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Simple Proofs of Occupancy Tail Bounds

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Abstract

We give short proofs of some occupancy tail bounds using the method of bounded differences in expected form and the notion of negative association.

1 Introduction

The purpose of this note is to give short, simple and natural proofs of some tail bounds on occupancy problems in [5]. The proofs also serve as advertisements for some very useful, but apparently not very well–known concepts and techniques:

- The *method of bounded differences* in the *expected* form [6, Cor. 6.10].
- A concept of negative dependence called *negative association* from the theory of multivariate probability inequalities, [3, 4, 9, 10].

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The first of these yields the Occupancy Bound 1 of [5, Theorem 2] by a direct plug–in substitution. The second gives a short and enlightening calculation–free proof of the (Chernoff) Occupancy Bound 2 in [5, Theorem 3].

The setting is the classical probabilistic experiment of throwing m balls independently and uniformly [1] into n bins (for positive integers m, n). The random variables of interest are defined as follows: for $i \in [n]$ [2], let Z_i be the indicator variable which is 1 if bin i is empty and 0 otherwise. Set $Z := \sum_i Z_i$ to be the number of empty bins. We are interested in tail bounds on the distribution of Z.

2 Bounded Differences

The "method of bounded differences" is usually stated and used in the following form [6, Lemma 1.2]

Proposition 1 (McDiarmid) *Let $X_1, \ldots, X_n$ be independent random variables, variable X_i taking values in a finite set A_i for each $i \in [n]$, and suppose the function $f : \prod_i A_i \to \mathsf{R}$ satisfies the following "bounded difference" condition: For each $i \in [n]$, there is a constant c_i such that*

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq c_i,$$

whenever the vectors $\mathbf{x}, \mathbf{x}'$ differ only in the kth co–ordinate. Then

$$\Pr[|f(\mathbf{X}) - E[f(\mathbf{X})]| > t] < 2\exp(-2t^2/\sum_i c_i^2).$$

This lemma has an attractive packaged form, but can be too weak for some applications. In this case, we may resort to the "method of bounded differences" in the expected form [6, Cor. 6.10]:

Proposition 2 (McDiarmid) *Let $X_1, \ldots, X_n$ be independent random variables, variable X_i taking values in a finite set A_i for each $i \in [n]$, and suppose the function f satisfies the following "bounded difference" conditions: For*

[1] The techniques apply equally well even if the uniformity assumption is dropped, but we retain it for simplicity.

[2] We denote $[n] := \{1, \ldots, n\}$

each $i \in [n]$, there is a constant c_i such that for any $x_k \in A_k, k \in [i-1]$ and for any $x_i, x_i' \in A_i$

$$\begin{aligned} |E[f(\mathbf{X}) \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = x_i] - \\ E[f(\mathbf{X}) \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = x_i']| \quad \leq c_i \quad . \end{aligned} \tag{1}$$

Then

$$\Pr[|f(\mathbf{X}) - E[f(\mathbf{X})]| > t] < 2\exp(-2t^2/\sum_i c_i^2).$$

We shall illustrate the "method of bounded differences" in the above form by applying it to study the occupancy statistics in the classical balls and bins experiment. First, we have by simple calculations, $E[Z_i] = (1-\frac{1}{n})^m$ and $E[Z] = \sum_i E[Z_i] = m(1-\frac{1}{n})^m$.

To get a tail probability estimate, regard $Z = Z(B_1, \ldots, B_m)$ where the random variables B_k take values in the set $[n]$ indicating which bin ball k occupies, for each $k \in [m]$. If one were to employ the "method of bounded differences" in the form of Proposition 1, it is easy to see that one must take $c_i := 1$ for each $i \in [m]$. Then one gets the tail probability bound:

$$\Pr[|Z - E[Z]| > t] < 2\exp(-2t^2/\sum_i c_i^2) = 2\exp(-2t^2/m).$$

This can be quite weak for large m and small t; compare the bound below.

An improved bound can be obtained by applying the expected version of Proposition 2. Fix some $i \in [m]$ and let us compute

$$\begin{aligned} |E[Z \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] - \\ E[Z \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i']| \quad , \end{aligned}$$

for fixed $b_1, \ldots, b_{i-1}, b_i, b_i' \in [n]$. Set $b := b_i \neq b_i' =: b'$. Let $I := \{b_1, \ldots, b_{i-1}\} \subseteq [n]$. Of course for $j \in I$,

$$\begin{aligned} E[Z_j \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] &= 0 = \\ E[Z_j \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i'] & \quad . \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} E[Z_b \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] &= 0 \\ E[Z_{b'} \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i'] &= 0. \end{aligned} \tag{3}$$

For $j \in [m] \setminus (I \cup \{b, b'\})$, we have

$$\begin{aligned} E[Z_j \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] &= (1 - \frac{1}{n})^{m-i} = \\ E[Z_j \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b'_i] & . \end{aligned} \tag{4}$$

Now suppose $b \in I$ but $b' \notin I$. Then with we have

$$E[Z_b \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b'_i] = 0, \tag{5}$$

whereas

$$E[Z_{b'} \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] = (1 - \frac{1}{n})^{m-i}. \tag{6}$$

