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Scalings in Linear Programming: Necessary and Sufficient Conditions for Invariance

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Abstract

We analyze invariance of the conclusion of optimality for the linear programming problem under scalings (linear, affine, . . .) of various problem parameters such as: the coefficients of the objective function, the coefficients of the constraint vector, the coefficients of one or more rows (columns) of the constraint matrix. Measurement theory concepts play a central role in our presentation and we explain why such approach is a natural one.

Keywords: Sensitivity Analysis, Scaling, Measurement

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1 Introduction

In this paper we analyze robustness of an optimal solution to the linear programming problem[1]

$$\begin{array}{l} \max P(\mathbf{x};\mathbf{w}) = \mathbf{w}^T\mathbf{x} \\ \text{subject to:} \\ \mathbf{x} \in F := \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq \mathbf{0} \;\&\; A\mathbf{x} = \mathbf{b}\} \end{array} \tag{1}$$

More precisely, we give conditions under which an optimal solution (or basic variables determining it) is invariant under *scaling* of various problem parameters such as the coefficients of the objective function $w_1, \ldots, w_n$, coefficients of the constraint matrix A (a_{ij}, $i, j = 1, \ldots, n$), and coefficients of the constraint vector $\mathbf{b} = (b_1, \ldots, b_m)^T$. By scaling of parameters $p_1, \ldots, p_k$ by a function Φ from some set of functions S, we mean replacing $p_1, \ldots, p_k$ by $\Phi(p_1), \ldots, \Phi(p_k)$. Although scaling would usually mean scaling by increasing linear functions (i.e., $S = \{\Phi : \mathbf{R} \to \mathbf{R} : \Phi(x) = \alpha x, \alpha > 0\}$), we also consider cases where various problem parameters are scaled by increasing affine functions $S = \{\Phi : \mathbf{R} \to \mathbf{R} : \Phi(x) = \alpha x + \beta, \alpha > 0\}$), and by even more general sets of functions. This approach to analyzing robustness of an optimal solution to the problem (1) is different than the standard sensitivity analysis approach where only the effects of small perturbations of problem parameters are considered. So, why should one be interested in the perturbations of data that arise from scalings? For mathematical purists, there is no justification needed since characterizing invariance of the conclusion of optimality under scaling of certain parameters by functions from a set S is an interesting mathematical question *per se*. Moreover, any characterization of this type might give better understanding of the linear programming problems. However, our motivation comes from situations that might arise in practice: whenever problem parameters represent data that can be measured in more than one acceptable way (e.g., costs/profits can be measured in US dollars, Danish kroner, Croatian kunas, or any other currency; time can be measured in seconds, minutes, hours,....). As will be shown (in Section 3), the way data is measured gives rise to a set of scaling functions in a natural way.

[1] Throughout we use a notational convention that boldfaced letters denote vectors, that is $\mathbf{x} = (x_1, \ldots, x_n)^T$. We also assume that $A \in M_{m,n}(\mathbf{R})$ is of full rank and we assume that $m \leq n$, i.e., $\text{rank}(A) = m$.

Example 1 Consider the production problem where n items could be produced and each produced unit of the j-th item brings the net profit/loss of w_j. Suppose that the production of any item requires m different resources (machines, raw materials, etc.) and suppose that one unit of the j-th item requires a_{ij} of the i-th resource. Further suppose that there is at most b_i of the i-th resource. Then the problem of maximizing the profit can be formulated as the linear programming problem (1)[2].

There are numerous problem parameters that can be presented in different but equally acceptable ways. For example, the coefficients of the objective function $w_1, \ldots, w_n$ can be expressed in US dollars but they can be expressed in German marks or any other currency. Note that any transformation from one monetary unit to another is an increasing linear function. If a flat fee needs to be paid for every currency conversion (say, price of an option to buy a currency at a given exchange rate), then a transformation from one monetary unit to another is an increasing affine function.

Similarly, for any i, there might be more than one way to represent $a_{i1}, \ldots, a_{in}, b_i$. For example, a_{ij} might represent time that the j-th item needs to spend on the i-th machine. This time can be measured in seconds, minutes, hours,.... In fact, relationship between any two acceptable ways to represent such data is described by an increasing linear transformation.[3]

Our presentation is built up on measurement theory concepts. In particular scales of measurement (see Section 2) and the concept of meaningfulness (see Section 3) are central notions of our presentation. The main reason for such approach is not only to expose a natural connection between meaningfulness and invariance under scalings, but also to show that invariance under scalings of problem parameters is a natural object of sensitivity analysis. This paper is an attempt to illustrate this type of analysis in the case of the linear programming problem which is the most widely used mathematical model for optimization problems arising in practice. However, investigation of meaningfulness of the conclusion of optimality is a theme that goes far beyond the scope of this paper. Roberts [7, 8] was first to point out a variety

[2]Note that we may assume that all constraints are in fact equalities. For example, any unused portion of the i-th resource can be viewed as another production item (it can be resold, stored, disposed,... at some per unit profit/loss w_{n+i})

[3]Of course, it is trivial to see that the conclusion of optimality is invariant under such transformations of problem parameters. The point of this example was to illustrate how measurement of problem parameters gives rise to scaling functions

of problems concerning meaningfulness of the statements that can be drawn from various mathematical models that are commonly used to solve operations research problems. Detailed treatment of the meaningfulness of the conclusion of optimality in combinatorial optimization problems can be found in [4] (in fact, this paper is an improved version of part of work presented there). Also, systematic analysis of the meaningfulness of the conclusion of optimality for single machine scheduling problems can be found in [3].

In most of the situations that will be analyzed here, it will be clear that there cannot be an optimal solution invariant under the given set of scaling functions. However, we will characterize situations in which the set of **basic variables** that defines the **optimal basis** and, hence, an optimal solution is invariant to such scalings. In some sense, the set of the basic variables determining an optimal solution is a natural object of our analysis. It describes only fundamental relationship between the set of feasible solutions F and the objective function P, eliminating actual numerical values of the problem parameters to the largest possible extent.

Let us introduce the notation by briefly defining basics about optimality of the problem (1):

$A\mathbf{x} = \mathbf{b}$ can be written as

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = \mathbf{b} \tag{2}$$

where A_i denotes the i-th column of A. For any m linearly independent columns of A: $A_{i_1}, \ldots, A_{i_m}$, there exists a unique solution $(x_{i_1}, \ldots, x_{i_m})$ to the system of m linear equalities:

$$A_{i_1} x_{i_1} + A_{i_2} x_{i_2} + \ldots + A_{i_m} x_{i_m} = \mathbf{b}$$

This solution can be extended to a solution of the system $A\mathbf{x} = \mathbf{b}$ by setting $x_k = 0$ for all $k \notin \{i_1, \ldots, i_m\}$. Such a solution is called a **basic solution** to the system $A\mathbf{x} = \mathbf{b}$ and variables $i_1, \ldots, i_m$ are called **basic variables** while the other $n - m$ variables are called **non-basic variables**. We say that $\mathbf{x}$ is a **basic feasible solution** if $\mathbf{x}$ is a basic solution and $\mathbf{x} \geq \mathbf{0}$[4].

An $m \times m$ non-singular matrix whose k-th column is A_{i_k} will be denoted by B. We will call B a **basis** since by giving B we actually prescribe basic variables. $\mathbf{w}_B$ will denote the vector $(w_{i_1}, w_{i_2}, \ldots, w_{i_m})^T$ and $\mathbf{x}_B$ will denote

[4]Every basic feasible solution corresponds to an extreme point of F and every extreme point of F corresponds to (one or more) basic feasible solutions (see, for example, [1, 2]).

the vector $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})^T$. In this notation, the basic variables for the basic feasible solution $\mathbf{x}$ corresponding to a basis B are given by $\mathbf{x}_B = B^{-1}\mathbf{b}$ (since $B\mathbf{x}_B = \mathbf{b}$). Moreover, we have $\mathbf{w}^T\mathbf{x} = \mathbf{w}_B{}^T\mathbf{x}_B$ since for every non-basic variable k, $x_k = 0$.

The notation

$$z_j := \mathbf{w}_B^T B^{-1} A_j$$

will be used throughout this paper

It is straightforward to show (see, for example, [1] or [2]) that basic feasible solution $\mathbf{x}^*$ corresponding to basis B is an optimal solution to problem (1) if and only if

$$z_j - w_j \geq 0, \quad j = 1, \ldots, n. \tag{3}$$

This characterization of an optimal solution to problem (1) is known as the **optimality criterion**. The basis B corresponding to an optimal solution $\mathbf{x}^*$ is called an **optimal basis**.

