Fostering the spirit of mathematical modelling in vocational education

PETER FREJD

This study explores the fostering of The Spirit of Mathematical Modelling in vocational education, within the Swedish Handicraft and Natural resource use programs. Through classroom observations, it identifies how two activities, hair colouring and sawing pattern, embody mathematical modelling aspects, highlighting collaboration, accountability, ample time, interdisciplinarity, situatedness and usefulness. The sawing pattern activity, in addition, emphasises agency, relevance, design and consultation. While fostering The Mathematical Modelling Spirit, the study notes the absence of justice and flow. It concludes by reflecting on the impact of these aspects on vocational mathematics education, their implications for career readiness and pedagogical opportunities. In addition, theoretical and methodological challenges are discussed.

There exists a widespread concurrence, both domestically and globally, among educational researchers that there is an imperative for investigative teaching and learning strategies aimed at connecting the mathematics students learn to their future occupation (e.g. FitzSimons, 2001; Frejd & Muhrman, 2022).Vocational mathematics education is an institution where learning takes place with an explicit aim to prepare individuals for future employment (FitzSimons, 2014). However, students in vocational education often lack interest in learning mathematics and do not see the relevance of learning mathematics (Muhrman, 2016). Dalby and Noyes (2016) argue that realistic workplace contexts motivate students to learn mathematics. Mathematical modelling in educational settings can be used to connect mathematics to practices outside the discipline of mathematics (Frejd & Vos, 2024; Frejd et al., accepted; Wake, 2014), such as workplace situations. Therefore, mathematical modelling could also be suitable in vocational education to change students´ negative atti-

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tudes towards the relevance of using mathematics in different situations. Mathematical modelling is a fundamental ability incorporated into curricula worldwide, both as a subject in itself and as a means of learning other content (Julie & Mudaly, 2007). The same applies to the Swedish vocational mathematics curriculum (Skolverket, 2012). Even though there seem to be many recommendations to include mathematical modelling in vocational education, Frejd and Ärlebäck (2024) found that ”we are missing large scale empirical studies, small interventions, overviews, theoretical research etc. on mathematical modelling with an emphasis on vocational education” (p. 135). As a response to the lack of both theoretical and empirical research, the aim of this paper is to explore theoretical principles for analysing classroom activities in vocational education to foster *The Spirit of Mathematical Modelling* (SoMM) (Frejd & Vos, 2024) and to apply these principles in a case study.

The following research question will guide my work: What aspects of SoMM relevant to vocational education can be discerned in different vocational education contexts?

The next few sections are structured as follows: First, I will discuss the concept of SoMM. Second, I develop analytical principles for analysing vocational education through SoMM. Thereafter, these principles will be applied to investigate and evaluate two vocational education contexts, namely two classroom activities in the Swedish upper secondary vocational programs—the Handicraft Program and the Natural Resource Use Program—to explore their potential to fostering SoMM.

The Spirit of Mathematical Modelling

Drawing on the discussions in research literature about mathematical modelling during the last 50 years Frejd and Vos (2024) analysed the term SoMM, which captures core aspects of mathematical modelling. With references to the levels of *Micro* (the level of the individual), *Meso* (the level of groups of individuals) and *Macro* (the level of broader social and cultural structures) Frejd and Vos (2024) found a collection of aspects that characterise the term and this collection was depicted in a thematic map illustrated as a dreamcatcher (figure 1)

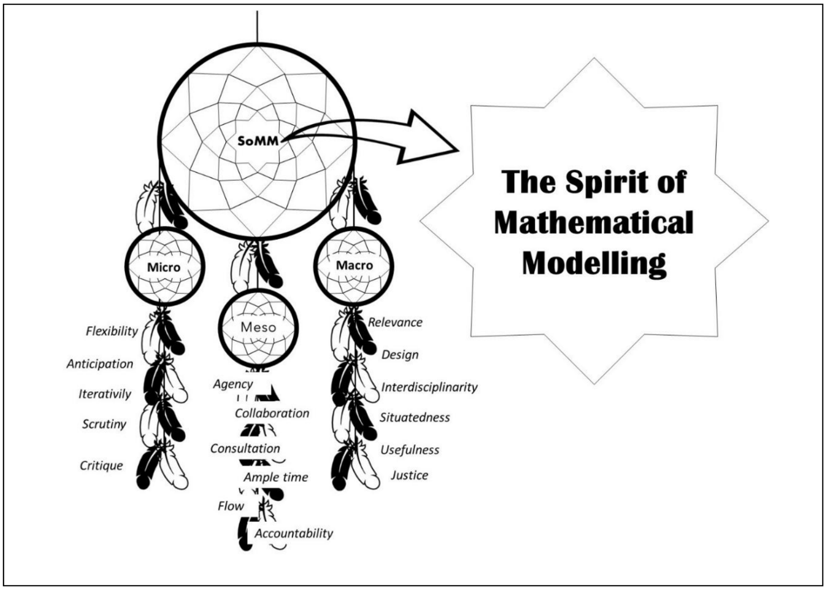

Figure 1. *The dreamcatcher depicting The Spirit of Mathematical modelling (Frejd & Vos, 2024, p. 286)*

