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PETRI NETS AND BISIMULATIONS

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Abstract

Several categorical relationships (adjunctions) between models for concurrency have been established, allowing the translation of concepts and properties from one model to another. The purpose of the present paper is twofold: firstly to present a central example of such a relationship (a coreflection between asynchronous transition systems and Petri nets), and secondly to illustrate its use by transferring to nets a general concept of bisimulation.

Introduction

Recently, category theory has been used to structure the seemingly confusing world of models for concurrency—see [24] for a survey. The general idea is to formalize that one model is more expressive than another in terms of an "embedding", typically taking the form of a (co)reflection, i.e. an adjunction in which the counit (unit) is an isomorphism. The models are equipped with behaviour preserving morphisms, to be thought of as kinds of simulations. Besides providing an abstract language for expressing relationships between seemingly very different models, category theory also allows the translation of constructions and properties between models via adjunctions. For instance, most process algebra constructs, like parallel and nondeterministic composition, may be understood in terms of universal constructions, like product and coproduct. The preservation properties of adjoints are helpful in showing and explaining why semantics is respected in moving from one model to another.

The purpose of this paper is twofold. First, we present in full detail one example of a central coreflection, embedding asynchronous transition systems, in the sense of Bednarczyk [1] and Shields [20], in Petri nets. Our category of nets is a little more general than that of 1-safe nets. Previously, a similar embedding of elementary transition systems into elementary net systems have been established, [15], based on a regional characterization of the case graphs of such net systems due to Ehrenfeucht and Rozenberg. This was generalized to an embedding of certain step-transition systems into general place transition nets by Mukund, [11]. Our result, obtained independently of [11], falls between these two, showing that for 1-safe nets, the appropriate notion of case graph is that of asynchronous

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transition systems. We present here the proof in full detail, partly because we feel the details provide useful insight (e.g. into the nature of the conditions of nets), partly because the proof is different from previously published proofs in that it first establishes an adjunction between general asynchronous transition systems and our nets, cutting down to the coreflection by imposing a few axioms on the objects of asynchronous transition systems.

The second purpose of this paper is to illustrate the translation of concepts between models, focussing here on the transference of the concept of bisimulation to Petri nets from other models . The notion of bisimulation was defined categorically in [5] in a form directly applicable to a wide range of models equipped with a notion of observations. This general definition takes the form of an existence of span of open maps. In [5] it was shown that in the special case of standard labelled transition systems with sequential observations, the definition agrees with the strong bisimulation of Milner, [10], and in the case of event structures with nonsequential observations in the form of pomsets, the definition yielded an interesting strengthening of the history-preserving bisimulation introduced by Rabinovitch and Trakhtenbrot [18]. Here we show how the coreflection from other models to nets combined with abstract properties of the general definition of bisimulation from [5], provides a notion of bisimulation on nets which automatically inherits a number of important properties.

The main message of this paper is that the categorical view of models for concurrency, like Petri nets, provides guidelines for definitions of concepts like behavioural equivalences, consistent across a range of models. We illustrate how a notion of bisimulation can be read off for nets, and that this comes automatically equipped with a number of essential properties. The categorical approach here contrasts with the more common alternative of searching for a sensible candidate for bisimulation on nets and, having found one of then checking it possesses these essential properties.

1 Models and a Coreflection

In this section we introduce the models of Petri nets and asynchronous transition systems, and present a coreflection between them. A category of transition systems plays a role in both.

1.1 Transition systems

Transition systems are a frequently used model of parallel processes. They consist of a set of states, with an initial state, together with transitions between states which are labelled to specify the kind of events they represent.

Definition: A *transition system* is a structure

$$(S, i, L, tran)$$

where

- S is a set of *states* with *initial state* i,
- L is a set of *labels*,
- $tran \subseteq S \times L \times S$ is the *transition relation*. As usual, a transition (s, a, s') is drawn as $s \xrightarrow{a} s'$.

It is convenient to introduce *idle* transitions, associated with any state. This has to do with our representation of partial functions. We view a partial function from a set L to a set L' as a (total) function $\lambda : L \cup \{*\} \to L \cup \{*\}$ such that $f(*) = *$, where $*$ is a distinguished element standing for "undefined". This representation is reflected in our notation $\lambda : L \rightarrow_* L'$ for a partial function λ from L to L'. It assumes that $*$ does not appear in the sets L and L', and more generally we shall assume that the reserved element $*$ does not appear in any of the sets appearing in our constructions.

Definition: Let $T = (S, i, L, tran)$ be a transition system. An *idle transition* of T typically consists of $(s, *, s)$, where $s \in S$. Define

$$tran_* = tran \cup \{(s, *, s) \mid s \in S\}.$$

Idle transitions help give a simple definition of morphism between transition systems.

Definition: Let

$$T_0 = (S_0, i_0, L_0, tran_0) \text{ and}$$
$$T_1 = (S_1, i_1, L_1, tran_1)$$

be transition systems. A *morphism* $f : T_0 \to T_1$ is a pair $f = (\sigma, \lambda)$ where

- $\sigma : S_0 \to S_1$
- $\lambda : L_0 \rightarrow_* L_1$ are such that $\sigma(i_0) = i_1$ and

$$(s, a, s') \in tran_0 \Rightarrow (\sigma(s), \lambda(a), \sigma(s')) \in tran_{1*}.$$

The intention behind the definition of morphism is that the effect of a transition with label a in T_0 leads to inaction in T_1 precisely when $\lambda(a)$ is undefined. In our definition of morphism, idle transitions represent this inaction, so we avoid the fuss of considering whether or not $\lambda(a)$ is defined. With the introduction of idle transitions, morphisms on transition systems can be described as preserving transitions and the initial state. It is stressed that an idle transition $(s,*,s)$ represents inaction, and is to be distinguished from the action expressed by a transition (s,a,s') for a label a.

Transition systems with morphisms form a category **T** in which the composition of two morphisms $f = (\sigma,\lambda) : T_0 \to T_1$ and $g = (\sigma',\lambda') : T_1 \to T_2$ is $g \circ f = (\sigma' \circ \sigma, \lambda' \circ \lambda) : T_0 \to T_2$ and the identity morphism for a transition system T has the form $(1_S, 1_L)$ where 1_S is the identity function on states and 1_L is the identity function on the labelling set of T.
(Here composition on the left of a pair is that of total functions while that on the right is of partial functions.)

1.2 Petri nets

A Petri net may be seen as a transition system with an explicit representation of (global) states as sets of (local) states (usually called conditions). The specific version adopted here was introduced in [9].

Definition: A *Petri net* consists of $(B, M_0, E, pre, post)$ where

B is a set of *conditions*, with *initial marking* M_0 a nonempty subset of B,
E is a set of *events*, and
$pre : E \to \mathcal{P}ow(B)$ is the *precondition* map such that $pre(e)$ is nonempty for all $e \in E$,
$post : E \to \mathcal{P}ow(B)$ is the *postcondition* map such that $post(e)$ is nonempty for all $e \in E$.

A Petri net comes with an initial marking consisting of a subset of conditions which are imagined to hold initially. Generally, a *marking*, a subset of conditions, formalizes a notion of global state by specifying those conditions which hold. Markings can change as events occur, precisely how being expressed by the transitions

$$M \xrightarrow{e} M'$$

events e determine between markings M, M'. In defining this notion it is convenient to extend events by an "idling event".

Definition: Let $N = (B, M_0, E, pre, post)$ be a Petri net with events E.
Define $E_* = E \cup \{*\}$.
We extend the pre and post condition maps to $*$ by taking

$$pre(*) = \emptyset, \quad post(*) = \emptyset.$$

Notation: Whenever it does not cause confusion we write ${}^\bullet e$ for the preconditions $pre(e)$ and $e^\bullet$ for the postconditions, $post(e)$, of $e \in E_*$. We write ${}^\bullet e^\bullet$ for ${}^\bullet e \cup e^\bullet$.

Definition: Let $N = (B, M_0, E, pre, post)$ be a net.
For $M, M' \subseteq B$ and $e \in E_*$, define

$$M \xrightarrow{e} M' \text{ iff } {}^\bullet e \subseteq M \ \&\ e^\bullet \subseteq M' \ \&\ M \setminus {}^\bullet e = M' \setminus e^\bullet.$$

Say $e_0, e_1 \in E_*$ are *independent* iff ${}^\bullet e_0^\bullet \cap {}^\bullet e_1^\bullet = \emptyset$.
A marking M of N is said to be *reachable* when there is a sequence of events, possibly empty, $e_1, e_2 \ldots e_n$ such that

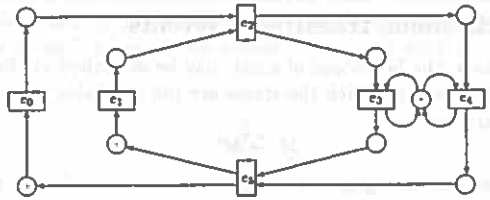
$$M_0 \xrightarrow{e_1} M_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} M_n = M.$$

in N. There is *contact* at a marking M when for some event e, all its preconditions are marked at M and yet e cannot ocur at M:

$${}^\bullet e \subseteq M \ \&\ e^\bullet \cap (M \setminus {}^\bullet e) \neq \emptyset.$$

A net is said to be *safe* when contact never occurs at any reachable marking.

Example: The following is an example of a graphical (following standard notation) representation of a safe net with six events and nine conditions. Notice in particular that events e_0 and e_1 are independent, whereas e_3 and e_4 are not. One of the essential properties of nets is this possibility of specifying independence amongst events in terms of pre- and postconditions.