As already mentioned, in the next section we will introduce the measurement theory terminology. Throughout the paper, this terminology will be used to describe and analyze invariance under various scalings. Section 3 is conceptually the most important part of this paper: in this section we define the concept of meaningfulness, overview our results, and give proofs for some simple cases. The cases that are technically more complicated will be analyzed in Section 4. Finally, we give some closing remarks in Section 5.

2 Background–Measurement Theory Terminology

Invariance of the conclusion of optimality under scaling of certain parameters of the problem (1) might not only be a nice property of an optimal solution, but it is often a necessary condition if the linear programming formulation is used to model a problem whose parameters are numerical representations of problem data. We will show how the information about the scale type of data often gives rise to a set of scaling functions S with the property that the conclusion of optimality must be invariant under scalings by functions from S. We first introduce some basic measurement theory concepts (following [6]).

Obviously, measurement has something to do with assigning numbers that correspond to or "preserve" certain observed relations. Formally, objects being measured together with relations that should be preserved define a **relational system**, i.e., an ordered $(p+q+1)$-tuple $(A, R_1, \ldots, R_p, \mathbf{o}_1, \ldots, \mathbf{o}_q)$, where A is a set of objects, $R_1, \ldots, R_p$ are (not necessarily binary) relations on A, and $\mathbf{o}_1, \ldots, \mathbf{o}_q$ are binary operations on A. The **type** of the relational system is a sequence $(r_1, \ldots, r_p; q)$ of length $p+1$ where r_i denotes that R_i is an r_i-ary relation (i.e., $R_i \subseteq A^{r_i}$). A **scale of measurement** is defined by a mapping (into some relational system on $\mathbf{R}$) that preserves relations and binary operations of the relational system whose elements are being measured, i.e., a scale of measurement is a triple $(\mathcal{A}, \mathcal{B}, f)$ where $\mathcal{A}$ and $\mathcal{B}$ are relational systems of the same type (and $\mathcal{B}$ is a relational system on $\mathbf{R}$.), and $f : A \rightarrow B$ is a **homomorphism** of relational systems (that is, for all $a_1, a_2, \ldots, a_{r_i} \in A$, $R_i(a_1, a_2, \ldots, a_{r_i}) \Leftrightarrow R'_i(f(a_1), f(a_2), \ldots, f(a_{r_i})) \quad i = 1, \ldots, p$, and for all $a, b \in A$, $f(a\mathbf{o}_i b) = f(a)\mathbf{o}'_i f(b), \quad i = 1, \ldots, q)$. Every scale of measurement $(\mathcal{A}, \mathcal{B}, f)$ defines a **representation** $\mathcal{A} \rightarrow \mathcal{B}$.

As already point out, usually there is more than one acceptable way to measure objects from $\mathcal{A}$ (i.e., there can be more than one representation $\mathcal{A} \rightarrow \mathcal{B}$). Let $\mathcal{S}_1 = (\mathcal{A}, \mathcal{B}, f)$ and $\mathcal{S}_2 = (\mathcal{A}, \mathcal{B}, g)$ be two scales of measurement. If there exists $\Phi : f(A) \rightarrow \mathbf{R}$ such that $\Phi \circ f = g$, then we say that Φ is an **admissible transformation** of $\mathcal{S}_1$ (since $g = \Phi \circ f$ defines $\mathcal{S}_2$, i.e., another acceptable way to measure objects from $\mathcal{A}$). The set of all admissible transformations of f (which defines $\mathcal{S}_1$) is denoted by $AT(f)$. f is a **regular** homomorphism (and $\mathcal{S}_1$ is a **regular** scale) if for any other scale $(\mathcal{A}, \mathcal{B}, h)$, there exists $\Phi_h \in AT(f)$ such that $h = \Phi_h \circ f$. Regular scales are important since $(\mathcal{A}, \mathcal{B}, f)$ and the corresponding $AT(f)$ define any other scale $(\mathcal{A}, \mathcal{B}, h)$. If any scale $(\mathcal{A}, \mathcal{B}, f)$ is regular then we say that $\mathcal{A} \rightarrow \mathcal{B}$ is a **regular representation**.

All the homomorphisms (scales, representations) that will be considered in the rest of this paper are assumed to be regular.[5]

The set $AT(f)$ defines the **scale type** of f. If all homomorphisms of a representation $\mathcal{A} \rightarrow \mathcal{B}$ have the same scale type (as will be the case with all scales that will be considered) then the scale type of a representation $\mathcal{A} \rightarrow \mathcal{B}$ is defined to be the scale type of any of its homomorphisms. We will refer to the following scale types that often appear in practice:

• **Absolute scales**. f is an absolute scale if $AT(f) = \{id\}$. The scale values are predetermined here and there is a unique way to measure data.

[5] This is a natural assumption since it can be shown that any homomorphism can be reduced to a regular one in a natural way [4, 6].

• **Ratio scales**. f is a ratio scale if $AT(f)$ is the set of all increasing linear transformations. It is not hard to see that in this case the scale value is determined up to choice of a unit. Examples of ratio scales are measurement of cardinal utility (i.e., monetary amounts), time, length, mass, ...
• **Interval scales**. f is an interval scale if $AT(f)$ is the set of all increasing affine transformations. In other words, the scale value is determined up to choice of a unit and choice of a zero point. For example, temperature is measured on an interval scale. Also, cardinal utility with one unknown parameter (for example per-unit production cost with the unknown fixed start-up cost), or time with one unknown parameter (e.g., calendar time, or per-unit processing time with unknown fixed start-up time) are examples interval scale type data.
• **Ordinal scales**. f is an ordinal scale if $AT(f)$ is the set of all increasing functions. Hence, the scale value is determined only up to order. For example, whenever only the ordering among objects being measured is known, we have an ordinal scale.

If $AT(f) \subseteq AT(g)$, we say that the scale of f is **stronger** than or equal to the scale of g (or the scale of g is **weaker** than or equal to the scale of f). For example, we can order scales introduced here from the strongest towards the weakest: absolute, ratio, interval, ordinal.

3 Meaningfulness of the conclusion of optimality

The central measurement theory concept that will be used throughout this paper is that of meaningfulness. We say that a statement involving scales of measurement is **meaningful** if its truth value is unchanged whenever every scale $(\mathcal{A}, \mathcal{B}, f)$ is replaced by another (acceptable) scale $(\mathcal{A}, \mathcal{B}, \Phi \circ f)$, $\Phi \in AT(f)$. In other words a statement is meaningful if it has the same truth value (always true or always false) regardless of the choice of the homomorphism $f : \mathcal{A} \to \mathcal{B}$. Therefore, meaningful statements are unambiguous in their interpretation and they say something about fundamental relations among the objects being measured. Statements which are not meaningful are **meaningless**. In other words, a statement involving scales is meaningless if there exists a scale $(\mathcal{A}, \mathcal{B}, f)$ and a scale $(\mathcal{A}, \mathcal{B}, \Phi \circ f)$, $\Phi \in AT(f)$, such that the statement is false. Obviously, we cannot put much weight behind such statements since their truth value is an accident of the choice of an (acceptable) way to measure the objects in question.

In this paper we analyze the meaningfulness of the conclusion of optimality for the linear programming problem (1). It should be noted that we may assume that *every* parameter of the problem (1) (i.e., coefficients of the objective function $w_1, \ldots, w_n$, coefficients of the constraint matrix $a_{11}, \ldots a_{mn}$ and coefficients of the resource vector $b_1, \ldots, b_m$) and the value of every variable $x_1, \ldots, x_n$ represent data measured on some scale of measurement (since everything is being measured on at least an absolute scale of measurement).

We say that $\mathbf{x}^*$ is a **meaningful optimal solution** to the problem (1) if and only if $\mathbf{x}$ is an optimal solution and the conclusion of optimality is a meaningful statement (with respect to all scales of measurement). More formally, suppose that there are k different scales of measurement involved in formulation of the problem (1) and suppose that $f_1, \ldots, f_k$ are regular homomorphisms representing all the data. Then an optimal solution $\mathbf{x}^*$ for problem (1) is meaningful if and only if, for any choice of $\Phi_1, \ldots, \Phi_k$ ($\Phi_i \in AT(f_i)$, $i = 1, \ldots, k$), $\bar{\mathbf{x}}^*$ is an optimal solution to the problem

$$\begin{array}{l} \max P(\mathbf{x}, \bar{\mathbf{w}}) = \bar{\mathbf{w}}^T \mathbf{x} \\ \text{subject to:} \\ \mathbf{x} \in \bar{F} := \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq \mathbf{0} \ \& \ \bar{A}\mathbf{x} = \bar{\mathbf{b}}\} \end{array}$$

where $\bar{x}_s := \Phi_{i(x_s)}(x_s)$, $\bar{w}_s := \Phi_{i(w_s)}(w_s)$, the coefficients of matrix $\bar{A}$ are $\bar{a}_{rs} := \Phi_{i(a_{rs})}(a_{rs})$, and $\bar{b}_r := \Phi_{i(b_r)}(b_r)$ for x_s, w_s, a_{rs}, b_r being data measured on the $i(x_s)$-th, $i(w_s)$-th, $i(a_{rs})$-th, and $i(b_r)$-th scale of measurement, respectively (that is, for any r and s $\Phi_{i(x_s)}$ is a regular homomorphism representing x_s, $\Phi_{i(w_s)}$ is a regular homomorphism representing w_s, $\Phi_{i(a_{rs})}$ is a regular homomorphism representing a_{rs}, and $\Phi_{i(b_r)}$ is a regular homomorphism representing b_r). In what follows we adopt the convention that if only the scale of some of the parameters is mentioned, it is assumed that the other parameters are fixed (i.e., measured on an absolute scale).