The cluster of feathers to the left in figure 1 are related to the Micro level and connects to the five aspects of *flexibility*, *anticipation*, *iteratively*, *scrutiny* and *critique*. Primarily, these aspects of SoMM are connected to the individuals' metacognition when it comes to coordinating and regulating different modelling processes. The aspect flexibility denotes the capacity for adaptive and innovative responses to unforeseen challenges, as well as the reflective modification of preliminary strategies. The aspect anticipation involves a proactive stance taken across all stages of modelling work and considering subsequent actions. The aspect iteratively refers to the recursive engagement with the stages of the modelling procedure, coupled with reflections of these engagements. The aspect scrutiny encompasses the modeler's analytical examination of an ill-defined task, and it includes the interrogation done by the modeller among peers, specialists, and the client, to discern the underlying objectives. The aspect critique involves rigorously assessing whether the outcomes will fulfill the quest for a satisfying solution and considerations on how similar challenges have been approached by other modelers.

In the centre of figure 1, the Meso level contains six aspects of SoMM. The aspect *agency* in the top of the feathers refers to the empowerment of modelers to exercise discretion and self-direction, which is intrinsically linked to autonomy. The aspect *collaboration* is a fundamental element of mathematical modelling, transcending simple group activities to encompass shared comprehension, joint responsibility for group advancement, and collective decision-making authority. The aspect *consultation* signifies an iterative dialogue to exchange insights and perspectives with a client throughout the modelling process. The aspect *ample time* denotes the provision of sufficient duration for in-depth exploration, which may span a single session or extend over multiple weeks. The aspect *flow* describes a state of deep engagement where students become oblivious to time due to the absorbing and stimulating nature of the task. Finally, the aspect *accountability* implies the obligation of modelers to provide sound advice to their clients, considering the context of the problem.

The right clusters of feathers in figure 1, the Macro level, list the following six aspects *relevance*, *design*, *interdisciplinarity*, *situatedness*, *usefulness* and *justice*. The aspect relevance refers to the practical urgency from a client for a solution, emphasising that the context should be genuine and not contrived merely to apply newly acquired theoretical concepts or algorithms. The aspect design involves the deliberate act of intervention and realization of objectives, such as employing models to determine necessary actions or to persuade policymakers or educational administrators. The aspect interdisciplinarity reflects the integration of mathematics with other scientific disciplines, such as medicine, environmental science, and natural sciences. The aspect situatedness emphasises authentic problems, thereby contextualizing mathematics within real-world scenarios where variables possess both quantitative dimensions (e.g., kilograms, euros) and qualitative significance (e.g., waste, healthcare costs). The aspect usefulness describes the recognized societal value of mathematics, acknowledging its role in scientific discourse for forecasting phenomena such as pandemics, ensuring structural safety, evaluating financial ventures, without necessitating detailed mathematical expertise. Finally, the aspect justice addresses the ethical considerations of mathematical modelling, particularly how it is employed in selection and assessment processes that may accidentally preserve or aggravate disparities based on gender, ethnicity, and other forms of discrimination. For more information on SoMM, see Frejd and Vos (2024).

The next sections connect relevant aspects of SoMM with vocational mathematics education research and discuss the term fostering.

Developing an analytic framework

Before identifying aspects of SoMM in research literature connected to elements of qualitative teaching in vocational mathematics education, the term *fostering* needs to be described. Fostering, in contrast to direct traditional instruction, is an educational strategy that prioritizes encouragement and support to facilitate students´ development (Kokkinos, 2009). This approach to teaching relies on building trust and engaging in meaningful interactions, often requiring time and a focus on specific goals. Fostering is associated with creativity and cooperative learning, and it embodies an indirect, investigative approach emphasising active engagement and self-motivation among learners. According to Frejd et al. (2024) this educational strategy develops a classroom environment that cherishes creativity, endurance, and open dialogue, which motivates students to tackle real-world challenges and develops their teamwork abilities.

For this study on developing analytical principles for classroom observations in vocational mathematics education, aspects of SoMM related to metacognition and cognition are excluded. Investigating these aspects typically requires methods beyond classroom observations, such as interviews, questionnaires, and tests (Robson, 2002). Thus, the literature review will focus solely on the macro and meso levels of SoMM

Vocational mathematics education through the lens of SoMM

From an international perspective, the strategies of embedding vocational education into the current education system show considerable heterogeneity, leading to diverse integrations of mathematics education (Frejd, 2018). Vocational education may transpire within formal or informal settings and is often embedded within different types of apprenticeship models. Despite the differences, research points in the direction of some general objectives within vocational mathematics education that should encompass the cultivation of a tripartite mathematical knowledge framework (e.g. Strässer, 2000; Wedege, 2000), see table 1.