As morphisms on nets we take:

Definition: Let $N = (B, M_0, E, pre, post)$ and $N' = (B', M_0', E', pre', post')$ be nets. A *morphism* $(\beta, \eta) : N \to N'$ consists of a relation $\beta \subseteq B \times B'$, such that β^{op} is a partial function $B' \to B$, and a partial function $\eta : E \to E'$ such that

$$\beta M_0 = M_0',$$
$$\beta\, {}^\bullet e = {}^\bullet \eta(e) \text{ and}$$
$$\beta e^\bullet = \eta(e)^\bullet.$$

Thus morphisms on nets preserve initial markings and events when defined. A morphism $(\beta, \eta) : N \to N'$ expresses how occurrences of events and conditions in N induce occurrences in N'. Morphisms on nets preserve behaviour:

Proposition 1 *Let $N = (B, M_0, E, pre, post)$, $N' = (B', M'_0, E', pre', post')$ be nets. Suppose $(\beta, \eta) : N \to N'$ is a morphism of net.*

- *If $M \xrightarrow{e} M'$ in N then $\beta M \xrightarrow{\eta(e)} \beta M'$ in N'.*
- *If ${}^\bullet e_1^\bullet \cap {}^\bullet e_2^\bullet = \emptyset$ in N then ${}^\bullet\eta(e_1)^\bullet \cap {}^\bullet \eta(e_2)^\bullet = \emptyset$ in N'.*

Proof: By definition,

$$^\bullet\eta(e) = \beta^\bullet e \text{ and } \eta(e)^\bullet = \beta e^\bullet$$

for e an event of N. Observe too that because β^{op} is a partial function, β in addition preserves intersections and set differences. These observations mean that $\beta M \xrightarrow{\eta(e)} \beta M'$ in N' follows from the assumption that $M \xrightarrow{e} M'$ in N, and that independence is preserved. □

Proposition 2 *Nets and their morphisms form a category in which the composition of two morphisms $(\beta_0, \eta_0) : N_0 \to N_1$ and $(\beta_1, \eta_1) : N_1 \to N_2$ is $(\beta_1 \circ \beta_0, \eta_1 \circ \eta_0) : N_0 \to N_2$ (composition in the left component being that of relations and in the right that of partial functions).*

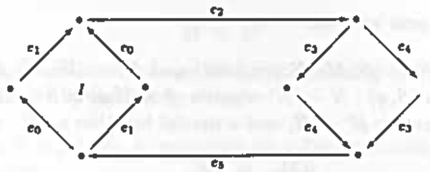
Definition: Let **N** be the category of nets described above.

1.3 Asynchronous transition systems

Following tradition, the behaviour of a net may be described via its *case graph*, i.e. a transition system in which the states are the reachable markings and the transitions are triples

$$M \xrightarrow{e} M'$$

as defined above. The case graph of our previous net example will be as follows:



Notice how the event pairs (e_0, e_1) and (e_3, e_4) give rise to the same kind of diamonds in the underlying transition system. Hence, in order to get a representation of the important distinction between the pairs in terms of independence, we need to add some structure to the notion of case graph, here indicated by the I in the independent diamond. This is exactly the motivation behind *asynchronous transition systems*, as introduced independently by Bednarczyk [1] and Shields [20]. The idea on which they are based is simple enough: extend transition systems by, in addition, specifying which transitions are independent of which. More accurately, transitions are to be thought of as occurrences of events which bear a relation of independence.

Definition: An *asynchronous transition system* consists of $(S, i, E, I, tran)$ where $(S, i, E, tran)$ is a transition system, $I \subseteq E^2$, the *independence relation* is an irreflexive, symmetric relation on the set E of *events* such that

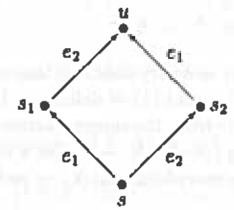
(1) $e \in E \Rightarrow \exists s, s' \in S.\ (s, e, s') \in tran$

(2) $(s, e, s') \in tran\ \&\ (s, e, s'') \in tran \Rightarrow s' = s''$

(3) $e_1 I e_2\ \&\ (s, e_1, s_1) \in tran\ \&\ (s, e_2, s_2) \in tran$
$\Rightarrow \exists u.\ (s_1, e_2, u) \in tran\ \&\ (s_2, e_1, u) \in tran$

(4) $e_1 I e_2\ \&\ (s, e_1, s_1) \in tran\ \&\ (s_1, e_2, u) \in tran$
$\Rightarrow \exists s_2.\ (s, e_2, s_2) \in tran\ \&\ (s_2, e_1, u) \in tran$

Axiom (1) says every event appears as a transition, and axiom (2) that the occurrence of an event at a state leads to a unique state. Axioms (3) and (4) express properties of independence: if two events can occur independently from a common state then they should be able to occur together and in so doing reach a common state (3); if two independent events can occur one immediately after the other then they should be able to occur with their order interchanged (4). Both situations lead to an "independence square" associated with the independence $e_1 I e_2$:



Morphisms between asynchronous transition systems are morphisms between their underlying transition systems which preserve the additional relations of independence.

Definition: Let $T = (S, i, E, I, tran)$ and $T' = (S', i', E', I', tran')$ be asynchronous transition systems. A *morphism* $T \to T'$ is a morphism of transition systems

$$(\sigma, \eta) : (S, i, E, tran) \to (S', i', E', tran')$$

such that

$$e_1 I e_2 \ \&\ \eta(e_1), \eta(e_2) \text{ both defined } \Rightarrow \eta(e_1) I' \eta(e_2).$$

Morphisms of asynchronous transition systems compose as morphisms between their underlying transition systems, and are readily seen to form a category.

Definition: Let **A** be the category of asynchronous transition systems.

1.4 Asynchronous transition systems and nets

1.4.1 An adjunction

There is an adjunction between the categories **A** and **N**. First, we note there is an obvious functor from nets to asynchronous transition systems, that of constructing the case graph of a net.

Definition: Let $N = (B, M_0, E, {}^\bullet(\ ), (\ )^\bullet)$ be a net. Define $na(N) = (S, i, E, I, tran)$ where

$$S = \mathcal{P}ow(B) \text{ with } i = M_0,$$
$$e_1 I e_2 \Leftrightarrow {}^\bullet e_1^\bullet \cap {}^\bullet e_2^\bullet = \emptyset,$$
$$(M, e, M') \in tran \Leftrightarrow M \xrightarrow{e} M' \text{ in } N, \text{ for } M, M' \in \mathcal{P}ow(B).$$

Let $(\beta, \eta) : N \to N'$ be a morphism of nets. Define

$$na(\beta, \eta) = (\sigma, \eta)$$

where $\sigma(M) = \beta M$, for any $M \in \mathcal{P}ow(B)$.

Proposition 3 *na is a functor* $\mathbf{N} \to \mathbf{A}$.

Proof: Letting N be a net, it is easily checked that $na(N)$ is an asynchronous transition system: properties (1) and (2) of definition 1.3 are obvious while properties (3) and (4) follow directly from the interpretation of independence of events e_1, e_2 as ${}^\bullet e_1^\bullet \cap {}^\bullet e_2^\bullet = \emptyset$. Letting $(\beta, \eta) : N \to N'$ be a morphism of nets, proposition 1 yields that $na(\beta, \eta)$ is a morphism $na(N) \to na(N')$. Clearly na preserves composition and identities. □

As a preparation for the definition of a functor from asynchronous transition systems to nets we examine how a condition of a net N can be viewed as a subset of states and transitions of the asynchronous transition system $na(N)$. Intuitively the *extent* $|b|$ of a condition b of a net is to consist of those markings and transitions at which b holds uninterruptedly. In fact, for simplicity, the extent $|b|$ of a condition b is taken to be a subset of $tran_*$, the transitions (M, e, M') and idle transitions $(M, *, M)$ of $na(N)$; the idle transitions $(M, *, M)$ play the role of markings M.

Definition: Let b be a condition of a net N. Let $tran$ be the transition relation of $na(N)$. Define the *extent* of b to be

$$|b| = \{(M, e, M') \in tran_* \mid b \in M \ \&\ b \in M' \ \&\ b \notin {}^\bullet e^\bullet\}.$$

Not all subsets of transitions $tran_*$ of a net N are extents of conditions of N. For example, if $(M, e, M') \notin |b|$ and $(M', *, M') \in |b|$ for a transition $M \xrightarrow{e} M'$ in N this means the transition starts the holding of b. But then $b \in e^\bullet$ so any other transition $P \xrightarrow{e} P'$ must also start the holding of b. Of course, a condition cannot be started or ended by two independent events because, by definition, they can have no pre- or postcondition in common. These considerations motivate the following definition of condition of a general asynchronous transition system. Notice that the definition is a generalization of the notion of regions for transition systems introduced by Ehrenfeucht and Rozenberg [15].

Definition: Let $T = (S, i, E, I, tran)$ be an asynchronous transition system. Its *conditions* are nonempty subsets $b \subseteq tran_*$ such that

(1) $(s, e, s') \in b \Rightarrow (s, *, s) \in b \ \&\ (s', *, s') \in b$

(2) (i) $(s, e, s') \in {}^\bullet b \ \&\ (u, e, u') \in tran \Rightarrow (u, e, u') \in {}^\bullet b$

(ii) $(s, e, s') \in b^\bullet \ \&\ (u, e, u') \in tran \Rightarrow (u, e, u') \in b^\bullet$

where for $(s, e, s') \in tran$ we define

$$(s, e, s') \in {}^\bullet b \Leftrightarrow (s, e, s') \notin b \ \&\ (s', *, s') \in b,$$
$$(s, e, s') \in b^\bullet \Leftrightarrow (s, *, s) \in b \ \&\ (s, e, s') \notin b \quad \text{and}$$
$${}^\bullet b^\bullet =^\bullet b \cup b^\bullet.$$

(3) $(s, e_1, s') \in^\bullet b^\bullet \ \&\ (u, e_2, u') \in^\bullet b^\bullet \Rightarrow \neg e_1 I e_2$.

Let B be the set of conditions of T. For $e \in E_*$, define

$$e^\bullet = \{b \in B \mid \exists s, s'. (s, e, s') \in {}^\bullet b\},$$
$${}^\bullet e = \{b \in B \mid \exists s, s'. (s, e, s') \in b^\bullet\}, \text{ and}$$
$${}^\bullet e^\bullet = {}^\bullet e \cup e^\bullet.$$

(Note that ${}^\bullet *^\bullet = \emptyset$.)
Further, for $s \in S$, define $M(s) = \{b \in B \mid (s, *, s) \in b\}$.

As an exercise, we check that the extent of a condition of a net is indeed a condition of its asynchronous transition system.

Lemma 4 *Let N be a net with a condition b. Its extent $|b|$ is a condition of $na(N)$. Moreover,*

(I) $(M,e,M') \in^{\bullet} |b| \Leftrightarrow b \in e^{\bullet}$

(II) $(M,e,M') \in |b|^{\bullet} \Leftrightarrow b \in^{\bullet} e.$

whenever $M \xrightarrow{e} M'$ *in* N.

Proof: We prove (I) (the proof of (II) is similar):

$$\begin{aligned}(M,e,M') \in^{\bullet} |b| &\Leftrightarrow (M,e,M') \notin |b| \;\&\; (M',*,M') \in |b| \\ &\Leftrightarrow \neg(b \in M \;\&\; b \in M' \;\&\; b \notin^{\bullet} e^{\bullet}) \;\&\; b \in M' \\ &\Leftrightarrow (b \notin M \;\&\; b \in M') \text{ or } (b \in^{\bullet} e^{\bullet} \;\&\; b \in M') \\ &\Leftrightarrow b \in e^{\bullet}, \text{ as } M \xrightarrow{e} M'.\end{aligned}$$

Using (I) and (II), it is easy to check that $|b|$ is a condition of $na(N)$. First we note $|b|$ is nonempty because it contains for instance $(\{b\},*,\{b\})$. We quickly run through the axioms required by definition 1.4.1:

(1) If $(M,e,M') \in |b|$ then $b \in M$ and $b \in M'$ whence $(M,*,M),(M',*,M') \in |b|$.