Remark. Suppose that $\mathbf{x}^*$ is an optimal solution to problem (1). Let $\{p_1, \ldots, p_l\}$ be some set of the problem parameters (i.e., each p_i is a coefficient of either $\mathbf{w}$ or A or $\mathbf{b}$). Let $\mathcal{S}$ be a scale of measurement with a property that there exists a set of functions G such that for every homomorphism f, the set of admissible transformations $AT(f) = G$. Then the following are equivalent:

(a) $\mathbf{x}^*$ is a meaningful optimal solution if $\{p_1, \ldots, p_l\}$ are measured on a

common scale of measurement $\mathcal{S}$.
(b) The conclusion that $\mathbf{x}^*$ is an optimal solution to problem (1) is invariant under scaling of $\{p_1, \ldots, p_l\}$ by any function from G.

For example, invariance of the conclusion of optimality of $\mathbf{x}^*$ under scaling of $\{p_1, \ldots, p_l\}$ by increasing linear (affine) transformations is equivalent to $\mathbf{x}^*$ being a meaningful optimal solutions if $\{p_1, \ldots, p_l\}$ represent data measured on a ratio (interval) scale of measurement.

In what follows we will discuss several situations:

- Coefficients of the objective function $(w_1, \ldots, w_n)$ are measured on a (common) scale of measurement (Theorem 2, and Corollary 3).
- Coefficients of the i-th row of the matrix A $(a_{i1}, \ldots, a_{in})$ together with b_i (i-th coordinate of the vector $\mathbf{b}$) are measured on a (common) scale of measurement $\mathcal{S}_i$ (Proposition 4 and Corollary 5).
- Coefficients of the i-th row of the matrix A $(a_{i1}, \ldots, a_{in})$ are measured on a (common) ratio scale of measurement **or** b_i is measured on some scale of measurement $\mathcal{S}_i$ (Proposition 6, Theorem 8, Corollary 9, Corollary 10, Theorem 12, and Corollary 13).
- Coefficients of the j-th column of the matrix A $(a_{1j}, \ldots, a_{mj})$ together with w_j (j-th coefficient of the objective function) are measured on a (common) scale of measurement $\mathcal{S}_i$ (Proposition 15).
- Coefficients of the j-th column of the matrix A $(a_{1j}, \ldots, a_{mj})$ are measured on a (common) ratio scale of measurement **or** w_j is measured on some scale of measurement $\mathcal{S}_i$ (Theorem 17, Corollary 18, Theorem 19, Corollary 20, and Theorem 21).

The following notation will be used in the proof of the next theorem and throughout the rest of the paper: for $\Phi : \mathbf{R} \to \mathbf{R}$, and $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbf{R}^n$, we denote $\Phi(\mathbf{x}) := (\Phi(x_1), \ldots, \Phi(x_n))^T$.

Theorem 2 *Let $\mathbf{x}^*$ be an optimal solution for problem (1).*

1. *$\mathbf{x}^*$ is a meaningful optimal solution if $w_1, \ldots, w_n$ are measured on a ratio scale.*
2. *Suppose that $w_1, \ldots, w_n$ are measured on an interval scale. Then $\mathbf{x}^*$ is a meaningful optimal solution if and only if $x_1 + x_2 + \ldots + x_n$ is a constant on F.*

Proof: 1. follows from the fact that $(\alpha\mathbf{w})^T\mathbf{x} = \alpha\mathbf{w}^T\mathbf{x}$, i.e., for any $\alpha > 0$, $(\alpha\mathbf{w})^T\mathbf{x}^* \geq (\alpha\mathbf{w})^T\mathbf{y}$ if and only if $\mathbf{w}^T\mathbf{x}^* \geq \mathbf{w}^T\mathbf{y}$.

In order to prove 2., first note that, for any $\Phi(t) = \alpha t + \beta$, $[\Phi(\mathbf{w})]^T\mathbf{x} = \alpha\mathbf{w}^T\mathbf{x} + \beta(1, \ldots, 1)\mathbf{x}$. Hence, for any $\alpha > 0$, $[\Phi(\mathbf{w})]^T\mathbf{x}^* \geq [\Phi(\mathbf{w})]^T\mathbf{y}$ if and only if

$$\mathbf{w}^T\mathbf{x}^* \geq \mathbf{w}^T\mathbf{y} + \frac{\beta}{\alpha}(1, \ldots, 1)(\mathbf{x} - \mathbf{y}).$$

If $x_1 + x_2 + \ldots + x_n$ is a constant on F (i.e., if for any $\mathbf{x}, \mathbf{y} \in f$, $(1, \ldots, 1)(\mathbf{x} - \mathbf{y}) = 0$), then $[\Phi(\mathbf{w})]^T\mathbf{x} \geq [\Phi(\mathbf{w})]^T\mathbf{y}$ if and only if $\mathbf{w}^T\mathbf{x} \geq \mathbf{w}^T\mathbf{y}$. Conversely, if $x_1 + x_2 + \ldots + x_n$ is not a constant on F, then there exists $\mathbf{y} \in F$ such that $(1, \ldots, 1)(\mathbf{x} - \mathbf{y}) \neq 0$ and we can find $\alpha > 0$ and β such that $[\Phi(\mathbf{w})]^T\mathbf{x}^* < [\Phi(\mathbf{w})]^T\mathbf{y}$. In other words, $\mathbf{x}^*$ is not a meaningful optimal solution. ▮

It should be noted that Theorem 2 holds as long as the objective function is linear, i.e., we need no assumptions whatsoever on the structure of the set F. (In fact, the theorem also holds for more general objective functions; see [4])

We use the special structure of the set of feasible solutions F to obtain

Corollary 3 *Suppose that $w_1, \ldots, w_n$ are measured on an interval scale and let $\mathbf{x}^*$ be an optimal solution for problem (1). Then $\mathbf{x}^*$ is a meaningful optimal solution if and only if there exists $\lambda \in \mathbf{R}^m$ such that $\lambda^T A = (1, 1, \ldots, 1)$.*

Proof: By Theorem 2, if $w_1, \ldots, w_n$ are measured on interval scale then $\mathbf{x}^*$ is a meaningful optimal solution if and only if $x_1 + x_2 + \ldots + x_n$ is a constant on $F := \{\mathbf{x} \geq \mathbf{0} : A\mathbf{x} = \mathbf{b}\}$. Note that any $\mathbf{x} \in F$ can be written as $\mathbf{x}_0 + \mathbf{y} \geq \mathbf{0}$ where $\mathbf{x}_0$ is some fixed element from F and $A\mathbf{y} = \mathbf{0}$. Moreover, $x_1 + x_2 + \ldots + x_n$ is a constant on F if and only if $y_1 + y_2 + \ldots + y_n = 0$ for every y such that $\mathbf{x}_0 + \mathbf{y} \in F$. Since $rank(A) = m$ all the rows of A are linearly independent and

$$\{\mathbf{y} : A\mathbf{y} = \mathbf{0}\} \tag{4}$$

is an $(n-m)$-dimensional subspace of $\mathbf{R}^n$. By adding an additional equality (an additional row in A) $y_1 + y_2 + \ldots + y_n = 0$, the set

$$\{\mathbf{y} : A\mathbf{y} = \mathbf{0}, y_1 + y_2 + \ldots + y_n = 0\}$$

will be the same as the set (4) if and only if $(1, \ldots, 1)$ can be expressed as a linear combination of the rows of A. Of course, this will be the case if and only if there exists a $\lambda \in \mathbf{R}^m$ such that $\lambda^T A = (1, 1, \ldots, 1)$ ▮

Proposition 4 *Suppose that $a_{i1}, a_{i2}, \ldots, a_{in}, b_i$ are measured on a scale of measurement $\mathcal{S}$.*

1. *If $\mathcal{S}$ is a ratio scale then every optimal solution $\mathbf{x}^*$ to problem (1) is meaningful.*

2. *If $\mathcal{S}$ is an interval scale and if*

$$x_1 + x_2 + \ldots + x_n = 1 \tag{5}$$

for all $\mathbf{x} \in F$, then then every optimal solution $\mathbf{x}^*$ to problem (1) is meaningful.