Finally (apart from the symmetric case of the previous one) if $b, b' \notin I$ then,

$$\begin{aligned} E[Z_{b'} \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] &= (1 - \frac{1}{n})^{m-i}. \\ E[Z_b \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b'_i] &= (1 - \frac{1}{n})^{m-i}. \end{aligned} \tag{7}$$

Comparing (2) through (7), for $i \in [m]$,

$$\begin{aligned} |E[Z \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b_i] - \\ E[Z \mid B_1 = b_1, \ldots, B_{i-1} = b_{i-1}, B_i = b'_i]| &\leq c_i, \end{aligned}$$

where

$$c_i = (1 - 1/n)^{m-i},$$

Thus we have from Theorem 2,

$$\Pr[|Z - E[Z]| > t] < 2\exp(-2t^2 / \sum_i c_i^2).$$

Since

$$\sum_i c_i^2 = \frac{n^2 - \mu^2}{2n - 1},$$

we get the occupancy bound Theorem 2 in [5]:

$$\Pr[|Z - \mu| \geq \theta\mu] \leq 2\exp\big(-\frac{\theta^2 \mu^2 (n - 1/2)}{n^2 - \mu^2}\big).$$

3 Negative Association

A very useful and robust notion of negative dependence between random variables called *negative association* was introduced by Joag–Dev and Proschan [4]:

Definition 3 (Negative Association) *Let* $\mathbf{X} := (X_1, \ldots, X_n)$ *be a vector of random variables.*

$(-A)$ *The random variables,* $\mathbf{X}$ *are* **negatively associated** *if for every two disjoint index sets,* $I, J \subseteq [n]$,

$$E[f(X_i, i \in I)g(X_j, j \in J)] \leq E[f(X_i, i \in I)]E[g(X_j, j \in J)],$$

for functions $f : \mathsf{R}^{|I|} \to \mathsf{R}$ *and* $g : \mathsf{R}^{|J|} \to \mathsf{R}$ *that are both non–decreasing (or both non–increasing) with respect to the usual co–ordinatewise ordering of Euclidean spaces.*

The next lemma list some properties that facilitate proofs of negative association. The properties themselves are immediate from the definition.

Lemma 4 *1. If* $\mathbf{X}$ *and* $\mathbf{Y}$ *satisfy* $(-A)$ *and are mutually independent, then the augmented vector* $(\mathbf{X}, \mathbf{Y}) = (X_1, \cdots, X_n, Y_1, \cdots, Y_m)$ *satisfies* $(-A)$.

2. Let $\mathbf{X} := (X_1, \cdots, X_n)$ *satisfy* $(-A)$*. Let* $I_1, \cdots, I_k \subseteq [n]$ *be disjoint index sets, for some positive integer* k*. For* $j \in [k]$*, let* $h_j : \mathsf{R}^{|I_k|} \to \mathsf{R}$ *be non–decreasing (or non–increasing) functions, and define* $Y_j := h_j(X_i, i \in I_j)$*. Then the vector* $\mathbf{Y} := (Y_1, \cdots, Y_k)$ *also satisfies* $(-A)$*. That is, non–decreasing (or non–increasing) functions of disjoint subsets of negatively associated variables are also negatively associated.*

Proposition 5 *The random variables* $Z_1, \ldots, Z_n$ *are negatively associated.*

Sketch of Proof. Introduce the indicator variables for $i \in [n], k \in [m]$,

$$B_{i,k} := \begin{cases} 1, & \text{if ball } k \text{ goes into bin } i; \\ 0, & \text{otherwise.} \end{cases}$$

For each $k \in [m]$, it is easy to show that the variables $(B_{i,k} \mid i \in [n])$ are negatively associated (observe that their joint distribution is simply a permutation distribution on $0, 0, \ldots, 0, 1$). Since each ball is thrown independently of the others, we conclude from Lemma 4(1) that the full set of variables $(B_{i,k} \mid i \in [n], k \in [m])$ are negatively associated. Finally observe that (using the Iverson symbol $[P]$ which is 1 if the boolean property P is true and 0 otherwise) $Z_i = [\sum_k B_{i,k} = 0]$ is a non–increasing function of $B_{i,k}, k \in [m]$. Hence the result follows from Lemma 4(2). ∎

Remark 6 In [2], a much fuller discussion of negative dependence in the balls and bins experiment can be found.

Proposition 7 ([2]) *The Chernoff–Hoeffding bound applies to sums of negatively associated variables.*

Sketch of Proof. From the definition of negative association, it follows by induction that if $X_1, \ldots, X_n$ are negatively associated, then

$$E[\prod_{i \in [n]} f_i(X_i)] \leq \prod_{i \in [n]} E[f_i(X_i)],$$

for any non–decreasing functions $f_i : \mathsf{R} \to \mathsf{R}$. Now we apply the usual proof of the Chernoff–Hoeffding bound (see for instance [1, 7]) using the above inequality to replace the equality $E[\prod_{i \in [n]} e^{X_i}] = \prod_{i \in [n]} E[e^{X_i}]$ (which holds when the variables are independent) by the above inequality with each $f_i(x) := e^x$. ∎

This yields directly, in a calculation–free manner, the (Chernoff) Occupancy Bound 2 in [5, Theorem 3].

Remark 8 It should be mentioned that [5] fail to mention [8] where this result is also obtained by arguments similar to theirs.

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