(2) (i) If $(M,e,M') \in^{\bullet} |b|$ then $b \in e^{\bullet}$, by (I) " $\Rightarrow$". Hence, if $P \xrightarrow{e} P'$ by (I) "$\Leftarrow$" we obtain $(P,e,P') \in^{\bullet} |b|$. The proof of (2)(ii) is similar.

(3) (i) If $(M,e_1,M'),(P,e_2,P') \in^{\bullet} |b|$ then $b \in e_1^{\bullet}$ and $b \in e_2^{\bullet}$, by (I) applied twice. Hence $\neg e_1 I e_2$. □

Definition: Let $(\sigma,\eta) : T \to T'$ be a morphism between asynchronous transition systems $T = (S,i,E,I,tran)$ and $T' = (S',i',E',I',tran')$. For $b \subseteq tran'_*$, define

$$(\sigma,\eta)^{-1}b = \{(s,e,s') \in tran_* \mid (\sigma(s),\eta(e),\sigma(s')) \in b\}$$

Lemma 5 *Let $(\sigma,\eta) : T \to T'$ be a morphism between asynchronous transition systems. Let b be a condition of T'. Then $(\sigma,\eta)^{-1}b$ is a condition of T provided it is nonempty. Furthermore,*

$$\begin{aligned}&(1)\quad (\sigma,\eta)^{-1}b \in^{\bullet} e \;\Leftrightarrow\; b \in^{\bullet} \eta(e) \\ &(2)\quad (\sigma,\eta)^{-1}b \in e^{\bullet} \;\Leftrightarrow\; b \in \eta(e)^{\bullet}\end{aligned}$$

for any event e of T.

Proof: We show (1), assuming $b \subseteq tran'_*$ and an event e of T. Observe

$$\begin{aligned}(\sigma,\eta)^{-1}b \in^{\bullet} e &\Leftrightarrow (s,e,s') \in (\sigma,\eta)^{-1}b^{\bullet}, \text{ for some states } s,s' \\ &\Leftrightarrow (s,*,s) \in (\sigma,\eta)^{-1}b \;\&\; (s,e,s') \notin (\sigma,\eta)^{-1}b \\ &\Leftrightarrow (\sigma(s),*,\sigma(s)) \in b \;\&\; (\sigma(s),\eta(e),\sigma(s')) \notin b \\ &\Leftrightarrow (\sigma(s),\eta(e),\sigma(s')) \in b^{\bullet} \\ &\Leftrightarrow b \in^{\bullet} \eta(e)\end{aligned}$$

The proof of (2) is similar. That $(\sigma,\eta)^{-1}b$ is a condition of T, if nonempty, follows straightforwardly from the assumption that b is a condition. □

Definition: Let $T = (S,i,E,I,tran)$ be an asynchronous transition system. Define $an(T) = (B,M_0,E,pre,post)$ by taking B to be the set of conditions of T, $M_0 = M(i)$, with pre and post condition maps given by the corresponding operations in T, *i.e.* $pre(e) = {}^{\bullet}e$ and $post(e) = e^{\bullet}$ in T. Let $(\sigma,\eta) : T \to T'$ be a morphism of asynchronous transition systems. Define $an(\sigma,\eta) = (\beta,\eta)$ where for conditions b of T and b' of T' we take

$$b\beta b' \text{ iff } b = (\sigma,\eta)^{-1}b'.$$

(Note that because of lemma 5,

$$b\beta b' \text{ iff } \emptyset \neq b = (\sigma,\eta)^{-1}b'$$

where we only assume b' is a condition of T'.)

The verification that $an(T)$ is indeed a net involves demonstrating that every event has at least one pre and post condition. This follows from the following lemma which indicates how rich an asynchronous transition system is in conditions (it says an arbitrary pairwise-dependent set of events can be made to be both the starting and ending events of a single condition):

Lemma 6 *Let $T = (S,i,E,I,tran)$ be an asynchronous transition system. Suppose X is a nonempty subset of E such that*

$$e_1,e_2 \in X \Rightarrow \neg e_1 I e_2.$$

Then, there is a condition b of T such that

$$X = \{e \mid b \in e^{\bullet}\} \;\&\; X = \{e \mid b \in^{\bullet} e\}.$$

Proof: Define

$$b = \{(s,e,s') \in tran_* \mid e \notin X\}.$$

It is simply checked that b is a condition with beginning and ending events X. □

Lemma 7 *Let $T = (S, i, E, I, tran)$ be an asynchronous transition system. Then $an(T)$ is a net. Moreover,*

$$e_1 I e_2 \Leftrightarrow {}^\bullet e_1^\bullet \cap {}^\bullet e_2^\bullet = \emptyset,$$

and

$$(s, e, s') \in tran \Rightarrow M(s) \xrightarrow{e} M(s') \text{ in } an(T).$$

Proof: For $an(T)$ to be a net it is required that its initial marking and pre and post conditions of events be nonempty. However, taking $b = tran$, yields a condition in the initial marking, while for an event e, letting X be $\{e\}$ in lemma 6 produces a pre and post condition of e.

If $e_1 I e_2$ then axiom (3) on conditions (definition 1.4.1) ensures ${}^\bullet e_1^\bullet \cap {}^\bullet e_2^\bullet = \emptyset$. Conversely, by lemma 6, if $\neg(e_1 I e_2)$ we can obtain a condition in ${}^\bullet e_1^\bullet \cap {}^\bullet e_2^\bullet$.

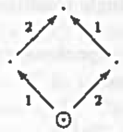
Suppose $(s, e, s') \in tran$. Then, letting B be the set of conditions of T,

$$\begin{aligned}
{}^\bullet e &= \{b \in B \mid (s, *, s) \in b \;\&\; (s, e, s') \notin b\} \subseteq M(s),\\
e^\bullet &= \{b \in B \mid (s, e, s') \notin b \;\&\; (s', *, s') \in b\} \subseteq M(s'), \text{ and}\\
M(s) \setminus {}^\bullet e &= \{b \in B \mid (s, *, s) \in b\} \setminus \{b \in B \mid (s, *, s) \in b \;\&\; (s, e, s') \notin b\}\\
&= \{b \in B \mid (s, e, s') \in b\}\\
&= \{b \in B \mid (s', *, s') \in b\} \setminus \{b \in B \mid (s, e, s') \notin b \;\&\; (s', *, s') \in b\}\\
&= M(s') \setminus e^\bullet.
\end{aligned}$$

Thus $M(s) \xrightarrow{e} M(s')$. □

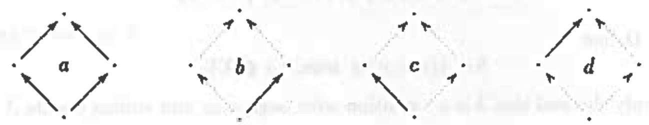
We illustrate how a net is produced from an asynchronous transition system.

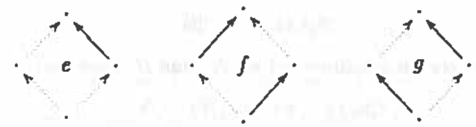
Example: Consider the following asynchronous transition system T with two independent events, 1 and 2:



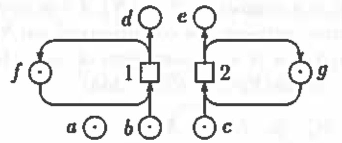
1 I 2

It has these conditions, where those transitions in the condition are represented by solid arrows:





Consequently the asynchronous transition system T yields this net $an(T)$:



Lemma 8 *an is a functor $\mathbf{A} \to \mathbf{N}$.*

Proof: The only difficulty comes in showing the well-definedness of an on morphisms. Let $(\sigma, \eta) : T \to T'$ be a morphism of asynchronous transition systems $T = (S, i, E, I, tran)$, $T' = (S', i', E', I', tran')$. We require that $an(\sigma, \eta) =_{def} (\beta, \eta)$ is a morphism of nets $an(T) \to an(T')$. Let $an(T) = (B, M_0, E, pre, post)$, $an(T') = (B', M_0', E', pre', post')$. We see β preserves initial markings by arguing:

$$\begin{aligned}
b' \in M_0' &\Leftrightarrow (i', *, i') \in b'\\
&\Leftrightarrow (\sigma(i), *, \sigma(i)) \in b'\\
&\Leftrightarrow (i, *, i) \in (\sigma, \eta)^{-1} b'\\
&\Leftrightarrow \beta^{op}(b') \in M_0.
\end{aligned}$$

The fact that $\beta e^\bullet = {}^\bullet \eta(e)$ and $\beta^\bullet e = \eta(e)^\bullet$ follows directly from (1) and (2) of lemma 5. □

In fact, an is left adjoint to na. Before proving this we explore the unit and counit of the adjunction. The unit of the adjunction:

Lemma 9 *Let T be an asynchronous system. Defining $\sigma_0(s) = M(s)$ for s a state of T and letting 1_E be the identity on the events of T, we obtain a morphism of asynchronous transition systems*

$$(\sigma_0, 1_E) : T \to na \circ an(T).$$

Proof: That $(\sigma_0, 1_E)$ is a morphism follows directly from lemma 7. □

The counit:

Lemma 10 *Let $N = (B, M_0, E, {}^\bullet(\ ), (\ )^\bullet)$ be a net. Let tran be the transitions of $na(N)$. For $b \in B$ and $c \subseteq tran_*$, taking*

$$c\beta_0 b \Leftrightarrow_{def} c = |b|$$

defines a relation between conditions of $na(N)$ and B, such that

$$(\beta_0, 1_E) : an \circ na(N) \to N$$

is a morphism of nets.

Proof: By lemma 4, $|b|$ is a condition of $na(N)$ if b is a condition of N. This ensures that β_0 is a relation between the conditions of $na(N)$ and B. We should check $(\beta_0, 1_E) : an \circ na(N) \to N$ is a morphism of nets. Let M_0' be the initial marking of $an \circ na(N)$: We see for any $b \in B$ that

$$\begin{aligned} \beta_0^{op}(b) \in M_0' \quad &\Leftrightarrow \quad (M_0, *, M_0) \in \beta_0^{op}(b) \\ &\qquad \text{by the definition of } an \text{ and } na, \\ &\Leftrightarrow \quad b \in M_0 \text{ by the definition of } \beta_0. \end{aligned}$$

From the equivalence

$$\beta_0^{op}(b) \in M_0' \Leftrightarrow b \in M_0$$

we deduce $\beta_0 M_0 = M_0'$, that β_0 preserves initial marking. In addition β_0 preserves pre and post conditions of events from II, I of lemma 4. □

Now we establish the adjunction between **A** and **N** in which an is left adjoint to na.