Proof: Obviously,

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$$

holds if and only if

$$\alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \ldots + \alpha a_{in}x_n = \alpha b_i$$

holds for any $\alpha > 0$ and this proves 1. Similarly, 2. follows directly from the obvious fact that for any choice of $\alpha > 0$ and β

$$(\alpha a_{i1} + \beta)x_1 + (\alpha a_{i2} + \beta)x_2 + \ldots + (\alpha a_{in} + \beta)x_n = \alpha b_i + \beta$$

if and only if

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i.$$

(This is because $x_1 + x_2 + \ldots + x_n = 1$.) ▮

Corollary 5 *Suppose that $a_{i1}, a_{i2}, \ldots, a_{in}, b_i$ are measured on a scale of measurement $\mathcal{S}_i$, $i = 1, \ldots, m$.*

1. *If for every $i = 1, \ldots, m$, $\mathcal{S}_i$ is a ratio scale or stronger, then every optimal solution $\mathbf{x}^*$ to problem (1) is meaningful.*

2. *If for every $i = 1, \ldots, m$, $\mathcal{S}_i$ is an interval scale and if (5) holds for all $\mathbf{x} \in F$, then then every optimal solution $\mathbf{x}^*$ to problem (1) is meaningful.*

Proof: Same as the proof of Proposition 4. ▮

Remark. Without the condition (5) the conclusion of feasibility of x is a meaningless statement in general and, consequently, the conclusion of optimality might be a meaningless statement. It might be the case that a particular $\mathbf{x}$ is in F for any choice of $\Phi(a_{i1}), \Phi(a_{i2}), \ldots, \Phi(a_{in}), \Phi(b_i)$. It is possible that there exists a meaningful optimal solution and that the conditions of Corollary 5 are not met. For example, consider problem (1) where

$$\mathbf{w}^T = (1,0,0,0),\ A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix},\ \mathbf{b}^T = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

It is easy to see that $\mathbf{x}^* = (1,0,0,0)^T$ is a meaningful optimal solution whenever a_{21}, a_{22}, a_{23}, a_{24}, b_2 are measured on an interval scale. This is because $\mathbf{x}^* \in F$ for any choice of $\alpha a_{21} + \beta$, $\alpha a_{22} + \beta$, $\alpha a_{23} + \beta$, $\alpha a_{24} + \beta$, $\alpha b_2 + \beta$ and because $x_1 \leq 1$ (this follows from the first equality and $x_i \geq 0$).

If we just change $\mathbf{w}^T$ in the example to be $(1,1,0,0)$, the conclusion of optimality won't be a meaningful statement anymore. $\mathbf{x}^* = (1,1,0,0)^T$ is not even feasible whenever $\beta \neq 0$.

4 Invariance of basic variables

In this section we will consider situations where the statement "$\mathbf{x} \in F$" will be obviously meaningless. However, one might be interested in determining if there exists a basis such that the corresponding $\mathbf{x}^*$ is optimal regardless of the choice of acceptable ways to measure problem data. In other words, we will investigate the meaningfulness of the statement "*the basic variables of the basis B determine an optimal basis*".

The first such situation is where only b_i are measured on scales of measurement $\mathcal{S}_i$. Any transformation $\Phi_i(b_i)$, can be viewed as replacing b_i with $b_i' := \alpha_i^{-1} b_i$ where $\alpha_i := b_i/\Phi(b_i)$. Obviously, in this case, the statement "$\mathbf{x} \in F$" is meaningful if and only if $b_i = b_i'$. However, we will be interested in determining if there exists a basis such that the corresponding $\mathbf{x}^*$ is optimal for any choice of $\Phi_i(b_i)$, where Φ_i is an admissible transformation for the scale of measurement of b_i. This question is equivalent to analyzing whether there exists a basis which is optimal whenever $a_{i1}, a_{i2}, \ldots, a_{in}$ are replaced by $\alpha_i a_{i1}, \alpha_i a_{i2}, \ldots, \alpha_i a_{in}$. For example, if b_i is measured on a ratio scale, then α_i can be any positive real number. Even if b_i (note that $\mathbf{b} \geq 0$ by the definition of problem (1)) were measured on an interval, ordinal or even nominal scale on $\mathbf{R}_+$ we still have $\alpha \in \mathbf{R}_+$ and we can view this problem as a problem where $a_{i1}, a_{i2}, \ldots, a_{in}$ are measured on a ratio scale.

The simplest possible example is when all b_i are measured on the same ratio scale $\mathcal{S}$.

Proposition 6 *Suppose that $b_1, \ldots, b_m$ are measured on a (common) ratio scale. Then the conclusion that B is an optimal basis for problem (1) is a meaningful statement.*

Proof: Clearly, for any $\alpha > 0$, $\mathbf{w}^T\mathbf{x}^{(1)} \leq w^T\mathbf{x}^{(2)}$ if and only if $\mathbf{w}^T(\alpha\mathbf{x}^{(1)}) \leq \mathbf{w}^T(\alpha\mathbf{x}^{(2)})$. Furthermore, for any $\alpha > 0$, B is a basis for a basic feasible solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ if and only if B is a basis for a basic feasible solution $\alpha\mathbf{x}$ of $A\mathbf{x} = \alpha\mathbf{b}$ ▮

We now turn our attention to the case where only one b_i is measured on some scale of measurement (i.e., $\mathcal{S}_k$, $k \neq i$, are absolute scales). As already mentioned, there is a natural correspondence between b_i being measured on a scale $\mathcal{S}$ and $a_{i1}, \ldots, a_{in}$ being measured on a (common) ratio scale.

Example 7 (Production problem revisited)
Consider the production problem presented in Example 1. It is possible that the exact amount of the i-th resource is not known exactly (for example, the total time machine i can be used might be proportional to the outside temperature) while the exact amount of other resources is known exactly. This can be viewed as $\mathbf{b}_i$ being measured on some scale of measurement $\mathcal{S}$ (if b_i is proportional to the outside temperature, then it is easy to see that $\mathcal{S}$ is an interval scale).

Also, for this fixed i, $a_{i1}, \ldots, a_{in}$ might be numerical representations of some measure of the i-th resource (which might be different than the measure of w_j or a_{kj}, $k \neq i$). For example, a_{ij} might represent the processing time of a unit of the j-th product on the i-th machine. These processing times might depend on the quality of some raw material used in the working process (the higher the quality, the shorter the processing times), so the numbers $a_{i1}, a_{i2}, \ldots, a_{in}$ are not known precisely. However, it is possible that (for a fixed i) all the ratios $r_{jk} = a_{ij}/a_{ik}$ are known (i.e., it might be known that the j-th item needs r_{jk} times more time than the i-th item). Hence, it is possible that $a_{i1}, \ldots, a_{in}$ are measured on a common (ratio) scale of measurement. ▮

The following notation will be used throughout: for any $\alpha \neq 0$ and $i = 1, \ldots, n$ we define

$$I(\alpha; i) := Diag(1, \ldots, 1, \alpha, 1, \ldots, 1)$$

where α is the (i, i) entry of $I(\alpha; i)$.

Theorem 8 *Let B be an optimal basis for problem (1). If $a_{i1}, a_{i2}, \ldots, a_{in}$ are measured on a ratio scale, then the statement that the basic variables of B define an optimal basis for problem (1) is meaningful if and only if $(B^{-1})_i \geq \mathbf{0}$ (the i-th column of B^{-1} has no negative entries) and*

$$\sum_{k=1}^{m} b_k (B^{-1})_k \geq b_i (B^{-1})_i. \tag{6}$$

Proof: Replacing $a_{i1}, a_{i2}, \ldots, a_{in}$ by $\alpha a_{i1}, \alpha a_{i2}, \ldots, \alpha a_{in}$ can be written in matrix form as replacing matrix A by matrix

$$A_\alpha := [I(\alpha; i)]A.$$

In this notation

$$B_\alpha = [I(\alpha; i)]B$$

and

$$B_\alpha^{-1} = B^{-1}[I(\alpha^{-1}; i)] = B^{-1} + B^{-1}[Diag(0, \ldots, 0, \frac{1}{\alpha} - 1, 0, \ldots, 0)].$$

B_α is an optimal basis if and only if the corresponding basic solution $\mathbf{x}_\alpha^*$ is feasible and the optimality criterion holds. We need to show that the statement "B_α is an optimal basis for problem (1) where A is replaced by A_α" is a true statement for any $\alpha > 0$ if and only if (6) holds. (Note that the statement is true for $\alpha = 1$ by hypothesis.)