Table 1. *Types of knowledge to be taught in vocational mathematics education*

Type of knowledge	Description
Mathematical	Formal mathematical principles and theories
Practical	The application of mathematical concepts within vocational settings
Reflective	Meta-cognitive insights regarding the nature and utility of mathematics

The prioritisation of the type of knowledge to be taught, as shown in table 1, in vocational mathematics education curricula depends on political, social and economic decisions (Frejd, 2018). In the Swedish context, the curriculum emphasises all three types of knowledge (Skolverket, 2012), but current teaching practice seems to prioritise mathematical knowledge in favour of practical and metacognitive knowledge (Muhrman, 2016).

Many suggestions for elements to include in vocational education for qualitative teaching are found in the research literature (e.g. LaCroix, 2014; Williams & Wake, 2007). One such element is *collaboration across disciplines*, encouraging mathematics teachers to work alongside other teachers and professionals. This teamwork is essential for understanding the unique challenges and requirements of different workplaces (Frejd & Muhrman, 2022), each with its distinct social and cultural practices (FitzSimons & Björklund Boistrup, 2017). Such interdisciplinary cooperation is beneficial for creating real-world problems that are relevant and meaningful for students, which is termed as relevance in the macro level of SoMM. The interdisciplinarity nature of mathematical modelling, also in the macro level of SoMM complements the call for collaboration across disciplines and highlighting the integration of mathematics with other fields.

Another element to consider is *authenticity*. Mouwitz (2013) emphasises that mathematics in the workplace is not a standalone subject but is deeply rooted in the cultural practices, traditions, and artifacts of various vocations. These factors are integral to the design of authentic classroom activities. Authenticity is a shared concern in SoMM, and addressed as situatedness in the macro level, emphasising that the modelling problems and the way to solve them should be relevant to professional practices. Strässer (2007) highlights that tools and artifacts used in the workplace contribute to the authenticity of vocational education. Often these tools function as 'black boxes' with specific inputs and outputs, necessitating the interpretation of these outputs, which is a common workplace task. Many operational decisions also depend on these numerical data outputs. Enhancing students' capacity to comprehend and critically evaluate the mathematical models that inform these decisions is a key objective of mathematics education, as highlighted by Skovsmose (1994). Consequently, vocational mathematics education plays a pivotal role in demystifying the mathematics embedded in workplace technology. This resonates with usefulness in the macro level of SoMM, which calls for developing students' understanding of the societal value of mathematics in different contexts.Wake (2007) suggests that students should explain the use of workplace tools to others, necessitating a deep understanding of both the workplace environment and the underlying mathematics.

In relation to authentic opportunities for learning, Frejd and Muhrman (2022) analysed the *learning space*, i.e. how students worked with mathematical problems in the mathematics classroom compared to how they worked with mathematical problems in the training hall. They concluded that the training hall environment significantly enhanced student motivation. Additionally, it was observed that the mathematical discourse within this setting was notably more substantive and engaging, which relates to the term flow in the meso level of SoMM, addressing when students get so engaged and absorbed by a task that they forget the time. Jensen (2022) also elucidated significant disparities in student engagement between workshop settings and the traditional mathematics classrooms. In addition, Jensen (2022) documented divergent conceptualizations of competence within these two distinct communities of practice. Engaging with vocational tasks in both the mathematics classroom and vocational training hall redefined classroom roles of teachers and students and altered perceptions of expertise (Sundtjønn, 2021). However, limiting tasks to the mathematics classroom often retains conventional roles, making it difficult for students to navigate different communities of practice (Sundtjønn, 2021)

The elements of *identity and self-esteem* are connected to where learning takes place. Mouwitz (2013) asserts that a thorough understanding of one's professional role is foundational to self-esteem and workplace identity. This comprehension typically fosters a commitment to excellence in one's work. It also frequently results in an individual assuming accountability for the quality of their work, ensuring it aligns with the expectations of clients or employers, which refers to the aspect accountability in the meso level of SoMM. Consequently, the viewpoints of clients and consumers should be a part of activities within vocational mathematics education to ensure relevance. In particular, the aspect consultation, describing the dialogue between a client and the modeller should be emphasised. Students' self-esteem is to some extent a product of an individual's agency in the meso level and autonomy (Ryan & Deci, 2002), meaning that students with low possibilities to regulate their own actions for learning in the classroom may develop low self-esteem.

Communication and cooperation skills are highly valued in the workplace. Mouwitz (2013) and Wake (2007) observe that a significant number of occupational tasks necessitate teamwork, which in turn demands proficient communication and collaboration abilities. Similar arguments are used in SoMM under the aspect of collaboration in the meso level. These joint ventures require planning and coordination. The integration of such collaborative tasks into vocational education curricula may serve to emphasise the authenticity and practical relevance of the educational experience.