Lemma 11 *Let $T = (S, i, E, I, tran)$ be an asynchronous transition system and $N = (B, M_0, E', pre, post)$ a net.*

For a morphism of nets $(\beta, \eta) : an(T) \to N$, defining $\sigma(s) = \beta \circ M(s)$, for $s \in S$, yields a morphism of asynchronous transition systems

$$\theta(\beta, \eta) =_{def} (\sigma, \eta) : T \to na(N).$$

For a morphism of asynchronous transition systems $(\sigma, \eta) : T \to na(N)$, defining

$$c\beta b \text{ iff } \emptyset \neq c = \{(s, e, s') \in tran_* \mid b \in \sigma(s) \ \&\ b \in \sigma(s') \ \&\ b \notin {}^\bullet\eta(e)^\bullet\},$$

yields a morphism

$$\varphi(\sigma, \eta) =_{def} (\beta, \eta) : an(T) \to N.$$

Furthermore, θ and φ are mutual inverses, establishing a bijection between morphisms

$$an(T) \to N$$

and

$$T \to na(N).$$

Proof: First note $\theta(\beta, \eta)$ and $\varphi(\sigma, \eta)$ above are morphisms because they are the compositions

$$\theta(\beta, \eta) : \quad T \xrightarrow{(\sigma_0, 1_E)} na \circ an(T) \xrightarrow{na(\beta, \eta)} na(N)$$

$$\varphi(\sigma, \eta) : \quad an(T) \xrightarrow{an(\sigma, \eta)} an \circ na(N) \xrightarrow{(\beta_0, 1_{E'})} N$$

with the "unit" and "counit" morphisms of lemmas 9, 10. We require that θ, φ form a bijection.

Letting $(\sigma, \eta) : T \to na(N)$, we require $\theta \circ \varphi(\sigma, \eta) = (\sigma, \eta)$. We know $\theta \circ \varphi(\sigma, \eta)$ has the form (σ', η). Writing $(\beta, \eta) =_{def} \varphi(\sigma, \eta)$ we have $\sigma'(s) = \beta \circ M(s)$ for any $s \in S$. Now note

$$\begin{aligned} b' \in \sigma'(s) \quad &\Leftrightarrow \quad b' \in \beta \circ M(s) \\ &\Leftrightarrow \quad \beta^{op}(b') \in M(s) \\ &\Leftrightarrow \quad (s, *, s) \in \beta^{op}(b') \\ &\Leftrightarrow \quad b' \in \sigma(s) \end{aligned}$$

where the final equivalence follows from the definition of φ, recalling $(\beta, \eta) = \varphi(\sigma, \eta)$. Thus $\sigma' = \sigma$ and hence $\theta \circ \varphi(\sigma, \eta) = (\sigma, \eta)$.

To complete the proof, it is necessary to show $\varphi \circ \theta(\beta, \eta) = (\beta, \eta)$ for an arbitrary morphism $(\beta, \eta) : an(T) \to N$. Then, write $(\beta', \eta) =_{def} \varphi \circ \theta(\beta, \eta)$. To show $\beta' = \beta$, consider an arbitrary $(s, e, s') \in tran_*$. Let $b \in B$. From the definitions of θ and φ,

$$(s, e, s') \in \beta'^{op}(b) \Leftrightarrow b \in \beta M(s) \ \&\ b \in \beta M(s') \ \&\ b \notin {}^\bullet\eta(e)^\bullet. \qquad (\dagger)$$

Note that

$$\begin{aligned} b \in \beta M(s) \quad &\Leftrightarrow \quad \beta^{op}(b) \in M(s) \\ &\Leftrightarrow \quad (s, *, s) \in \beta^{op}(b). \end{aligned}$$

Note too that, as (β, η) is a morphism,

$$b \in {}^\bullet\eta(e)^\bullet \Leftrightarrow \beta^{op}(b) \in {}^\bullet e^\bullet.$$

Hence, rewriting (†),

$$(s, e, s') \in \beta'^{op}(b) \Leftrightarrow (s, *, s) \in \beta^{op}(b) \ \&\ (s', *, s') \in \beta^{op}(b) \ \&\ \beta^{op}(b) \notin {}^\bullet e^\bullet.$$

However, under the assumption that $(s, *, s)$ and $(s', *, s')$ belong to $\beta^{op}(b)$ we have

$$\beta^{op}(b) \notin {}^\bullet e^\bullet \Leftrightarrow (s, e, s') \in \beta^{op}(b).$$

(Recall the definition of ${}^\bullet e$ and $e^\bullet$ in $an(T)$.)

Thus

$$(s, e, s') \in \beta'^{op}(b) \Leftrightarrow (s, e, s') \in \beta^{op}(b).$$

Consequently, $\beta' = \beta$, and we conclude $\varphi \circ \theta(\beta, \eta) = (\beta, \eta)$. □

Theorem 12 *The functors* $an : \mathbf{A} \to \mathbf{N}$ *and* $na : \mathbf{N} \to \mathbf{A}$ *form an adjunction with an left adjoint to na; the components of the units and counits of the adjunction are the morphisms given in lemmas 9, 10.*

Proof: Let T be an asynchronous transition system and N a net. Let $(\sigma_0, 1_E) : T \to na \circ an(T)$ be the morphism described in lemma 9. Let $(\sigma, \eta) : T \to na(N)$ be a morphism in $\mathbf{A}$. Then, because of the bijection, $\varphi(\sigma, \eta)$ is the unique morphism $h : an(T) \to N$ such that

$$(\sigma, \eta) = \theta(h) = na(h) \circ (\sigma_0, 1_E)$$

—as remarked in the proof of lemma 11, $\theta(h)$ is this composition. □

1.4.2 A coreflection

Neither $\mathbf{A}$ nor $\mathbf{N}$ embeds fully and faithfully in the other category via the functors of the adjunction. This accompanies the facts that neither unit nor counit is an isomorphism (see [8] p. 88); in passing from a net N to $an \circ na(N)$ extra conditions are most often introduced; the net $an \circ na(N)$ is always safe, as we will see. While passing from an asynchronous transition system T to $na \circ an(T)$ can, not only blow-up the number of states, but also collapse states which cannot be separated by conditions.

A (full) coreflection between asynchronous transition systems and nets can be obtained at the cost of adding three axioms. Let $\mathbf{A}^0$ be the full subcategory of asynchronous transition systems $T = (S, i, E, I, tran)$ satisfying the following:

Axiom 1 Every state is reachable from the initial state, i.e. for every $s \in S$ there is a chain of events $e_1, \ldots, e_n$, possibly empty, for which $i \xrightarrow{e_1 \cdots e_n} s$, where i is the initial state.

Axiom 2 $M(u) = M(s) \Rightarrow u = s$, for all $s, u \in S$.

Axiom 3 ${}^\bullet e \subseteq M(s) \Rightarrow \exists s'. (s, e, s') \in tran$, for all $s \in S, e \in E$.

There is a close similarity to the regional axioms characterizing the case graphs of elementary net systems in terms of the regional axioms of Ehrenfeucht and Rozenberg, as presented in [15]. Axioms 2 and 3 enforce two separation properties. The contraposition of Axiom 2 says

$$u \neq s \Rightarrow M(u) \neq M(s)$$

i.e. that if two states are distinct then there is a condition of T holding at one and not the other. In fact, Axiom 2 is equivalent to

$$u \neq s \Rightarrow \exists b.\, b \in M(u) \;\&\; b \notin M(s)$$

though we postpone the justification of this till after we have treated *complementation* of conditions. We can recast Axiom 3 into the following form when it becomes more clearly a separation axiom: If (u, e, u') is a transition and s is a state from which there is no e-transition then there is a condition b of T such that

$$b \in M(u) \;\&\; (u, e, u') \notin b \;\&\; b \notin M(s).$$

Axioms 2 and 3 hold for any asynchronous transition system $na(N)$ got from a net N. The proof that Axiom 3 holds uses the operation of *complementation* on conditions of an asynchronous transition system. The properties of complementation are listed below:

Proposition 13 *Let b be a condition of an asynchronous transition system $T = (S, i, E, I, tran)$. Define*

$$\bar{b} = \{(s, e, s') \in tran_* \mid (s, e, s') \notin b \;\&\; (s, *, s) \notin b \;\&\; (s', *, s') \notin b\}.$$

If nonempty, $\bar{b}$ is a transition of T. It has the following properties:

$$\begin{aligned} (s, *, s) \in \bar{b} &\Leftrightarrow (s, *, s) \notin b, \text{ for any } s \in S, \\ \bar{b} \in {}^\bullet e &\Leftrightarrow b \in e^\bullet \;\&\; b \notin {}^\bullet e \\ \bar{b} \in e^\bullet &\Leftrightarrow b \in {}^\bullet e \;\&\; b \notin e^\bullet \text{ for any } e \in E. \end{aligned}$$

Let $(\sigma, \eta) : T' \to T$ be a morphism of asynchronous transition systems and b be a condition of T. Then

$$(\sigma, \eta)^{-1}\bar{b} = \overline{(\sigma, \eta)^{-1}b}.$$

Suppose u, s are two distinct markings of a net N. Then certainly there is a condition b of the net in one but not the other.

Suppose for instance $b \notin u$ and $b \in s$. Then, from the way the extent of a condition is defined,

$$|b| \notin M(u) \text{ and } |b| \in M(s).$$

With complementation we can separate the other way:

$$\overline{|b|} \in M(u) \text{ and } \overline{|b|} \notin M(s).$$

This justifies our earlier remark that that Axiom 2 is equivalent to the seemingly stronger axiom:

$$u \neq s \Rightarrow \exists b.\, b \in M(u) \;\&\; b \notin M(s)$$

We return to the verification that the asynchronous transition system $na(N)$ of a net N satisfies Axioms 2 and 3.

Proposition 14 *Let $N = (B, M_0, E, pre, post)$ be a net. Then $na(N)$ satisfies the Axioms 2 and 3 above.*

Proof: If u, s are distinct states of $na(N)$ they are distinct markings of N and hence only one contains some condition b. But then $|b|$ can only be an element of one of $M(u)$ and $M(s)$ which are therefore unequal. This demonstrates (the contraposition of) Axiom 2.

Now we show $na(N)$ satisfies the contraposition of Axiom 3. Supposing $u \xrightarrow{e} u'$ and $s \stackrel{e}{\not\to}$ in N, we are required to exhibit a condition c of $na(N)$ such that

$$c \in{}^{\bullet}e \;\&\; c \notin M(s).$$

There are two ways in which the marking s can fail to enable event e. Either

(i) $pre(e) \not\subseteq s$ or

(ii) $post(e) \cap (s \setminus pre(e)) \neq \emptyset$.