Note that $(A_\alpha)_j = [I(\alpha; i)]A_j$ and we have

$$B_\alpha^{-1}(A_\alpha)_j = B^{-1}[I(\alpha^{-1}; i)][I(\alpha; i)]A_j = B^{-1}A_j.$$

Consequently,

$$\mathbf{w}_B{}^T B_\alpha^{-1}(A_\alpha)_j - w_j = \mathbf{w}_B{}^T B^{-1}A_j - w_j$$

and, for any $\alpha > 0$, the optimality criterion holds for B_α since it holds for B by hypothesis.

Feasibility corresponds to the condition

$$(\mathbf{x}^*_\alpha)_{B_\alpha} = B_\alpha^{-1}\mathbf{b} \geq \mathbf{0},$$

since $B(\mathbf{x}^*)_B = \mathbf{b}$.

$$\begin{aligned} B_\alpha^{-1}\mathbf{b} &= B^{-1}b + B^{-1}[Diag(0,\ldots,0,\frac{1}{\alpha}-1,0\ldots,0)]\mathbf{b} \\ &= \mathbf{x}^*_B + (B^{-1})_i(\frac{1}{\alpha}-1)b_i \\ &= \mathbf{x}^*_B + b_i\frac{1-\alpha}{\alpha}(B^{-1})_i. \end{aligned}$$

Therefore, $B_\alpha^{-1}\mathbf{b} \geq \mathbf{0}$ if and only if

$$\mathbf{x}^*_B \geq b_i\frac{\alpha-1}{\alpha}(B^{-1})_i. \tag{7}$$

Hence, it remains to show that (7) holds for any $\alpha > 0$ if and only if both $(B^{-1})_i \geq \mathbf{0}$ and (6) hold. Inequality (7) is required to hold for any $\alpha > 0$. Since $f(\alpha) := C\frac{\alpha-1}{\alpha}$ is a monotone function on $\mathbf{R}_+$ we only need to check that Inequality (7) holds when $\alpha \to 0^+$ and when $\alpha \to +\infty$.

When $\alpha \to 0^+$ then $\frac{\alpha-1}{\alpha} \to -\infty$ and Inequality (7) holds if and only if $(B^{-1})_i \geq \mathbf{0}$. This is because $\mathbf{x}^*_B \geq \mathbf{0}$ (by feasibility) and $b_i \geq 0$ (by definition of problem (1)).

When $\alpha \to +\infty$ then $\frac{\alpha-1}{\alpha} \to 1^-$ and we need to show that $\mathbf{x}^*{}_B \geq b_i(B^{-1})_i$. Note that

$$\mathbf{x}^*{}_B = B^{-1}\mathbf{b} = b_1(B^{-1})_1 + \ldots + b_m(B^{-1})_m.$$

Therefore, we need

$$b_1(B^{-1})_1 + \ldots + b_m(B^{-1})_m \geq b_i(B^{-1})_i,$$

which is exactly (6) ▮

Corollary 9 *Let B be an optimal basis for problem (1). Suppose that b_i from problem (1) is measured on a scale of measurement $(\mathcal{A}, \mathcal{B}, f)$ where $\mathcal{B}$ is a relational system on $\mathbf{R}_+$ If $(B^{-1})_i \geq \mathbf{0}$ and if (6) holds, then the statement that the basic variables of a basis B define an optimal basis for problem (1) is meaningful.*

Proof: This follows directly from Theorem 8. As mentioned just before Theorem 8, replacing b_i by $\Phi(b_i)$ ($\Phi \in AT(f)$) is equivalent to replacing a_{i1}, $a_{i2}, \ldots, a_{in}$ by $\alpha a_{i1}, \alpha a_{i2}, \ldots, \alpha a_{in}$ where $\alpha = b_i/\Phi(b_i)$. $\alpha \geq 0$ since both b_i and $\Phi(b_i)$ are in $\mathbf{R}_+$. ▮

The following corollary is the converse of Corollary 9.

Corollary 10 *Let B be an optimal basis for problem (1). Suppose that b_i from problem (1) is measured on a scale of measurement $(\mathcal{A}, \mathcal{B}, f)$ where $\mathcal{B}$ is a relational system on $\mathbf{R}_+$. Further suppose that the set*

$$S := \{ \frac{b_i}{\Phi(b_i)} : \ \Phi \in AT(f)\} \subseteq \mathbf{R}_+$$

does not have an infimum or supremum in $\mathbf{R}_+$. If the statement that the basic variables of a basis B define an optimal basis for problem (1) is meaningful, then $(B^{-1})_i \geq \mathbf{0}$ and (6) holds.

Proof: This follows from the proof of Theorem 8. Note that (7) must hold for every $\alpha \in S$. Since $\inf S = 0$ and $\sup S = \infty$, we use the same argument as in the proof of Theorem 8 ▮

Example 11 Consider problem (1) where

$$\mathbf{w}^T = (2, -1, 0, 0), \ A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \ \mathbf{b}^T = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

It is easy to see that $\mathbf{x}^* = (1, 0, 0, 1)^T$ is the optimal solution and the basic variables are 1 and 4. The corresponding optimal basis B and its inverse B^{-1} are

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \ B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

If b_1 is measured on some scale of measurement, then the conclusion that the basic solution corresponding to basic variables 1. and 4. is optimal is meaningless by Corollary 10 since the first column of B^{-1}, $(B^{-1})_1$, is not a nonnegative vector. Indeed, whenever b_1 is replaced by $\Phi(b_1)$ we can consider problem (1) where instead of replacing b_1 by $\Phi(b_1)$, we replace A by

$$\begin{pmatrix} \frac{1}{\Phi(b_1)} & 0 & \frac{1}{\Phi(b_1)} & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Note that the basic solution in which 1 and 4 are basic variables is not feasible whenever $\Phi(\mathbf{b}_1) > 2$ since

$$\begin{pmatrix} \frac{1}{\Phi(b_1)} & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \end{pmatrix}^T = \begin{pmatrix} \Phi(b_1) & 0 \\ -\Phi(b_1) & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}^T = \begin{pmatrix} \Phi(b_1) & 2-\Phi(b_1) \end{pmatrix}.$$

Therefore, such a basic solution cannot be optimal (since is not even feasible) whenever $\Phi(b_1) > 2$. ∎

Theorem 12 *Let B be an optimal basis for problem (1). Suppose that for every $i = 1, \ldots, m$, b_i from problem (1) is measured on a scale of measurement $(\mathcal{A}_i, \mathcal{B}_i, f_i)$, where $\mathcal{B}_i$ is a relational system on $\mathbf{R}_+$. Further suppose that for*

$$S_i := \{\frac{b_i}{\Phi_i(b_i)} : \ \Phi_i \in AT(f_i)\} \subseteq \mathbf{R}_+$$

$\inf(S_i) = 0$, *$i = 1, \ldots, m$. Then the statement that the basic variables of a basis B define an optimal basis for problem (1) is meaningful if and only if $B^{-1} \geq \mathbf{0}$.*

Proof: We again note that replacing b_i by $\Phi_i(\mathbf{b}_i)$ is equivalent to replacing $a_{i1}, \ldots, a_{in}$ by $\alpha_i a_{i1}, \ldots, \alpha_i a_{in}$ where $\alpha_i = b_i/\Phi_i(b_i)$.

We follow the proof of Theorem 8. For any choice of $\Phi_i \in AT(f_i)$, $i = 1, \ldots, m$, replacing b_i by $\Phi(b_i)$ can be viewed as replacing matrix A by matrix

$$A_\alpha := (\prod_{i=1}^{m}[I(\alpha_i; i)])A = [Diag(\alpha_1, \ldots, \alpha_m)]A.$$

We define

$$B_\alpha := (\prod_{i=1}^{m}[I(\alpha_i; i)])B = [Diag(\alpha_1, \ldots, \alpha_m)]B.$$

We want to show that the statement "B_α is an optimal basis for problem (1) where A is replaced by A_α" is a true statement for any choice of $(\alpha_1, \ldots, \alpha_n)^T = \alpha > \mathbf{0}$. (Note that the statement is true for $\alpha = (1, \ldots, 1)^T$ by hypothesis.) As in the proof of Theorem 8, we conclude that the optimality criterion holds for a basic solution $\mathbf{x}^*_\alpha$ (corresponding to the basis B_α) if and only if it holds for $\mathbf{x}^*$ (corresponding to the basis B). This is because matrices $I(\alpha_i; i)$ and $I(\alpha_j; j)$ commute for any $\alpha_i, \alpha_j \in \mathbf{R}_+$ and any

$i, j \in [m]$). Hence, for any $\alpha > \mathbf{0}$, the optimality criterion holds for $\mathbf{x}^*_\alpha$ (because, by hypothesis, the optimality criterion holds for $\mathbf{x}^*$ corresponding to the basis B)