Within the limited sample of research literature examined above, the aspects ample time, design and justice in the meso or the macro level were not identified. However, one may argue that these aspects could also be productively integrated into vocational mathematics education. The culture of the workplace demands that the product or service provided is of high quality, and this may take a long time to complete (Mouwitz, 2013), in other words, ample time. In SoMM design refers to an act of accomplishing or shaping something with the support of mathematical models. In professional vocational practices it may be interpreted as the act of practically constructing or developing something or convincing a client based on a mathematical model. For example, a carpenter might use formulas or artefacts to calculate angles for cutting wood pieces that fit together perfectly, or a carpenter may use mathematical models to estimate the cost of a project and possibly convince a client that it is a good offer. Concerns about justice arise when mathematical models are used in selection and evaluation processes, as they can sometimes intentionally exacerbate gender, ethnic, and other forms of discrimination. In workplace situations, justice is also an aspect to consider, such as if a company uses mathematical models to determine credit scores. These models can sometimes unfairly penalize individuals from certain socioeconomic backgrounds if the data used reflects historical bias.

In summary, all, except three, aspects of SoMM in the Meso and Macro perspective were linked to a sample of previous literature on vocational mathematics education, and the three remaining aspects could easily be argued to be included as analytical principles. The table below summarises this section and describes the analytical principles for analysing classroom activities within vocational education through the aspects of SoMM.

Table 2. *Analytical principles for analysing vocational mathematics education through SoMM*

Aspects of SoMM	Related elements in Vocation. Ed.	Descriptions of analytical principles for analysing activities in vocational mathematics education.
Agency	Identity and self-esteem	Agency in SoMM is in line with the aims of vocational mathematics education to develop students' workplace identity and self-esteem. Students should be given autonomy in learning activities to make decisions and apply their knowledge independently in different situations.
Collaboration	Communication and cooperation skills	Collaboration is an aspect emphasised both in the modelling work and in workplace-related situations. Many jobs in the professions require teamwork, so students need to practise cooperation and communication skills during their training.
Consultation	Identity and communication	Considering a client perspective in teaching, which includes educational experiences about consultation, is a common focus in modelling and in vocational mathematics education. It emphasises authenticity and develops students' identity in relation to their future working practices.
Ample time	-	One aspect of SoMM is ample time, which can also be part of vocational mathematics education. The development of high quality products or services requires students to practise for long periods of time, and learning is usually not a quick process.
Flow	The learning space	An engaging environment is crucial for deep learning, so in order to develop flow, the learning space in vocational mathematics education needs to be considered.
Accountability	Identity	Part of the students' identity process is practising giving sound advice to clients and considering the consequences of possible mistakes, which is described in the SoMM as accountability.
Relevance	Collaboration across disciplines	The emphasis on ensuring that mathematics education is relevant to vocational practice, which often requires close cooperation between mathematics and vocational teachers.
Design	-	Workplace tasks often involve practical tasks based on underlying assumptions, so considering how models are put into practice in vocational mathematics education is essential for students to learn.
Interdisciplinary	Collaboration across disciplines	Integrating mathematics with other subjects can be done in different ways, but research in mathematics education suggests that collaboration between teachers of different subjects makes it more interesting and relevant for students.
Situatedness	Authenticity	Authenticity is a shared concern, reflecting an emphasis on ensuring that mathematics education is relevant to the professional practices, cultural practices and traditions of different professions.
Usefulness	Authenticity	To highlight the cultural practices and traditions of different professions and to explain the practicality of mathematics.
Justice	-	Provide students with educational experiences where they can discuss the consequences of using workplace-related mathematical models that may be unfair to different groups of people.

Method

In the following sections, two classroom activities in vocational education, *Hair colouring* and *Sawing patterns*, will be analysed. These activities were observed by the author within a research project about mathematics in vocational education financed by the Swedish Research Council. The data collected from the observations consists of classroom notes, instructions, worksheets, pictures and some student solutions. Based on this data, narrative accounts were developed considering the nine ”dimensions of descriptive observations”: space, actors, activities, objects, acts, events, time goals, and feelings (Robson, 2002, p. 320). These narrative accounts will in the next few sections be described and analysed with a ”Template approach” (Robson, 2002, p. 458). The Template approach, as described by Robson (2002), is an analytic approach which involves creating a set of codes or categories (a ”template”) that are applied to the data. For the purpose of evaluating the extent to which the observed classroom activities foster different aspects of SoMM, the template for analysis consists of the analytical principles in table 2.

Hair colouring

The instructional design, a joint effort by mathematics and hairdressing teachers, integrated both subjects in an activity about hair colouring. The activity was conducted over two one-hour lessons, spanning over two days, the first lesson in a mathematics classroom and the second one in a salon. Both teachers taught the final lesson.

Sixteen girls attended the first lesson, and the mathematics teacher began by explaining the aim of the lesson. ”You are going to develop a tool that will automatically calculate how much colour and how much hydrogen peroxide you should use to mix for different hair types and lengths. When you are done with the task, all you need to do is estimate the length of the hair to get the amount of colour you need,” the teacher said before asking the students to bring their laptops forward. The teacher opened a spreadsheet and showed it on the whiteboard at the front of the classroom and started talking about rows and columns.