In the case of (i), there is a condition $b \in B$ of the net such that

$$b \in pre(e) \;\&\; b \notin s.$$

Hence

$$|b| \in{}^{\bullet}e \;\&\; |b| \notin M(s).$$

In the case of (ii), there is a condition $b \in B$ of the net such that

$$b \in post(e) \;\&\; b \in s \;\&\; b \notin pre(e)$$

Hence

$$|b| \in e^{\bullet} \;\&\; |b| \in M(s) \;\&\; |b| \notin{}^{\bullet}e.$$

But then, taking the complement of $|b|$,

$$\overline{|b|} \in{}^{\bullet}e \;\&\; \overline{|b|} \notin M(s),$$

by proposition 13.

In either case, (i) or (ii), we obtain a condition c of $na(N)$ for which

$$c \in{}^{\bullet}e \;\&\; c \notin M(s).$$

□

Recall a net is *safe* if for each reachable marking M and event e

$${}^{\bullet}e \subseteq M \Rightarrow e^{\bullet} \cap (M \setminus{}^{\bullet}e) = \emptyset.$$

As we now see, if T is an asynchronous transition system which satisfies Axioms 2 and 3 then $an(T)$ is a safe net whose behaviour is seen to be isomorphic to that of T on reachable states.

Lemma 15 *Assume $T = (S, i, E, I, tran)$ is an asynchronous transition system satisfying Axioms 2 and 3 above. Then*

1. $e_1 I e_2 \Leftrightarrow{}^{\bullet}e_1^{\bullet} \cap{}^{\bullet}e_2^{\bullet} = \emptyset$ in $an(T)$, for any events e_1, e_2,

2. $(s, e, s') \in tran \Leftrightarrow M(s) \xrightarrow{e} M(s')$ in $an(T)$ for any $s, s' \in S$ and $e \in E$,

3. $an(T)$ is a safe net in which every reachable marking has the form $M(s)$ for some state s of T.

Proof: By lemma 7,

$$e_1 I e_2 \Leftrightarrow{}^{\bullet}e_1^{\bullet} \cap{}^{\bullet}e_2^{\bullet} = \emptyset,$$
$$(s, e, s') \in tran \Rightarrow M(s) \xrightarrow{e} M(s') \text{ in } an(T).$$

This yields (1) and (2)"$\Rightarrow$". To establish the converse, (2)"$\Leftarrow$", with the assumption of Axioms 2 and 3, suppose $M(s) \xrightarrow{e} M(s')$. Then ${}^{\bullet}e \subseteq M(s)$ so $(s, e, s_1) \in tran$ from some state s_1 by Axiom 3. Thus $M(s) \xrightarrow{e} M(s_1)$ and so $M(s') = M(s_1)$. Now by Axiom 2 we deduce $s' = s_1$, and hence the converse

$$M(s) \xrightarrow{e} M(s') \Rightarrow (s, e, s') \in tran.$$

We now show (3). Any reachable marking of $an(T)$ has the form $M(s)$ for some $s \in S$ by the following argument: Assuming $M(s) \xrightarrow{e} M_1$ we necessarily have ${}^{\bullet}e \subseteq M(s)$ whereupon, as above, there is a transition (s, e, s_1) of T with $M_1 = M(s_1)$; thus, by induction along any reachability chain, any reachable marking of $an(T)$ is of the form $M(s)$ for some state s of T. Because the two sets

$$e^{\bullet} = \{b \in M(s') \mid (s, e, s') \notin b\},$$
$$M(s) \setminus{}^{\bullet}e = \{b \in M(s) \mid (s, e, s') \in b\}$$

are clearly disjoint, the net $an(T)$ is safe. □

Corollary 16 *For any net N, the net $an \circ na(N)$ is safe.*

The coreflection between $\mathbf{A}^0$ and $\mathbf{N}$ is defined using a simple coreflection between the full subcategory of $\mathbf{A}$, consisting of objects, where all states are reachable, and $\mathbf{A}$.

Definition: Let $\mathbf{A}^R$ be the full subcategory of $\mathbf{A}$ consisting of asynchronous transition systems $(S, i, E, I, tran)$ satisfying Axiom 1, *i.e.* so that all states s are reachable.

Let $\mathcal{R}$ act on an asynchronous transition system $T = (S, i, E, I, tran)$ as follows:

$$\mathcal{R}(T) = (S', i', E', I', tran')$$

where

$$\begin{aligned} S' & \quad \text{consists of all reachable states of } T \\ E' &= \{e \in E \mid \exists s, s' \in S'.\, (s,e,s') \in tran\} \\ I' &= I \cap (E' \times E') \\ tran' &= tran \cap (S' \times E' \times S'). \end{aligned}$$

For a morphism $(\sigma,\eta) : T \to T'$ of asynchronous transition systems, define $\mathcal{R}(\sigma,\eta) = (\sigma',\eta')$ where σ' and η' are the restrictions of σ and η to the states, respectively events, of $\mathcal{R}(T)$.

We note that a morphism from an asynchronous transition system in which all states are reachable is determined by how it acts on events:

Proposition 17 *Suppose (σ,η) and (σ',η) are morphisms $T \to T'$ between asynchronous systems where each state of T is reachable. Then $\sigma = \sigma'$.*

Proof: An obvious consequence of the determinacy property

$$(s,e,s_1) \in tran \ \&\ (s,e,s_2) \in tran \Rightarrow s_1 = s_2$$

of asynchronous transition systems. □

Proposition 18 *The operation $\mathcal{R}$ is a functor $\mathbf{A} \to \mathbf{A}^R$ which is right adjoint to the inclusion functor $\mathcal{I} : \mathbf{A}^R \to \mathbf{A}$. The unit of the adjunction at $T \in \mathbf{A}^R$ is the identity on T, making the adjunction a coreflection. The counit at $T \in \mathbf{A}^R$ is given by $(j_S, j_E) : R(T) \to T$ where j_S and j_E are the inclusion maps on states and events respectively. Moreover, $\mathcal{R}$ preserves Axioms 2 and 3 in the sense that if T satisfies Axiom 2 (or 3) then $\mathcal{R}(T)$ satisfies Axiom 2 (or 3).*

Proof: We omit the straightforward proof that $\mathcal{R}$ is a right adjoint to the inclusion of categories with counit as claimed. Let $j : \mathcal{R}(T) \to T$ be a component of the counit. The transitions $tran'$ of $\mathcal{R}(T)$ are a subset of those of T. If b is a condition of T then $j^{-1}b = b \cap tran'$ is a condition of $\mathcal{R}(T)$ provided it is nonempty. Suppose s_1 and s_2 are two distinct states of $\mathcal{R}(T)$. If T satisfies Axiom 2 then there is a condition b of T such that one and only one of $(s_1,*,s_1) \in b$, $(s_2,*,s_2) \in b$ holds. But then $j^{-1}b$ is a condition of $\mathcal{R}(T)$ separating s_1, s_2. Thus $\mathcal{R}$ preserves Axiom 2, and by a similar argument, Axiom 3. □

We show the adjunction, with *an* left adjoint to $\mathcal{R} \circ na$, obtained as the composition forms a coreflection. Its counit is given by the notion of *reachable extent* of a condition. This consists essentially of the reachable markings and transitions at which b holds uninterruptedly.

Definition: Let N be a net. Let $tran_*$ be the transitions and idle transitions of $\mathcal{R} \circ na(N)$. Define

$$|b|^R = |b| \cap tran_*.$$

Theorem 19 *Defining $na_0 = \mathcal{R} \circ na$, the composition of functors, yields a functor $na_0 : \mathbf{N} \to \mathbf{A}^0$ which is right adjoint to $an_0 : \mathbf{A}^0 \to \mathbf{N}$, the restriction of an to $\mathbf{A}^0$.*

The unit at $T = (S,i,E,I,tran) \in \mathbf{A}^0$ is an isomorphism

$$(\sigma, 1_E) : T \to na_0 \circ an_0(T)$$

where $\sigma(s) = M(s)$ for $s \in S$, making the adjunction a coreflection.

The counit at a net N is

$$(\beta, 1_E) : an_0 \circ na_0 \to N$$

where

$$c\beta b \text{ iff } \emptyset \neq c = |b|^R$$

between conditions c of $na_0(N)$ and b of N.

Proof: The adjunctions compose to give $\mathcal{R} \circ na : \mathbf{N} \to \mathbf{A}^R$ a right adjoint to $\mathcal{I} \circ an : \mathbf{A}^R \to \mathbf{N}$. However, the image $\mathcal{R} \circ na(N)$ of a net N always satisfies Axioms 2 and 3 as well as 1. This is because $na(N)$ satisfies Axioms 2 and 3, and $\mathcal{R}$ preserves these axioms. Thus the adjunction cuts down to one where $na_0 : \mathbf{N} \to \mathbf{A}^0$ is right adjoint to $an_0 : \mathbf{A}^0 \to \mathbf{N}$. It is an adjunction with unit at $T = (S,i,E,I,tran) \in \mathbf{A}^0$ a morphism in $\mathbf{A}^0$

$$(\sigma, 1_E) : T \to na_0 \circ an_0(T)$$

where $\sigma(s) = M(s)$ for $s \in S$.

That the unit $(\sigma, 1_E) : T \to na_0 \circ an_0(T)$ is an isomorphism follows from lemma 15. Hence the functors an_0, na_0 form a coreflection with an_0 left adjoint to na_0.

That the counit has the form claimed follows by composing the natural bijections of the adjunctions given by proposition 18 and lemma 11. □

One consequence of the coreflection is that any net N can be converted to a safe net $an_0 \circ na_0(N)$ with the same behaviour, in the sense that that there is as an isomorphism between reachable asynchronous transition systems the two nets induce under na_0. Another is that $\mathbf{A}^0$ has products and coproducts given by the same constructions as those of $\mathbf{A}$.

The coreflection $\mathbf{A}^0 \hookrightarrow \mathbf{N}$ cuts down to an equivalence of categories by restricting to the appropriate full subcategory of nets.

Definition: Let $\mathbf{N_0}$ be the full subcategory on nets such that

$$b \mapsto |b|^R$$

is a bijection between conditions of N and those of $na_0(N)$.