In order to show feasibility of $\mathbf{x}^*_\alpha$ we need to show that $(B_\alpha)^{-1}\mathbf{b} \geq \mathbf{0}$. We have

$$\begin{aligned} B_\alpha^{-1}\mathbf{b} &= B^{-1}b + B^{-1}[Diag(\frac{1}{\alpha_1} - 1, \ldots, \frac{1}{\alpha_m} - 1)\mathbf{b}] \\ &= \mathbf{x}^*_B + \sum_{i=1}^m (B^{-1})_i(\frac{1}{\alpha_i} - 1)b_i \\ &= \mathbf{x}^*_B + \sum_{i=1}^m b_i \frac{1-\alpha_i}{\alpha_i}(B^{-1})_i \end{aligned}$$

It remains to prove that

$$[\forall \alpha > \mathbf{0} : \ \mathbf{x}^*_B + \sum_{i=1}^m b_i \frac{1-\alpha_i}{\alpha_i}(B^{-1})_i \geq \mathbf{0}\,] \Leftrightarrow B^{-1} \geq \mathbf{0}.$$

Proof of ($\Rightarrow$):
Note that we can set any of $\alpha_i := 1$ by choosing Φ_i to be the identity function (which is certainly an admissible transformation). Therefore, by setting $\alpha_j = 1$ for all $j \neq i$, we see that the condition $(B_\alpha)^{-1}\mathbf{b} \geq \mathbf{0}$ implies

$$\mathbf{x}^*_B \geq b_i \frac{\alpha_i - 1}{\alpha_i}(B^{-1})_i, \ i := 1, \ldots, m. \tag{8}$$

As in the proof of Theorem 8 (when $\alpha_i \to 0^+$), using the fact that $\inf S_i = 0$, we conclude that $(B^{-1})_i \geq \mathbf{0}$ for every $i = 1, \ldots, m$. Hence, if $\mathbf{x}^*_\alpha$ is feasible then $B^{-1} \geq \mathbf{0}$.
Proof of ($\Leftarrow$):
if $B^{-1} \geq \mathbf{0}$ then

$$B_\alpha^{-1}\mathbf{b} = \sum_{i=1}^m \frac{1}{\alpha_i}(B^{-1})_i b_i \geq \mathbf{0}$$

for any choice of $\alpha_1, \ldots, \alpha_m \in \mathbf{R}_+$ (since $b_i \geq 0$ by definition of problem (1)).
▮

Corollary 13 *Let B be an optimal basis for problem (1) such that $B^{-1} \geq \mathbf{0}$. If for every $i = 1, \ldots, m$, b_i from problem (1) is measured on a scale of measurement $(\mathcal{A}_i, \mathcal{B}_i, f_i)$, where $\mathcal{B}_i$ is a relational system on $\mathbf{R}_+$, then the statement that the basic variables of a basis B define an optimal basis for problem (1) is meaningful.*

Proof: This is just one direction (sufficiency) of Theorem 12. Note that in the proof of sufficiency, we did not need the assumption $\inf S_i = 0$. ∎

Now we turn our attention to the case when $w_j, a_{1j}, a_{2j}, \ldots, a_{nj}$ are measured on the same scale of measurement.

Example 14 We consider the production problem from Example 1 again. Changing the definition of the unit of the j-th item will change all of w_j, a_{1j}, a_{2j},..., a_{nj}. (Of course, it is possible that all n items are measured on the same scale of measurement. In this case we would need to change $\mathbf{w}$ and A completely.) In the simplest possible case, if the new unit of the j-th item is $\alpha\times$(old unit), then w_j, a_{1j}, a_{2j},..., a_{nj} need to be replaced by αw_j, αa_{1j}, αa_{2j},..., αa_{nj}. In matrix form, this change can be written as multiplying both $\mathbf{w}^T$ and A from the right by the matrix $I(\alpha; j)$. ∎

Proposition 15 *Suppose that, for every $j = 1, \ldots, n$, $w_j, a_{1j}, a_{2j}, \ldots a_{nj}$ are measured on a scale $\mathcal{S}_j$ which is stronger than or equal to a ratio scale. Then the statement that the basic variables of a basis B define an optimal basis for problem (1) is meaningful. Moreover, $(x_1, \ldots, x_n)^T$ is an optimal solution for the original problem if and only if*

$$\bar{\mathbf{x}} = (\frac{1}{\alpha_1}(x_1), \ldots, \frac{1}{\alpha_n}(x_n))^T$$

is an optimal solution for problem (1) where w_j, a_{1j}, a_{2j}, ..., a_{nj} are replaced by $\alpha_j(w_j)$, $\alpha_j(a_{1j})$, $\alpha_j(a_{2j})$, ..., $\alpha_j(a_{nj})$, $j = 1, \ldots n$.

Proof: Note that for any $\mathbf{y} \in \mathbf{R}^n$, $(\alpha_1 w_1, \ldots, \alpha_n w_n)\bar{\mathbf{y}} = \mathbf{w}^T\mathbf{y}$. Also note that replacing $a_{1j}, a_{2j}, \ldots, a_{nj}$ by $\alpha_j(a_{1j}), \alpha_j(a_{2j}), \ldots, \alpha_j(a_{nj})$, $j = 1, \ldots n$, can be written in matrix form as replacing matrix A by matrix

$$A_\alpha := A\prod_{j=1}^{n}[I(\alpha_j; j)] = A[Diag(\alpha_1, \ldots, \alpha_n)].$$

Since $\bar{y} = [Diag(\alpha_1, \ldots, \alpha_n)]^{-1}\mathbf{y}$, we have $A\mathbf{y} = \mathbf{b} \Leftrightarrow A_\alpha\bar{\mathbf{y}} = b$. Finally, $\alpha_1 > 0, \ldots, \alpha_n > 0$ implies $\mathbf{y} \geq \mathbf{0} \Leftrightarrow \bar{\mathbf{y}} \geq 0$. ∎

Finally, we consider situations where w_j, $j = 1, \ldots, n$, are not known precisely or where w_j are measured on some scale of measurement. In other words, $\Phi(w_j)$ is an acceptable replacement for w_j whenever Φ is an admissible transformation of the scale of measurement of w_j. We will make the (reasonable) assumption that $\Phi(w_j)$ and w_j have the same sign, i.e., $w_j\Phi(w_j) > 0$ for all Φ. Similar to our analysis of the admissible transformations of b_i, here we can change the unit of the j-th item by replacing $\Phi(w_j), a_{1j}, a_{2j}, \ldots, a_{nj}$ by $w_j = \alpha\Phi(w_j), \alpha a_{1j}, \alpha a_{2j}, \ldots, \alpha a_{nj}$ where $\alpha = w_j/\Phi(w_j)$ (> 0 by assumption).

Similar to our analysis of the meaningfulness of the conclusion of optimality in the case when b_i are measured on some scale of measurement, we will first give a detailed analysis of the case when just one w_j is measured on some scale of measurement (i.e., w_k, $k \neq j$, are measured on an absolute scale) and then extend these results to the case when all w_j's are measured on (possibly different) scales of measurement (Theorem 21).

Example 16 We turn once more to the production problem described in Example 1. The exact profit per unit of the j-th item, w_j, might not be known exactly (for example it might be proportional to the current market price). This can be viewed as w_j being measured on some scale of measurement $\mathcal{S}$.

Also, for a fixed j, the numbers $a_{1j}, a_{2j}, \ldots, a_{nj}$ might not be known precisely but all the ratios $q_{kl} = a_{jk}/a_{kl}$ might be known. For example, the production of the the j-th item might require a different amount of resources depending on the outside temperature (provided that the production process must be kept at the constant temperature) but it is known that the j-th item contains q_{kl} times more of the k-th ingredient(resource) than l-th ingredient(resource). In such a case, $a_{1j}, a_{2j}, \ldots, a_{nj}$ represent data measured on a common ratio scale. ▮

The first step is to analyze the situation when $a_{1j}, a_{2j}, \ldots, a_{nj}$ are measured on a ratio scale of measurement.

Theorem 17 *Let $\mathbf{x}^*$ be an optimal solution for problem (1). Furthermore, let $\mathbf{x}^*$ be a basic feasible solution of $A\mathbf{x} = b$ and let j be a non-basic variable for $\mathbf{x}^*$. Suppose that $a_{1j}, a_{2j}, \ldots, a_{nj}$ are measured on a ratio scale of measurement. Then $\mathbf{x}^*$ is a meaningful optimal solution to problem (1) if and only if $z_j \geq 0 \geq w_j$.*

Proof: $\mathbf{x}^*$ is a meaningful optimal solution if and only if $\mathbf{x}^*$ is feasible and satisfies the optimality criterion whenever A is replaced by $A[I(\alpha; j)]$, $\alpha > 0$.