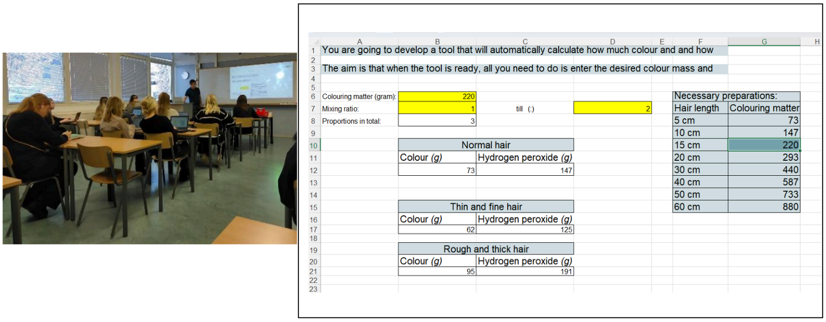

Figure 2. *To the left the mathematics classroom and to the right the digital tool (translated version)*

The teacher demonstrated, using proportionality, how to complete the table on hair length and colouring matter (see figure 2, right). For 15 cm of hair, 220 g colouring matter is required , the teacher argued and to calculate the amount of colouring matter for 5 cm we write in Excel ”=220/3” or ” = G10/3” in cell H10. Students then created their own tables for hair length and colouring matter, with the teacher guiding and explaining the benefits of using cell names. Thereafter, the teacher redirected focus to additional Excel tables (the tables for Normal hair, Thin and fine hair, and a table declaring Colouring matter, Mixing ratio and Proportions in total) and showed how to use Excel for calculating the required amounts of colour and hydrogen peroxide for some customers´ hair. To motivate the students, the teacher said, ”You have done this in your VET-class”. He continued with the example of 220 g of colouring matter and a mixing ratio of 1:2. He said it makes 3 parts and showed the formula ”=220/3” which yields 73 g of colour and ”2 * 73” gives 146 g of hydrogen peroxide. He inserted the formula =B6/B8 (B6= the colouring matter and B8=total number of parts) for calculating the amount of colour in cell B12. The teacher also showed how to adjust colour quantities for thin hair (15 % less) and thick hair (30 % more) using formulas like ”=B12 - 0.15 * B12” for thin hair (the teacher had not introduced change factor). Students then practiced these calculations and completed a few exercises before the lesson ended.

The second lesson was situated in a training hall, a nail styling make-up studio, with small tables, movable chairs, and table lamps for manicure, and fifteen students attended. The VET-teacher introduce the lesson by saying ”we are going to work with different cases that deal with colour”, and the mathematics teacher adds ”you should use the digital tool you developed during the last lesson”. The mathematics teacher continued and asked the students to log in to get access to a document with different client cases (see an example of the task in figure 3, right). The students' tasks are about estimating the length of the hair from pictures and calculating the amount of Colour (g) and Hydrogen peroxide (g). The students work together in groups of three to four students around the small tables with their laptops open (figure 3, left). Both teachers walk around the training hall and support the students.

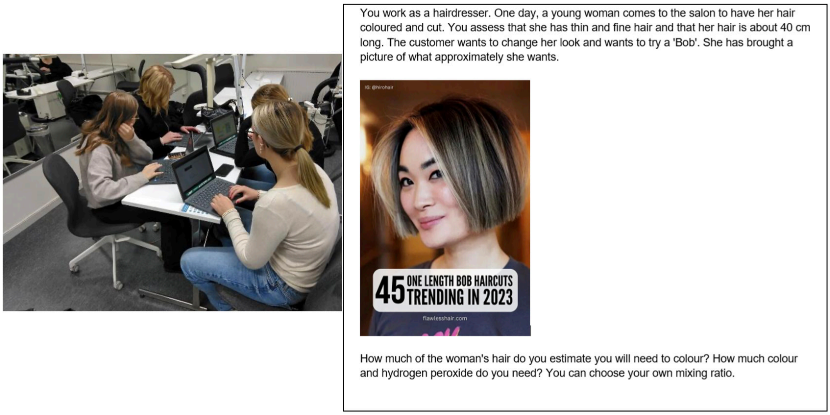

Figure 3. *To the left the vocational hall and to the right a translated version of one of the tasks*

The mathematics teacher focused on guiding students through Excel, while the vocational teacher addressed practical queries. For example, the VET teacher used her 5 cm hair length to help a student estimate hair length, likening it to the size of a thumb. For "lighter toning," she clarified to a student that no mixing is needed – only colour amount. The mathematics teacher guided students in Excel. For example, explained that an output of 19 g of colour and 38 g of hydrogen peroxide was incorrect, when the colouring matter was 267 g (as 19 g + 38 g does not equal 267 g) and that thick hair increases the colouring matter by 30 % compared to normal hair.

The students worked for about an hour. In the end of the lesson, both teachers asked the students if they think they may continue to use the developed tools in the future, and the students answered with YES.