Theorem 20 *The functor an restricts to a functor* $an_0 : \mathbf{A^0} \to \mathbf{N_0}$. *The functor* $\mathcal{R} \circ na$ *restricts to a functor* $na_0 : \mathbf{N_0} \to \mathbf{A^0}$. *The functors* an_0, na_0 *form an equivalence of categories.*

Proof: Recall the coreflection of theorem 19: $na_0 = \mathcal{R} \circ na : \mathbf{N} \to \mathbf{A^0}$ is right adjoint to $an_0 : \mathbf{A^0} \to \mathbf{N}$, the restriction of an to $\mathbf{A^0}$. The counit of the coreflection, at a net N,

$$(\beta, 1_E) : an_0 \circ na_0(N) \to N$$

has $c\beta b$ iff $c = |b|^R$, between condition. This is an isomorphism iff $N \in \mathbf{N_0}$. We thus obtain an equivalence of categories. □

Nets in $\mathbf{N_0}$ are saturated with conditions in the sense that they have as many conditions as is allowed by their reachable behaviour and independence (regarded as an asynchronous transition system). Nets in $\mathbf{N_0}$ cannot however have more than one copy of a condition with particular starting and ending events (they are *condition-extensional*). This is because:

Proposition 21 *Let T be an asynchronous transition system for which each state is reachable. If b_1, b_2 are conditions of T for which*

$$^{\bullet}b_1 = {}^{\bullet}b_2 \quad \text{and} \quad b_1^{\bullet} = b_2^{\bullet}$$

then

$$b_1 = b_2.$$

Proof: Suppose $^{\bullet}b_1 = {}^{\bullet}b_2$ and $b_1^{\bullet} = b_2^{\bullet}$ for conditions b_1, b_2 of T. Inductively along a chain of transitions

$$(i, e_1, s_1), (s_1, e_2, s_2), \ldots, (s_{n-1}, e_n, s_n)$$

the membership of (s_{i-1}, e_i, s_i) (or $(s_i, *, s_i)$) in b_1 and in b_2 must agree. □

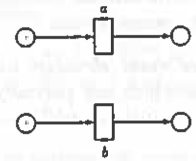
If on the other hand an asynchronous transition system T has a state which is not reachable then there will be distinct conditions of T with the same end points. Suppose T has states which are not reachable let $tran_0$ be all transitions, including idle ones, which are not reachable. If b_1 is a condition, say consisting solely of reachable transitions of T, then so is $b_2 = b_1 \cup tran_0$ a condition, necessarily distinct from b_1, but with $^{\bullet}b_1 = {}^{\bullet}b_2$ and $b_1^{\bullet} = b_2^{\bullet}$.

2 Labelled Models and Bisimulation

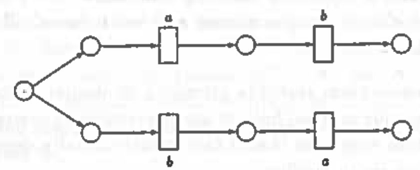
The coreflection presented in the previous section is just one example of many categorical relationships between models for concurency—see [24] for a survey. We shall now put the coreflection into a wider picture, allowing us to apply to nets the general notion of bisimulation obtained from a span of open maps, suggested in [5].

2.1 Labelled models and their relationship

Like most models for concurrency, nets [16] and asynchronous transition systems [12], or more precisely their labelled versions, have been used as models for process languages like CCS, [10]. As an illustration, following [16], the CCS expression $a.nil|b.nil$ is represented by the labelled net:



In contrast the (strongly bisimilar) expression $a.b.nil + b.a.nil$ is represented by:



There is a general way of introducing labels to models in such a way that one may carry over adjunctions between unlabelled models to their labelled counterparts. We refer to [24] for details. Here we sketch the idea, applicable to the categories of nets and asynchronous transition systems:

- Add to structures X an extra component of a (total) labelling function $l : E \to L$ from the structure's set of events E to a set of labels L; we obtain labelled structure as pairs (X, l).
- Assume morphisms $f : X \to X'$ of unlabelled structures include a component η between sets of events. A morphism of labelled structures $(X, l) \to (X', l')$ is a pair (f, λ) where $f : X \to X'$ is a morphism on the underlying

unlabelled structures and $\lambda : L \rightarrow_* L'$ is a partial function on the label sets such that $\lambda \circ l = l' \circ \eta$.

Morphisms between labelled structures are of this generality in order to obtain operations of process calculi as universal constructions. However, for our purpose of studying bisimulation on nets, it suffices to work with structures having a common set of labels L, and define morphisms as before, but with the extra condition that the component λ is the identity on L—this implies that the event component η is total. (In fact, this subcategory is the fibre over L with respect to the obvious functor projecting labelled structures to their label sets.)

Rather than going through the tedious and simple definitions of the labelled versions of nets and asynchronous transition systems, we illustrate the idea by giving the definition of labelled event structures. The events of an event structure are to be thought of as representing individual occurrences of actions of a system. The structural parts of an event structure are intended to capture the causal and nondeterministic aspects of such computations:

Definition: Define an *L-labelled event structure* to be a structure $(E, \leq, \#, l)$ consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, a binary, symmetric, irreflexive relation $\# \subseteq E \times E$, the *conflict relation*, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite,}$$
$$e\#e' \leq e'' \Rightarrow e\#e''$$

for all $e, e', e'' \in E$, and a surjective labelling function $l : E \rightarrow L$.

Say two events $e, e' \in E$ are *concurrent*, and write $e \; co \; e'$, iff $\neg(e \leq e' \text{ or } e' \leq e \text{ or } e\#e')$.

The finiteness assumption restricts attention to discrete processes where an event occurrence depends only on finitely many previous occurrences. The axiom on the conflict relation expresses that if two events causally depend on events in conflict then they too are in conflict.

To understand the "dynamics" of an event structure $(E, \leq, \#, l)$ we show how an event structure determines a labelled asynchronous transition system $(S, i, E, I, tran, l)$. Guided by our interpretation we can formulate a notion of computation state of an event structure $(E, \leq, \#, l)$. Taking a computation state of a process to be represented by the set x of events which have occurred in the computation, we expect that

$$e' \in x \;\&\; e \leq e' \Rightarrow e \in x$$

—if an event has occurred then all events on which it causally depends have occurred too—and also that

$$\forall e, e' \in x. \neg(e\#e')$$

—no two conflicting events can occur together in the same computation. Let $C(E, \leq, \#, l)$ denote the subsets of events satisfying these two conditions, traditionally called the ***configurations*** of the event structure. We let S be the set of finite configurations and i the empty configuration.

Events manifest themselves as atomic jumps from one configuration to another. For configurations x, x' define

$$(x, e, x') \in tran \Leftrightarrow e \notin x \;\&\; x' = x \cup \{e\}.$$

It is easy to see that this indeed defines a labelled asynchronous transition system, and that the construction extends to a functor with the following definition of morphisms for labelled event structures:

Definition: Let $ES = (E, \leq, \#, l)$ and $ES' = (E', \leq', \#', l')$ be event structures labelled with L. A *morphism* from ES to ES' consists of a total function $\eta : E \rightarrow E'$ on events which satisfies

(i) if $x \in C(ES)$ then $\eta x \in C(ES')$ &
$\forall e_0, e_1 \in x. \eta(e_0) = \eta(e_1) \Rightarrow e_0 = e_1$.

(ii) if $e \in E$ then $l(e) = l'(\eta(e))$

Definition: Let $\mathbf{E}_L$ denote the category of event structures labelled with L with (fibre) morphisms defined as above.

Note that Pratt's *pomsets* can be identified with special kinds of event structures, those without any conflict, and that Milner's *synchronization trees* can be identified with those event structures having empty *co*-relation.

Let us denote the labelled versions of our categories of nets and asynchronous transition systems with (fibred) morphisms by $\mathbf{N}_L$, $\mathbf{A}_L$ and $\mathbf{A}_L^0$ respectively. Similarly the category of transition systems over label set L, with morphisms having the identity as label component, will be denoted $\mathbf{T}_L$, and its full subcategory of synchronizaton trees $\mathbf{S}_L$.

It follows for general reasons that the adjunction and coreflection between nets and asynchronous transition systems lift to a coreflection between the labelled versions. The modified adjoints are essentially the adjoints presented in the previous section, simply carrying the label parts across from one model to the other. Furthermore, this coreflection is part of a small collection of coreflections as in the diagram below.

$$\begin{array}{ccccc} \mathbf{S}_L & \xrightarrow{st} & \mathbf{T}_L & & \\ \downarrow{\scriptstyle se} & & & & \\ \mathbf{E}_L & \xrightarrow{ea_0} & \mathbf{A}_L^0 & \xrightarrow{an_0} & \mathbf{N}_L \end{array}$$

When specifying a functor of one of the coreflections above, we adopt a convention; for example the left adjoint from $\mathbf{S}_L$ to $\mathbf{T}_L$ is denoted st while its right adjoint is ts.

The left adjoints, drawn above, embed one model in another. We have deliberately used the notation an_0 also for the labelled version of the embedding of $\mathbf{A}^0_L$ into $\mathbf{N}_L$. For details of the other coreflections we refer to [24]. The functor ea_0 is basically described above, and its right adjoint is an unfolding of labelled asynchronous transition systems into labelled event structures, generalizing the well known unfolding of transition systems into synchronization trees. The composition $ae_0 \circ na_0$ yields the unfolding of nets into event structures, familiar from [14]. For readers familiar with net theory, it is worth mentioning that for a net N, $en \circ ne(N)$ is simply the saturated version of the net unfolding of N as defined in [14]. There are not coreflections from transition systems $\mathbf{T}_L$ to the categories of labelled nets $\mathbf{N}_L$ or asynchronous transition systems $\mathbf{A}_L$ or $\mathbf{A}^0_L$. There are not for the irritating reason that, unlike transition systems, these two models allow more than one transition with the same label between two states. This stops the natural bijection required for the "inclusion" of transition systems to be a left adjoint.

2.2 Path-lifting morphisms

In this section we briefly present some of the main ideas, definitions and results from [5], providing a general notion of bisimulation applicable to a wide range of models. For the missing proofs we refer to [5].

Informally, a computation path should represent a particular run or history of a process. For transition systems, a computation path is reasonably taken to be a sequence of transitions. Let's suppose the sequence is finite. For a labelling set L, define the category of branches $\mathbf{Bran}_L$ to be the full subcategory of transition systems, with labelling set L, with objects those finite synchronisation trees with at most one maximal branch. A computation path in a transition system T, with labelling set L, can then be represented by a morphism

$$p : P \to T$$

in $\mathbf{T}_L$ from an object P of $\mathbf{Bran}_L$. How should we represent a computation path of a net or an event structure? To take into account the explicit concurrency exhibited by an event structure, it is reasonable to represent a computation path as a morphism from a partial order of labelled events, that is from a *pomset*. Define the category of pomsets $\mathbf{Pom}_L$, with respect to a labelling set L, to be the full subcategory of $\mathbf{E}_L$ whose objects consist exclusively of finite pomsets. A computation path in an event structure E, with labelling set L, is a morphism

$$p : P \to E$$

in $\mathbf{E}_L$ from an object P of $\mathbf{Pom}_L$. Because labelled event structures and so pomsets embed in nets $\mathbf{N}_L$, via the coreflection $\mathbf{E}_L \to \mathbf{N}_L$, the idea extends: a computation path in a net N, with labelling set L, is represented by a morphism

$$p : P \to N$$

in $\mathbf{N}_L$ from the image P of an object of $\mathbf{Pom}_L$ under the coreflection, the saturated labelled net corresponding to P. In future, when discussing nets, we will deliberately confuse pomsets with their image in $\mathbf{N}_L$ under the embedding.

