusion–exclusion principle holds in this ranked lattice.∎

Example 4 (Partitions) Let m be a positive integer and let $\mathcal{P}_m$ denote the set of all *partitions* of the set $[m]$ ordered by (inverse) *refinement.* Thus $\Pi_1 \leq \Pi_2$ if the partition Π_1 refines the partition Π_2, that is, each block of the partition Π_2 is the union of some blocks of partition Π_1. The bottom element is the partition into singletons and the top element is the partition in which all elements are in a single block. The join of partitions Π_1 and Π_2, is the partition which groups two elements together in a block if they are in the same block in one of the two partitions and the meet is the partition that groups two elements together in a block if they are in the same block in both partitions.

The rank of a partition Π is $n-k$ where k is the number of blocks in Π. The inclusion–exclusion formula fails in this ranked lattice. To see this, take the lattice $\mathcal{P}_3$ and check the formula for $n=3$ with the partitions $\{23,1\},\{31,2\}$ and $\{12,3\}$. ∎

Example 5 (Vector Spaces) Let V be a vector space over a field k and let $L(V,k)$ denote the family of all finite-dimensional k-subspaces of V ordered by the subspace relation. The meet operation is just set intersection. The join of the subspaces U and W is given by the *sum* $U+W := \{u+w \mid u \in U, w \in W\}$; this is the least subspace that contains both U and W. (Note that the set theoretic union of two subspaces is not necessarily a subspace!) The bottom element is the trivial space $\{0\}$.

The rank of a subspace W is its dimension, dim W. The inclusion–exclusion formula does *not* hold in general in this ranked lattice. To see this, consider the plane R^2 regarded as a vector space over the reals. Let X be the one dimensional space spanned by $(1,0)$, namely the x axis, let Y be the one dimensional space spanned by $(0,1)$, namely the y axis, and let Z be the one dimensional space spanned by $(1/2,1/2)$. Then, $X+Y+Z$ is the whole plane, and has dimension 2. However, each pairwise intersection is trivial, so the right–hand side of the inclusion–exclusion formula gives 3 and not 2. ∎

In each of the examples, the rank function is indeed modular. So the reason for the failure in the last example must lie elsewhere. Indeed, it is a property of the lattice that is responsible, as we shall see in the next section.

3 Generalised Inclusion–Exclusion

In this section, we give a complete characterisation of the ranked lattices for which the general inclusion–exclusion formula (3) holds. The (perhaps surprising) implication (3) → (2) is the reason for the title of this note.

Theorem 6 *Let* (L,r) *be a ranked lattice. The following are equivalent:*

1. *L is a distributive lattice and r is modular.*

2. *The general inclusion–exclusion formula (3) for every positive integer n holds for (L,r).*

3. *The inclusion–exclusion formula (3) for $n=3$ holds for (L,r).*

Proof. (1) → (2) This is essentially the standard proof of the inclusion–exclusion formula (3), using induction on n. For $n=1$ the result is trivial and for $n=2$, the modular law is given to hold by hypothesis. For $n \geq 3$, we have

$$\begin{aligned}
r(\bigvee_{i\in[n]} A_i) &= r(\bigvee_{i\in[n-1]} A_i \vee A_n) \\
&= r(\bigvee_{i\in[n-1]} A_i) + r(A_n) - r((\bigvee_{i\in[n-1]} A_i) \wedge A_n), \quad \text{using the modular law} \\
&= r(\bigvee_{i\in[n-1]} A_i) + r(A_n) - r(\bigvee_{i\in[n-1]} (A_i \wedge A_n)), \quad \text{using distributivity of L} \\
&= \sum_{\emptyset\neq I\subseteq[n-1]} (-1)^{|I|-1} r(\bigwedge_{i\in I} A_i) + r(A_n) + \sum_{\emptyset\neq J\subseteq[n-1]} (-1)^{|J|-1} r(\bigwedge_{j\in J}(A_j \wedge A_n)) \\
&\qquad \text{by induction} \\
&= \sum_{\emptyset\neq I\subseteq[n], n\notin I} (-1)^{|I|-1} r(\bigwedge_{i\in I} A_i) + \sum_{\emptyset\neq I\subseteq[n], n\in I} (-1)^{|I|-1} r(\bigwedge_{i\in I} A_i) \\
&= \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|-1} r(\bigwedge_{i\in I} A_i).
\end{aligned}$$

(2) → (3) is trivial.

(3) → (1) First, applying the inclusion–exclusion formula (3) for $n=3$ with $A_3 := \emptyset$ yields the modular law (4). Now, applying the modular law twice, we get for any $A,B,C \in L$:

$$\begin{aligned}
r(A\vee B\vee C) &= r(A\vee B) + r(C) - r((A\vee B)\wedge C) \\
&= r(A) + R(B) + r(C) - r(A\wedge B) - r((A\vee B)\wedge C) \quad (5)
\end{aligned}$$

Also, applying the inclusion–exclusion formula for $n=3$ directly and then the modular law gives

$$\begin{aligned}
r(A\vee B\vee C) &= r(A) + r(B) + r(C) - r(B\wedge C) - r(C\wedge A) - r(A\wedge B) + r(A\wedge \\
&= r(A) + r(B) + r(C) - r((B\wedge C)\vee(C\wedge A)) - r(A\wedge B).
\end{aligned}$$

Comparing equations (5) and (6), we see that

$$r((A \vee B) \wedge C) = r((A \wedge C) \vee (B \wedge C)). \tag{7}$$

Since $(A \wedge C) \vee (B \wedge C) \leq (A \vee B) \wedge C$, equation (7) implies that in fact $(A \wedge C) \vee (B \wedge C) = (A \vee B) \wedge C$, that is, the lattice is distributive. ∎

A corollary of possible independent interest is:

Corollary 7 *A lattice is distributive if there exists a rank function on it which satisfies the inclusion–exclusion formula (3) for $n = 3$.*

References

[1] R.P. Stanley, *Enumerative Combinatorics Vol I*, Wadsworth and Brooks/Cole, 1986.

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