An analysis of the activity Hair colouring

The two lessons in the activity differ significantly. In the first lesson, the teacher provides direct instructions for creating a mathematical model in Excel to calculate the amounts of colour and hydrogen peroxide. The second lesson focuses on applying the developed tool by discussing client cases in a more fostering manner.

From the top of table 2, the first lesson lacks the aspect of agency, as it involves replicating the teacher's work, and the Excel model is somewhat pre-defined. The second lesson also has a weak agency aspect, with straightforward tasks, but it does feature clear collaboration among students and teachers. The client cases simulate a consultation aspect of SoMM, but resembling more of a one-way communication than an iterative dialogue. Most students completed the tasks within the two lessons, suggesting that the ample time was sufficient. However, a critical view suggests that more hands-on experience in colour mixing is necessary for deeper expertise. Despite recognizing the tool's usefulness by the end of the second lesson, students did not opt to stay beyond the scheduled time, which is a characteristic of lack of flow. It is possible that the activity did not provide enough opportunities for further discussion. The presence and evaluation by the VET teacher in the second lesson fostered accountability, as students had to justify their decisions.

The aspects of interdisciplinarity and situatedness in table 2 were evident. The integration of mathematics and hairdressing discourse, and the realistic scenario of a client bringing in a haircut photo, demonstrate the practical application of mathematics in hair colouring and the use of technology in workplace problem-solving. The relevance of using an Excel file for these calculations in a real-world context is debatable. To address the design aspect and make a tangible impact, the activity could be expanded to include the application of mathematical models on actual clients visiting the salon. The justice aspect, related to modelling and discrimination, does not apply to this activity.

Sawing patterns

This three-hour lesson with focus on wood processing was a part of the course wood science 100p (Skolverket, 2022) and organized by a VET teacher. The aim of the lesson was exploring sawing patterns for maximizing volume yield and value. The VET-teacher initiated an activity where ten students paired up, received A3 papers, pencils, and tapes, and went outside to a sawmill on school grounds (see figure 4). Each pair chose a log, measured its length, and traced the smaller end's circumference onto their paper.

Figure 4. *The sawmill*

After this outdoor activity, the class reconvened indoors where the teacher outlined sawing techniques and patterns on the whiteboard, making sketches, noting a 5 mm saw seam for planks. Students then received worksheets and price lists (see figure 5) for various timber sizes of planed and rough saw timber. The worksheet consisted of a table with the following labels: top diameter of the log, length of the log, volume of the log (m3fub [cubic meters solid under bark]), class, value, volume of timber (m3), value of the timber, and saw yield. The pricelists were taken from a nearby sawmill and from a builder's merchant. The teacher instructed the pairs to first calculate the value of the log (fill in the first four cells in the table) and then draw the "best" sawing pattern on their A3 paper applying the dimensions of boards and plank found in the price list, thereafter calculate the volume of timber, value of the timber, and saw yield.

Top diameter of the log	
Length of the log	
Volume of the log (m3fub)	
Class	
Value	
Volume of the timber (m3)	
Value of the timber	
Saw yield	

Sjöbo Såg - Göran Hägnestig

Prislista 2021

Brädor

Dimension	Kr/lpm
25x50	3,5
25x75	5
25x100	6,5
25x125	8
25x150	9,5
25x175	11,5
25x200	13
25x225	15

Regelvirke

Dimension	Kr/lpm
50x50	6,5
50x75	10
50x100	13
50x125	16
50x150	19
50x175	22
50x200	26
50x225	30

Kvistfri tall är dubbelt pris.
Priser är inkl. moms.

AB Backabo Byggvaror

Datum 210928 **Nummer** 571079
Leverensdatum 210928 **Orderdatum** 210928 / **Ordernr** 571079
Vår referens **Beställningsnr/Er referens**

Godsmottagare Mailfaktura!

Köpare HERMANSSON FELIX
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518 32 SANDARED

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Vi förbehåller oss äganderätten av
materialet tills likvid erlagts av Er.
Anmärkningar ej gjorda inom 8 dagar godkännes ej.

Betalningsvillkor
10 dagar netto
Vår säljare

Artikelnr	Benämning	Antal	Enhet	á-pris	%	Belopp
87221095	FINSÅG.GRAN 22*95	1	M	17.70		17.70
87221120	FINSÅG.GRAN 22*120	1	M	28.00		28.00
87222145	FINSÅG.GRAN 22*145	1	M	31.00		31.00
87222170	FINSÅG.GRAN 22*170	1	M	35.00		35.00
87222195	FINSÅG.GRAN 22*195	1	M	49.00		49.00
87222220	FINSÅG.GRAN 22*220MM	1	M	49.00		49.00
88045045	RÅPLAN 45*45MM os	1	M	20.00		20.00
88045070	RÅPLAN 45*70MM	1	M	34.00		34.00
88045095	RÅPLAN 45*95MM c24	1	M	46.00		46.00
88045120	RÅPLAN 45*120MM c24	1	M	56.00		56.00
88045145	RÅPLAN 45*145MM c24	1	M	63.00		63.00
88045170	RÅPLAN 45*170MM C24	1	M	82.00		82.00
88045195	RÅPLAN 45*195MM c24	1	M	90.00		90.00
88045222	RÅPLAN 45*220MM c24	1	M	96.00		96.00