More precisely, assume a category of models $\mathbf{M}$ (this can be any of the labelled categories of models we are considering) and a choice of path category, a subcategory $\mathbf{P} \hookrightarrow \mathbf{M}$ consisting of path objects (these could be branches, or pomsets) together with morphisms expressing how they can be extended. Define a *path* in an object X of $\mathbf{M}$ to be a morphism

$$p : P \to X,$$

in $\mathbf{M}$, where P is an object in $\mathbf{P}$. A morphism $f : X \to Y$ in $\mathbf{M}$ takes such a path p in X to the path $f \circ p : P \to Y$ in Y. The morphism f expresses the sense in which Y simulates X; any computation path in X is matched by the computation path $f \circ p$ in Y.

We might demand a stronger condition of a morphism $f : X \to Y$ expressed succinctly in the following *path-lifting condition:*

Whenever, for $m : P \to Q$ a morphism in $\mathbf{P}$, a "square"

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow{q} & Y \end{array}$$

in $\mathbf{M}$ commutes, *i.e.* $q \circ m = f \circ p$, meaning the path $f \circ p$ in Y can be extended via m to a path q in Y, then there is a morphism p' such that in the diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle m}\downarrow & \nearrow{\scriptstyle p'} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow{q} & Y \end{array}$$

the two "triangles" commute, *i.e.* $p' \circ m = p$ and $f \circ p' = q$, meaning the path p can be extended via m to a path p' in X which matches q. When the morphism f satisfies this condition we shall say it is $\mathbf{P}$-*open*.

It is easily checked that $\mathbf{P}$-open morphisms include all the identity morphisms (in fact, all isomorphisms) of $\mathbf{M}$ and are closed under composition there; in other words they form a subcategory of $\mathbf{M}$.

For the well-known model of transition systems open morphisms are already familiar:

Proposition 22 *With respect to a labelling set L, the $\mathbf{Bran}_L$-open morphisms of $\mathbf{T}_L$ are the "zig-zag morphisms" of [21], the "p-morphism" of [19], the "abstraction homomorphisms" of [4], and the "pure morphisms" of [2], i.e. those label-preserving morphisms $(\sigma, 1_L) : T \to T'$ on transition systems over labelling set L with the property that for all reachable states s of T*

if $\sigma(s) \xrightarrow{a} s'$ in T' then $s \xrightarrow{a} u$ in T and $\sigma(u) = s'$, for some state u of T.

Let us return to the general set-up, assuming a path category $\mathbf{P}$ in a category of models $\mathbf{M}$. Say two objects X_1, X_2 of $\mathbf{M}$ are $\mathbf{P}$*-bisimilar* iff there is a *span* of $\mathbf{P}$-open morphisms f_1, f_2:

$$X_1 \xleftarrow{f_1} X \xrightarrow{f_2} X_2$$

For the interleaving models of transition systems and synchronisation trees with path category $\mathbf{P}$ taken to be branches, $\mathbf{P}$-bisimulation coincides with Milner's strong bisimulation:

Theorem 23 *Two transition systems (and so synchronisation trees), over the same labelling set L, are $\mathbf{Bran}_L$-bisimilar iff they are strongly bisimilar in the sense of [10].*

Clearly, in general, the relation of $\mathbf{P}$-bisimilarity between objects is reflexive (identities are $\mathbf{P}$-open) and symmetric (in the nature of spans). It is also transitive provided $\mathbf{M}$ has pullbacks, and so an equivalence relation on objects, by virtue of the following fact:

Proposition 24 *Pullbacks of $\mathbf{P}$-open morphisms are $\mathbf{P}$-open.*

Transitivity of $\mathbf{P}$-bisimilarity is clear for $\mathbf{M}$ with pullbacks; two spans of open morphisms combine to form a span by pulling back from their vertices, as we can do for all the models we consider:

Proposition 25 *The categories $\mathbf{T}_L, \mathbf{S}_L, \mathbf{N}_L, \mathbf{A}_L^0, \mathbf{A}_L$, and $\mathbf{E}_L$ have pullbacks.*

Proof: We show that $\mathbf{N}_L$ has pullbacks. There are coreflections from all categories $\mathbf{S}_L, \mathbf{E}_L, \mathbf{A}_L^0$ into $\mathbf{N}_L$. Using the fact that right adjoints preserve limits, and pullbacks in particular, we obtain pullbacks in any of $\mathbf{S}_L, \mathbf{E}_L, \mathbf{A}_L^0$ as images under the right adjoints of the pullback in $\mathbf{N}_L$ of diagrams transported into $\mathbf{N}_L$ by the left adjoints. Because there is not a coreflection from the category of transition systems into nets, $\mathbf{T}_L$ requires a separate (though simple) treatment (or see [5]).

We construct pullbacks in $\mathbf{N}_L$ explicitly in the following way. Suppose $f_1 = (\sigma_1, \eta_1) : N_1 \to N_0$ and $f_2 = (\sigma_2, \eta_2) : N_2 \to N_0$ are morphisms in $\mathbf{N}_L$ where

$$N_i = (B_i, M_i, E_i, pre_i, post_i, l_i), i = 0, 1, 2.$$

We want to construct a pullback $N = (B, M, E, pre, post, l), \pi_1, \pi_2$:

$$\begin{array}{ccc} N & \xrightarrow{\pi_2} & N_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ N_1 & \xrightarrow[f_1]{} & N_0 \end{array}$$

The construction of the events of N, E, is based on pullbacks in the category of sets:

$$E = \{(e_1, e_2) \in E_1 \times E_2 \mid \eta_1(e_1) = \eta_2(e_2)\}$$

The construction of the conditions of N, B, is based on pushouts in the category of sets with partial functions. Let R denote the equivalence relation on $B_1 \cup B_2$ generated by R_0, where

$$b_1 \; R_0 \; b_2 \quad \text{iff} \quad \text{there exists } b_0 \text{ in } B_0 \text{ such that}$$
$$\beta_1(b_0) = b_1 \text{ and } \beta_2(b_0) = b_2$$

We define

$$B \;=\; \text{the equivalence classes, } c, \text{ of } R, \text{satisfying}$$
$$\beta_1^{op}(c) = \beta_2^{op}(c).$$

And with these events and conditions of N we let:

$$\begin{aligned} M &= \{c \in B \mid c \subseteq M_1 \cup M_2\} \\ pre((e_1, e_2)) &= \{c \in B \mid c \subseteq pre_1(e_1) \cup pre_2(e_2)\} \\ post((e_1, e_2)) &= \{c \in B \mid c \subseteq post_1(e_1) \cup post_2(e_2)\} \\ l((e_1, e_2)) &= l_1(e_1)(= l_2(e_2)) \end{aligned}$$

And finally we define the components $\pi_1 = (\bar{\beta}_1, \bar{\eta}_1)$ and $\pi_2 = (\bar{\beta}_2, \bar{\eta}_2)$ of the pullback as follows:

$$\begin{aligned} \bar{\eta}_i((e_1, e_2)) &= e_i \\ \bar{\beta}_i(b_i) &= \text{the } R\text{-equivalence class of } b_i \text{ if this belongs to } B, \\ &\quad\ \text{undefined otherwise.} \end{aligned}$$

We leave it to the reader to check that these constructions indeed define a pullback in $\mathbf{N}_L$ as required. All the required properties follow by simple calculations. □

Corollary 26 *For all the model categories mentioned in previous proposition, and for all subcategories of observations, $\mathbf{P}_L$, the relation of $\mathbf{P}_L$-bisimilarity is an equivalence.*

And, finally, a few general facts about how open morphisms are preserved and reflected by functors, especially as part of a coreflection. For notational simplicity we shall assume the left adjoints of the coreflections are inclusions. It follows that for the coreflections of Section 2.1, open morphisms, with respect to a choice of path category, are preserved in both directions of the adjunction.

Proposition 27 *Let* $\mathbf{M}$ *be a full subcategory of* $\mathbf{N}$, *and* $\mathbf{P}$ *a subcategory of* $\mathbf{M}$. *A morphism* f *of* $\mathbf{M}$ *is* $\mathbf{P}$*-open in* $\mathbf{M}$ *iff* f *is* $\mathbf{P}$*-open in* $\mathbf{N}$.

Proof: Directly from the definition of open morphism. □

Lemma 28 *Let* $\mathbf{M}$ *be a coreflective subcategory of* $\mathbf{N}$ *with* R *right adjoint to the inclusion function* $\mathbf{M} \hookrightarrow \mathbf{N}$ *and* $\mathbf{P}$ *a subcategory of* $\mathbf{M}$. *Then:*

(i) A morphism f *of* $\mathbf{M}$ *is* $\mathbf{P}$*-open in* $\mathbf{M}$ *iff* f *is* $\mathbf{P}$*-open in* $\mathbf{N}$.

(ii) The components of the counit of the adjunction $\varepsilon_X : R(X) \to X$ *are* $\mathbf{P}$*-open in* $\mathbf{M}$.

(iii) A morphism f *is* $\mathbf{P}$*-open in* $\mathbf{N}$ *iff* $R(f)$ *is* $\mathbf{P}$*-open in* $\mathbf{M}$.

2.3 $\mathbf{Pom}_L$-bisimulation for Nets

We have already seen (Lemma 22, Theorem 23) that for the well-known model of transition systems, the general definition of $\mathbf{P}$-open morphism and $\mathbf{P}$-bisimilarity coincide with familiar notions; in particular, we recover the equivalence of strong bisimilarity central to Milner's work. Here we explore how the general definitions specialise to the models of event structures and nets, with nonsequential observations in the form of pomsets.

We start by characterising $\mathbf{Pom}_L$-open morphisms on labelled asynchronous transition systems. Following our convention, we shall identify pomsets with their image under the embedding $\mathbf{E}_L \to \mathbf{A}_L$.