Since j is non-basic, $x_j^* = 0$ and $\mathbf{x}^* = [I(\alpha; j)]^{-1}\mathbf{x}^*$. Therefore, for any α, $A[I(\alpha; j)]\mathbf{x}^* = \mathbf{b}$.

Hence it remains to show that the optimality criterion holds whenever A is replaced by $A[I(\alpha; j)]$ if and only if $z_j \geq 0 \geq w_j$. We first note that B remains unchanged since the j-th column of A is not a column of B. For any $k \neq j$, A_k remains unchanged and consequently $z_k - w_k$ remains unchanged and $z_k - w_k \geq 0$ since $\mathbf{x}^*$ is an optimal solution to problem (1). It remains to check the optimality criterion for $k = j$:

$$\mathbf{w}_B{}^T B^{-1}(\alpha A_j) - w_j \geq 0.$$

This is equivalent to

$$z_j = \mathbf{w}_B{}^T B^{-1} A_j \geq \frac{1}{\alpha} w_j. \tag{9}$$

Note that (9) holds for all $\alpha \geq 0$ (i.e., $\mathbf{x}^*$ is a meaningful optimal solution) if and only if $z_j \geq 0 \geq w_j$. ▮

Corollary 18 *Let $\mathbf{x}^*$ be an optimal solution for problem (1). Furthermore, let $\mathbf{x}^*$ be a basic feasible solution of $A\mathbf{x} = b$ and let j be a non-basic variable for $\mathbf{x}^*$. Suppose that w_j is measured on a scale of measurement $\mathcal{S}$. Further suppose that the set*

$$S := \{\frac{w_j}{\Phi(w_j)} : \ \Phi \in AT(f)\} \subseteq \mathbf{R}_+$$

is unbounded in $\mathbf{R}_+$ (i.e., $\inf S = 0$ and $\sup S = \infty$). Then $\mathbf{x}^*$ is a meaningful optimal solution to problem (1) if and only if $z_j \geq 0 \geq w_j$

Proof: This follows directly from the proof of Theorem 17 and the fact that replacing w_j by $\Phi(w_j)$ is equivalent to replacing $a_{1j}, \ldots, a_{nj}$ by $\alpha a_{1j}, \ldots, \alpha a_{nj}$ where $\alpha = w_j/\Phi(w_j)$. Note that $\mathbf{x}^*$ is a meaningful optimal solution if and only if (9) holds for all $\alpha \in S$. The latter is equivalent to $z_j \geq 0 \geq w_j$ since $\inf S = 0$ and $\sup S = \infty$. ▮

In what follows, $(B^{-1})_{(r)}$ will denote the r-th row of the matrix B^{-1}.

Theorem 19 *Let B be an optimal basis for problem (1) and let j be a basic variable for the corresponding optimal solution $\mathbf{x}^*$. Suppose that a_{1j},*

$a_{2j}, \ldots, a_{nj}$ are measured on a (common) ratio scale of measurement. Then the statement that the basic variables of a basis B define an optimal basis for problem (1) is meaningful if and only if

$$z_k - w_k \geq w_j((B^{-1})_{(j)})A_k \geq 0 \tag{10}$$

holds for all non-basic variables $k \in [n]$.

Proof: Let $\Phi(t) = \alpha t$, $\alpha > 0$, be an admissible transformation of a_{1j}, $a_{2j}, \ldots$, a_{nj}. For any $\mathbf{x} \in \mathbf{R}^n$ we define

$$\mathbf{x}(\alpha) := [I(\alpha; j)]^{-1}\mathbf{x} = (x_1, \ldots, x_{j-1}, \frac{1}{\alpha}x_j, x_{j+1}, \ldots, x_n)^T.$$

Then $\mathbf{x} \geq 0$ if and only if $\mathbf{x}(\alpha) \geq \mathbf{0}$ (note that $\alpha > 0$) and $A\mathbf{x} = \mathbf{b}$ if and only if $A[I(\alpha; j)]\mathbf{x}(\alpha) = b$. Furthermore, if $\mathbf{x}$ is a basic feasible solution, then $\mathbf{x}(\alpha)$ is also a basic feasible solution with the same basic variables.

Therefore, the meaningfulness is equivalent to the statement that, for any $\alpha > 0$, $\mathbf{x}^*(\alpha)$ is an optimal solution for problem (1) where A is replaced by $A[I(\alpha; j)]$.

Note that $\mathbf{x}^*$ is feasible for the original problem, so $\mathbf{x}^*(\alpha)$ is feasible for problem (1) where A is replaced by $A[I(\alpha; j)]$. Therefore, we only need to check that (10) holds if and only if $\mathbf{x}^*(\alpha)$ satisfies the optimality criterion for any $\alpha > 0$. Since j is a basic variable for $\mathbf{x}^*$, the matrix consisting of columns of $A[I(\alpha; j)]$ indexed by basic variables is just $B[I(\alpha; j)]$. In other words, the basis matrix for $\mathbf{x}^*(\alpha)$ is just a basis matrix for $\mathbf{x}^*$ where the j-th column is multiplied by α. Note that

$$(B[I(\alpha; j)])^{-1} = [I(\alpha; j)]^{-1}B^{-1} = [I(\alpha^{-1}; j)]B^{-1}.$$

The optimality criterion trivially holds for basic variables. The optimality criterion for a non-basic variable k,

$$\mathbf{w}_B{}^T(B[I(\alpha; j)])^{-1}(A[I(\alpha; j)])_k - w_k \geq 0,$$

becomes

$$\sum_{i=1}^{m}(w_B)_i([I(\alpha^{-1}; j)]B^{-1}A_k)_i - w_k \geq 0$$

since $(AI(\alpha; j))_k = A_k$ for every non-basic variable k. Also note that

$$(I(\alpha; j)^{-1}B^{-1}A_k)_i = \begin{cases} ((B^{-1})_{(i)})A_k & \text{if} \quad i \neq j \\ \frac{1}{\alpha}((B^{-1})_{(j)})A_k & \text{if} \quad i = j \end{cases}$$

Now we have

$$\begin{aligned}
\mathbf{w}_B{}^T(B[I(\alpha^{-1};j)](A[I(\alpha;j)])_k - w_k &= \sum_{i=1}^{m}(w_B)_i((B^{-1})_{(i)})A_k - w_k \\
&\quad + w_j(\frac{1}{\alpha} - 1)((B^{-1})_{(j)})A_k \\
&= z_k - w_k + \frac{1-\alpha}{\alpha} w_j((B^{-1})_{(j)})A_k
\end{aligned}$$

Therefore, the optimality criterion holds for all α if and only if for all non-basic k,

$$z_k - w_k \geq \frac{\alpha - 1}{\alpha} w_j((B^{-1})_{(j)})A_k \tag{11}$$

for all $\alpha > 0$. It remains to show that (11) holds for all $\alpha > 0$ if and only if (10) holds. As in the proof of Theorem 8 we observe that $f(\alpha) := C\frac{\alpha-1}{\alpha}$ is a monotone function on $\mathbf{R}_+$ and we only need to check that (11) holds when $\alpha \to 0^+$ and when $\alpha \to +\infty$.

When $\alpha \to 0^+$ then $\frac{\alpha-1}{\alpha} \to -\infty$ and (11) holds if and only if $w_j((B^{-1})_{(j)})A_k \geq \mathbf{0}$. This is because $z_k - w_k \geq 0$ by the optimality criterion for $\mathbf{x}^*$.

When $\alpha \to +\infty$ then $\frac{\alpha-1}{\alpha} \to 1^-$ and (11) is equivalent to $z_k - w_k \geq w_j((B^{-1})_{(j)})A_k$. ∎

Corollary 20 *Let B be an optimal basis for problem (1) and let j be a basic variable. Suppose that w_j is measured on a scale of measurement $(\mathcal{A}, \mathcal{B}, f)$ where $\mathcal{B}$ is a relational system on $\mathbf{R}$ such that the set*

$$S = \{\frac{w_j}{\Phi(w_j)} : \Phi \in AT(f)\} \subseteq \mathbf{R}_+$$

is unbounded in $\mathbf{R}_+$ (i.e., $\inf S = 0$ and $\sup S = \infty$) Then the statement that the basic variables of the basis B define an optimal basis for problem (1) is meaningful if and only if

$$z_k - w_k \geq w_j((B^{-1})_{(j)})A_k \geq 0$$

holds for all non-basic variables $k \in [n]$.