Figure 5. *Top left the worksheet (translated), bottom left the price list for sawed timber (rough) and to the right the price list for sawed timber (planed)*

Some pairs had forgotten how to calculate the volume of the log in the unit of m3fub. The teacher refreshed the students' memory by providing an example on the whiteboard. I observed a pair calculating the log's volume using the formula 0.17/2*0.17/2*pi*5*1.23 = 0.14 m3fub, with a top diameter of 17 cm and a length of 5 m. The coefficient 1.23 is a conversion factor that converts the volume (m^3 toub) of a cylinder, calculated using the top diameter under bark multiplied by the log's length, into m^3 fub (cubic meters of solid wood under bark). The pair classified the spruce log as quality class 1 due to its straightness and determined its value to be 14* 500= 70 SEK based on the pricelist and their calculated volume.

The students' next step was to select suitable sawing patterns, comparing dimensions from the pricelist to maximize the number of boards and planks they could fit onto their A3 paper. The observed pair devised three distinct sawing patterns, as illustrated in figure 6.

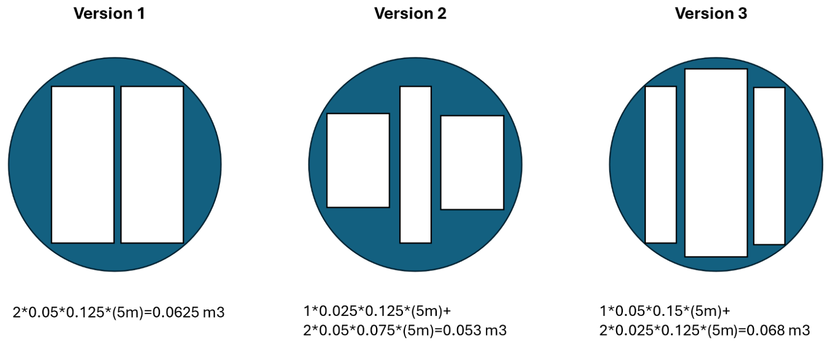

Figure 6. *A pair of students' sawing pattern with calculated timber volume*

The students assessed three sawing patterns, calculating timber volume, value, and yield for each. Version 1 resulted in 0.0625 m^3 and 160 SEK (rough) or 560 SEK (planed) with a 45 % yield. Version 2 had 0.053 m^3, 140 SEK (rough), 480 SEK (planed), with a yield uncalculated. Version 3 produced 0.068 m^3, 175 SEK (rough), 595 SEK (planed), with a 49 % yield. With some detectives' work, their solutions can be found in figure 7.

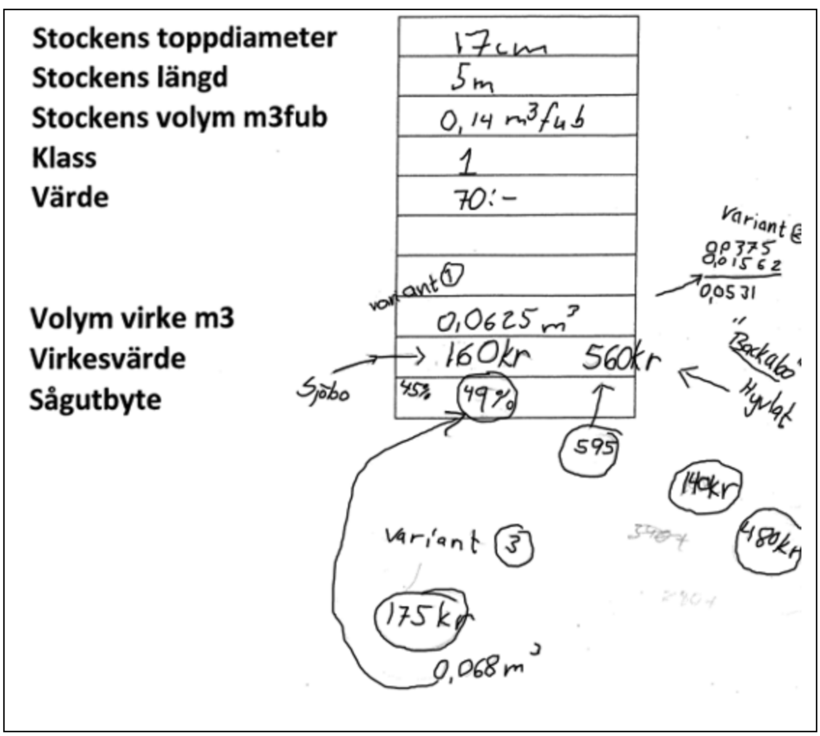

Figure 7. *The pair of students' solution*

After completing their tables, students shared their sawing patterns and yields on the whiteboard, noting that larger log diameters yield better results. The lesson concluded with a practical demonstration at the sawmill, where the teacher and students sawed logs using the students' patterns.