Proposition 29 *The* $\mathbf{Pom}_L$*-open morphisms of* $\mathbf{A}_L$ *are precisely those which satisfy the "zig-zag" condition of Proposition 22 and which, in addition, reflect consecutive independence, i.e. morphisms satisfying:*

η is total and label preserving

whenever $(\sigma(s), e', u') \in tran_2$ *then there exists* $(s, e, u) \in tran_1$ *such that* $\eta(e) = e'$ *and* $\sigma(u) = u'$

whenever $(s, e, u), (u, e', v) \in tran_1$, *with* s *reachable, and* $\eta(e) I_2 \eta(e')$ *in* T_2, *then* $e I_1 e'$ *in* T_1.

Proof: Just like the proofs of the other results of this section, the proof of this proposition is a more or less straightforward modification of the proof of the corresponding result from [5]. However, we are going to refer to parts of this proof later on, and hence we present the modified proof here in some detail.

Let $f = (\sigma, \eta) : T \to T'$ be an open morphism in $\mathbf{A}_L$. The function η is total and label preserving from definition of morphisms in $\mathbf{A}_L$, and by considering linear pomsets, where causal dependency is a total order, it is clear as in Proposition 22, that f satisfies the "zig-zag" condition. The only nontrivial part is the reflection of consecutive independence.

Suppose

$$s \xrightarrow{e} u \text{ and } u \xrightarrow{e'} v,$$

with s reachable, are two consecutive transitions in T for which

$$\sigma(s) \xrightarrow{\eta(e)} \sigma(u) \text{ and } \sigma(u) \xrightarrow{\eta(e')} \sigma(v)$$

and assume $\eta(e)$ and $\eta(e')$ are independent in T'. Assume further $l(e) = l(\eta(e)) = a$ and $l(e') = l(\eta(e')) = a'$.

Because s is reachable there is a chain of transitions

$$i = s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} s_n = s$$

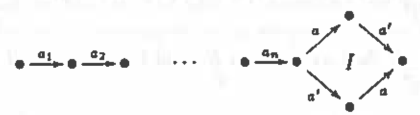
in T from its initial state i. Assume $l(e_i) = a_i$. Let P be the linear pomset with $n+2$ elements, ordered and labelled as indicated in the following associated labelled asynchronous transition system (only labels indicated for the transitions):

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \quad \cdots \quad \bullet \xrightarrow{a_n} \bullet \xrightarrow{a} \bullet \xrightarrow{a'} \bullet$$

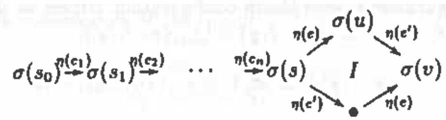
Let $p : P \to T$ be that morphism in $\mathbf{A}_L$ which maps this chain of transitions to

$$s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} \quad \cdots \quad \xrightarrow{e_n} s \xrightarrow{e} u \xrightarrow{e'} v.$$

in T. Let Q be the pomset differing from P only in that the a and a' labelled elements are unordered, i.e. the pomset associated with the following labelled asynchronous transition system:



Let $q : Q \to T'$ be that morphism in $\mathbf{A}_L$ mapping these transitions to



in T'. Letting $m : P \to Q$ be the obvious morphism of pomsets, we observe the commuting diagram:

$$\begin{array}{ccc} P & \xrightarrow{p} & T \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow{q} & T' \end{array}$$

But f is open, so we obtain a morphism $p' : Q \to T$ such that the two "triangles" commute in:

$$\begin{array}{ccc} P & \xrightarrow{p} & T \\ {\scriptstyle m}\downarrow & \nearrow^{p'} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow{q} & T' \end{array}$$

Because p' preserves independence, we see that e and e' are independent in T. So because f is open it satisfies the "zig-zag" condition and reflects consecutive independence.

For the proof in the other direction we refer to [5].

□

And now to the question of bisimulations. In [5] it was shown that in the case of event structures taking the path category **P** to be pomsets one gets a reasonable strengthening of a previously studied equivalence, that of *history-preserving bisimulation*. Its definition depends on the simple but important remark, that a configuration of an event structure can be regarded as a pomset, with causal dependency relation and labelling got by restricting that of the event structure.

Definition: (Rabinovitch-Trakhtenbrot [18], van Glabeek-Goltz [6])

A *history-preserving bisimulation* between two event structures E_1, E_2 consists of a set H of triples (x_1, f, x_2) where x_1 is a configuration of E_1, x_2 a configuration of E_2 and f is a isomorphism between them (regarded as pomsets), such that $(\emptyset, \emptyset, \emptyset) \in H$ and, whenever $(x_1, f, x_2) \in H$

(i) if $x_1 \xrightarrow{a} x_1'$ in E_1 then $x_2 \xrightarrow{a} x_2'$ in E_2 and $(x_1', f', x_2') \in H$ with $f \subseteq f'$, for some x_2' and f'.

(ii) if $x_2 \xrightarrow{a} x_2'$ in E_2 then $x_1 \xrightarrow{a} x_1'$ in E_1 and $(x_1', f', x_2') \in H$ with $f \subseteq f'$, for some x_1' and f'

We say a history-preserving bisimulation is *coherent* when it further satisfies

(iii) for all configuration-isomorphism-configuration triples as above, (x, f, y), (x_i, f_i, y_i), $i \in I$ such that $(x, f, y) = \bigcup_{i \in I}(x_i, f_i, y_i)$,

$$(x, f, y) \in H \text{ iff } \forall i \in I.\ (x_i, f_i, y_i) \in H.$$

Notice, that the only if-part of condition (iii) is exactly the notion of strongness introduced in [5], where it was shown that this extra assumption is indeed a strengthening of history-preserving bisimulation, and that it provides a characterization of $\mathbf{Pom}_L$-bisimilarity for a category of (noncoherent) labelled event structures. Presently, we do not know whether or not the if-part of (iii) is a further strengthening of the notion of bisimilarity. However, it turns out that because we are working with coherent models here ($\mathbf{E}_L$, $\mathbf{A}_L$ and $\mathbf{N}_L$), the modified proofs from [5] lead naturally to characterizations in terms of coherent history-preserving bisimulation.

Theorem 30

(i) Two event structures, with labelling sets L, are $\mathbf{Pom}_L$-bisimilar iff they are coherent history-preserving bisimilar.

(ii) Two nets, with label sets L, are $\mathbf{Pom}_L$-bisimilar iff their case graphs as labelled asynchronous transition systems are $\mathbf{Pom}_L$-bisimilar iff their unfoldings to event structures are coherent history-preserving bisimilar.

Proof: For the proof of (i) we refer to [5]. There a proof is provided for a chatacterization of $\mathbf{Pom}_L$-bisimilarity for a noncoherent version of $\mathbf{E}_L$ in terms of strong history-preserving bisimulation. The proof here is basically a repeat with a few extra arguments dealing with coherence. In the proof construction of history-preserving bisimulation from a span of $\mathbf{Pom}_L$-open maps, coherence follows from the coherence of models in $\mathbf{E}_L$. And assuming coherence of a history-preserving bisimulation allows you to conclude that the construction of span of $\mathbf{Pom}_L$-open maps is indeed within (the coherent) $\mathbf{E}_L$.

For the proof of (ii), assuming N_1, N_2 are $\mathbf{Pom}_L$-bisimilar, there is a span of open morphisms in $\mathbf{N}_L$ whose image under na_0 is a span of open morphisms in $\mathbf{A}_L^0$ (by Lemma 28). This ensures the case graphs $na_0(N_1)$, $na_0(N_2)$ are $\mathbf{Pom}_L$-bisimilar in $\mathbf{A}_L$ by Proposition 27. From the same reasoning, this in turn implies that the unfoldings $ae_0 \circ na_0(N_1)$, $ae_0 \circ na_0(N_2)$ are $\mathbf{Pom}_L$-bisimilar as event structures, and hence coherent history-preserving bisimilar from (i).

On the other hand, by the proof of part (i), assuming the unfoldings of N_1 and N_2 are coherent history-preserving bisimilar we obtain a span of open morphisms in $\mathbf{N}_L$:

$$\begin{array}{ccccc} & & N & & \\ & {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} & \\ en \circ ne(N_1) & & & & en \circ ne(N_2) \end{array}$$

Composing with components of the counit

$$\varepsilon_1 : en \circ ne(N_1) \to N_1,$$
$$\varepsilon_2 : en \circ ne(N_2) \to N_2,$$

which are open by Lemma 28, we obtain a span of open morphisms relating N_1, N_2. □

So, for general reasons, the notion of bisimulation for nets agrees with the notion of bisimulation for the associated case graphs and unfoldings. These are properties which probably would be required by any notion of bisimulation, and which normally require individual proofs.

Many attempts have been made of defining bisimulation for noninterleaving models like Petri nets. Also the idea of parameterizing such definitions on a notion of observation is not new, see e.g. [3]. However, there are major differences. To bring out one, we briefly address the question of robustness of our notion of bisimulation. The question is how sensitive our notion of $\mathbf{Pom}_L$-bisimilarity for nets is to the particular choice of category of observations $\mathbf{Pom}_L$. In particular the notion may seem questionable to everybody holding the view that pomsets in their full generality are not observable at all.

However, let us define a pomset to be an *Almost Totally Ordered Multiset* iff it is of one of the two simple forms considered in the proof of Proposition 29, i.e. allowing at most two (maximal) elements to be unordered. Note that in the range of subclasses of *pomsets* considered in the literature, [17], this class is as close to $\mathbf{Bran}_L$ as one can get! Let us denote the full subcategory of $\mathbf{Pom}_L$ consisting of object of this simple form by $\mathbf{Atom}_L$.

Corollary 31

(i) A morphism in $\mathbf{N}_L$ *is* $\mathbf{Pom}_L$*-open iff it is* $\mathbf{Atom}_L$*-open.*

(ii) Two nets are $\mathbf{Pom}_L$*-bisimilar iff they are* $\mathbf{Atom}_L$*-bisimilar.*

Proof: Clearly (ii) follows from (i), so we concentrate on a proof of (i).
The *"only if"* part of (i) follows immediately from definition of open maps.
The *if* part in the category $\mathbf{A}_L$ follows from the proof of Proposition 29. But then it also holds in the category $\mathbf{A}_L^0$ from Proposition 27, and hence also in $\mathbf{N}_L$ by Lemma 28. □

3 Concluding remarks

We have illustrated how to introduce bisimulation for Petri nets following a general pattern, a pattern which automatically guarantees consistency with bisimulation on a number of related models. But, this just sets the scene, and many questions are left open. For instance, it is desirable to have a more operational characterization of $\mathbf{Pom}_L$-bisimulation for nets, e.g. in the spirit of Milner's original definition of bisimulation for transition systems. One obvious idea would be to modify the game theoretic characterization for $\mathbf{Pom}_L$-bisimulation for transition systems with independence given in [13] to nets. Another challenging issue is the decidability of our equivalences, about which very little is known at present.

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