Proof: This follows from the proof of Theorem 19 (in the same way as Corollary 18 follows from Theorem 17). Since $\inf S = 0$ and $\sup S = \infty$, we use the same argument as in the proof of Theorem 19 ▮

Theorem 21 *Let B be an optimal basis for problem (1). Suppose that for every $j = 1, \ldots, n$, w_j from problem (1) is measured on a scale of measurement $(\mathcal{A}_j, \mathcal{B}_j, f_j)$ where $\mathcal{B}_j$ is a relational system on $\mathbf{R}$. Further suppose that, for any $j = 1, \ldots, n$,*

$$S_j := \{\frac{w_j}{\Phi_j(w_j)} : \ \Phi_j \in AT(f_j)\} \subseteq \mathbf{R}_+$$

is such that $\inf(S_j) = 0$. Then the statement that the basic variables of the basis B define an optimal basis for problem (1) is meaningful if and only if

$$(\mathbf{w}_B)_k (B^{-1})_{(k)} A_j \geq 0 \geq w_j \tag{12}$$

for every basic variable k and every non-basic variable j.

Proof: We again note that replacing w_j by $\Phi_j(w_j)$ is equivalent to replacing $a_{1j}, \ldots, a_{mj}$ by $\alpha_i a_{1j}, \ldots, \alpha_i a_{mj}$ where $\alpha_i = w_j/\Phi_j(w_j)$. For any $\mathbf{x}$ we define $\bar{\mathbf{x}}$ as in Proposition 15. Let $\mathbf{x}^*$ be an optimal solution to problem (1) corresponding to B. Note that $\bar{\mathbf{x}}^*$ corresponds to a basis

$$B_\alpha := B[Diag(\alpha_1, \ldots, \alpha_n)] = B(\prod_{j=1}^{n} I(\alpha_j; j)).$$

We also use the notation

$$A_\alpha := A[Diag(\alpha_1, \ldots, \alpha_n)] = A \prod_{j=1}^{n} I(\alpha_j; j).$$

Since B is an optimal basis by hypothesis, we need to prove that the statement "B_α is an optimal basis for problem (1) where A is replaced by A_α" is true for any $\alpha > \mathbf{0}$ if and only if (12) holds. Note that the basic feasible solution $\bar{\mathbf{x}}^* \geq \mathbf{0}$ corresponding to B_α is always a feasible solution since $A_\alpha \bar{\mathbf{x}}^* = \mathbf{b}$. Therefore, we just need to check that
(i) $\bar{\mathbf{x}}^*$ satisfies the optimality criterion for any choice of $\alpha_j \in S_j$, $j = 1, \ldots, n$.
is equivalent to

(ii) $(\mathbf{w}_B)_k(B^{-1})_{(k)}A_j \geq 0 \geq w_j$ for every basic variable k and every non-basic variable j.
Proof of (i) $\Rightarrow$ (ii):
Note that $1 \in S_j$ for every j since the identity function is always an admissible transformation. For any j, we set $\alpha_i := 1$, $i \neq j$, and let $\alpha_j \to 0^+$. If j is a non-basic variable then, as in the proof of Theorem 17, we conclude that $0 \geq w_j$. If j is a basic variable, then, as in the proof of Theorem 19, we conclude that $(\mathbf{w}_B)_j(B^{-1})_{(j)}A_i \geq 0$ must hold for any non-basic variable i. Hence, (12) holds for every basic variable k and every non-basic variable j.
Proof of (ii) $\Rightarrow$ (i):
Suppose that (12) holds for every basic variable k and every non-basic variable j and we need to show that $\bar{\mathbf{x}}^*$ satisfies the optimality criterion:

$$(\mathbf{w}_B)^T(B_\alpha^{-1})(A_\alpha)_j - w_j \geq 0$$

for every non-basic variable j (note that the optimality criterion always holds trivially for any basic variable j). Now,

$$\begin{aligned}(\mathbf{w}_B)^T(B_\alpha^{-1})(A_\alpha)_j &= \sum_{i=1}^{m}(\mathbf{w}_B)_k(B_\alpha^{-1})_{(k)}(A_\alpha)_j \\ &= \sum_{i=1}^{m}(\mathbf{w}_B)_k\frac{\alpha_j}{\alpha_k}(B^{-1})_{(k)}A_j \\ &\geq 0 \\ &\geq w_j\end{aligned}$$

where the first inequality holds because $\frac{\alpha_j}{\alpha_k} > 0$ and by using (12) for every basic variable k. The last inequality also holds by (12). ▮

Example 22 Consider the problem from Example 11. Note that B, which is the optimal basis (the first and fourth variable are basic) satisfies (12). Hence, it is meaningful to say that the first and fourth variable are the basic variables of an optimal solution whenever w_i is measured on a scale S_i, $i = 1, 2, 3, 4$, satisfying the conditions of Theorem 21. ▮

Example 23 To illustrate Theorem 21 we consider problem (1) where

$$\mathbf{w}^T = (1, -1, 0, 0),\ A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ \mathbf{b}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

It is easy to see that $\mathbf{x}^* = (3, 1, 0, 0)^T$ is the optimal solution and the basic variables are 1 and 2. The corresponding optimal basis B and its inverse B^{-1} are

$$B = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Note that (12) does not hold for $k = 1$ and $j = 3$. Hence, if w_1 is measured on some scale of measurement satisfying the conditions of Theorem 21, the conclusion that the first and second variable are basic variables of an optimal solution is meaningless. For example, if we replace w_1 with $\Phi(w_1)$, then

$$z_4 - w_4 = (\Phi(w_1), -1) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}) - 0 = 2\Phi(w_1) - 1.$$

Hence, whenever $\Phi(w_1) < \frac{1}{2}$, the basic feasible solution with basic variables 1 and 2 is not optimal since the optimality criterion does not hold. ∎

5 Closing Remarks

We again point out (see Remark at the beginning of Section 3) that all results presented here can be easily reformulated without referring to scales of measurement and meaningfulness. For example, one could state Theorem 12 and Theorem 21 as follows:

Theorem 12 (reformulated) *Let B be an optimal basis for problem (1). Suppose that for every $i = 1, \ldots, m$, G_i is a group of real-valued functions such that*

$$S_i := \{\frac{b_i}{\Phi_i(b_i)} : \ \Phi_i \in G_i\} \subseteq \mathbf{R}_+$$

and such that $\inf(S_i) = 0$. *Then, $B^{-1} \geq \mathbf{0}$ if and only if, for any choice of $f_1 \in G_1, \ldots, f_n \in G_n$ the basic variables of B define an optimal basis for problem (1) where $\mathbf{b} = (b_1, \ldots, b_m)^T$ is replaced by $(\Phi_1(b_1), \ldots, \Phi_m(b_m))^T$ (i.e., the basic variables of an optimal basis are invariant under scaling of b_i by functions from G_i, $i = 1, \ldots, m$).*

Theorem 21 (reformulated) *Let B be an optimal basis for problem (1). Suppose that for every $j = 1, \ldots, n$, G_j is a group of real-valued functions such that*

$$S_j := \{\frac{w_i}{\Phi_j(w_j)} : \ \Phi_j \in G_j\} \subseteq \mathbf{R}_+$$

and such that $\inf(S_j) = 0$. *Then*

$$(\mathbf{w}_B)_k (B^{-1})_{(k)} A_j \geq 0 \geq w_j$$

holds for every basic variable k *and every non-basic variable* j *if and only if basic variables of* B *define an optimal basis for problem (1) where* $\mathbf{w} = (w_1, \ldots, w_n)^T$ *is replaced by* $(\Phi_1(w_1), \ldots, \Phi_n(w_n))^T$ *(i.e., the basic variables of an optimal basis are invariant under scaling of* w_j *by functions from* G_j, $j = 1, \ldots, n$*).*

It should be noted that the striking similarity of these two theorems as well as the presented proofs is no accident but a consequence of the duality in linear programming (more about duality can be found in any standard book on linear programming, e.g., see [1]).

Our analysis was limited to scalings by increasing linear (affine) transformations and scalings that can be related to these. It would be interesting to consider scalings by some other sets of functions and determine necessary and/or sufficient conditions for invariance of the conclusion of optimality. (Meaningfulness of the conclusion of optimality for optimization problems where the parameters of the objective function are measured on an ordinal scale is discussed in [5]). Sort of an inverse approach to the problem would also be interesting: given a problem (1) and a set of its parameters $\{p_1, \ldots, p_l\}$, find the largest set of scaling functions, S, such that the conclusion of optimality is invariant under scaling of $\{p_1, \ldots, p_l\}$ by any function from S.

Finally, we note that characterizing invariance under scalings is not a problem that is limited to linear programming problems. In fact, such analysis can be applied to any mathematical model (invariance of the output with respect to scalings of input parameters). Of course, it is hopeless to expect some sensible results if the problem is stated in such generality. The main reason why we were able to characterize invariance of the conclusion of optimality for problem (1) under certain scalings is because of the simplicity of the objective function and the simplicity of the structure of the set of feasible solutions.

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