An analysis of the sawing pattern activity

In this activity, the teacher combined direct instruction with fostering techniques. He guided students in measuring log lengths, transferring the smallest end of the log onto A3 paper, and calculating the log's value using a pre-defined table. Particularly, when students established optimal sawing patterns to maximize yield and value, they had the opportunity to collaboratively in pairs to try without any directed instructions to create, explore and evaluate different sawing patterns, while the teacher focused on fostering.

Compared to the hairdressing salon activity, agency was more pronounced in the sawing pattern. Students chose their logs and independently developed geometric mathematical models to determine the best sawing patterns, considering timber prices to assess value. Collaboration was central; students worked in pairs and shared their findings, with teachers facilitating rather than instructing. This could be seen as a form of consultation, involving iterative dialogues between students and teachers. The activity placed students in a role-playing scenario, selecting logs at the sawmill, fostering a sense of ownership and professional responsibility. All pairs completed their calculations and presentations within the allotted time, indicating ample time for the task. The teacher's application of students' saw patterns to actual timber sawing reinforced accountability and engagement, though not to the extent of experiencing flow.

All the following aspects in table 2 were evident except justice, as the task focused on sawing yield and log value without ethical considerations. The activity was interdisciplinary, leaning more towards vocational than mathematical education. It exemplified the practicality of mathematics in deriving optimal solutions in receiving the best profit. For farmers with forests and small sawmills, the task's authenticity in determining sawing patterns was relevant, addressing both situatedness and relevance. However, in larger sawmills, optimal patterns are computer-generated using X-ray and 3D scanning data (Skog, 2013). The situatedness is also present in the different types of units used, such as m3fub, m3toub and the conversion factor 1,23. The students also witnessed how their mathematical model of sawing pattern turned their log into sawed timber, realizing an object and making some impact of their model, fulfilling the design aspect of SoMM.

Summary of results

In this paper, I explored the presence of the aspects of SoMM in two classroom activities in vocational education and identified a number of aspects, which are presented in table 3.

Table 3. *Summary of identified aspects of SoMM in the two activities*

	Aspects of SoMM											
	Agency	Collaboration	Consultation	Ample of time	Flow	Accountability	Relevance	Design	Interdisciplinary	Situatedness	Usefullness	Justice
Activity 1	-	x	-	x	-	x	-	-	x	x	x	-
Activity 2	x	x	x	x	-	x	x	x	x	x	x	-

The two classroom activities analysed showcased collaboration, ample time, accountability, interdisciplinary connections, situatedness, and usefulness. These aspects are consistent with effective vocational mathematics teaching practices, particularly because the activities were, to some extent, conducted in the students' training hall, and the VET teacher played an active role (Dalby & Noyes, 2016; Frejd & Muhrman, 2022). The sawing pattern activity further incorporated agency, consultation, relevance, and design, offering students strategic freedom and practical insights into workplace mathematics. Sawing patterns and timber yield are not only relevant for workers in sawmills but also constitute general knowledge for everyone in the forestry industry. Seeing the results of a mathematical model and understanding its impact, as highlighted by the design aspect, is what modelling education aims to achieve (Frejd & Vos, 2024). The analysed activities reflect the SoMM's goals, though justice and flow were not observed.

Discussion

This article is a first attempt to develop analytical principles for analysing vocational education through SoMM and apply these principles to analyse some classroom activities. The extent to which the activities aim to foster as opposed to teach mathematical modelling could be questioned. Both activities have pre-defined mathematical models that the students have to reinvent either in Excel or geometrically on A3 paper. Most of the instructions are direct and the calculations are procedural. However, this is a common strategy for vocational mathematics education, as workplace mathematics consists of mathematical models in some kind of technology with an input and output system, which can be digital or paper-based. Both the mathematical models that the students developed in Excel (figure 2) and the structured worksheet (figure 5) were recognised as making the mathematics visible as the students had to do the calculations. Making mathematics visible, thus demystifying workplace mathematics, is central for vocational mathematics education (Williams & Wake, 2007). None of the activities had the explicit aim of teaching mathematical modelling, instead the focus was linked to 1) *mathematical knowledge* of how to create formulas in Excel during the first lesson about developing digital calculating tool useful for bleaching and toning clients' hair, 2) *practical knowledge* of how the application of mathematics is used within professional settings (hairdressing and sawing), and 3) *reflective knowledge* about meta-cognitive insights on the nature and utility of mathematics.

The role of the vocational teachers is central to many of the aspects found, such as developing relevance, accountability, consultation, design, etc. They have the power to make the activity authentic for the students and to explain the norms and expectations of the workplace, which helps the students to build identity and self-esteem (Mouwitz, 2013). Recommendations for the use of two-teacher systems, where mathematics and vocational teachers work together and where part of the mathematics education takes place in the students' training rooms, are found in previous research literature (Frejd & Muhrman, 2022). The findings of this study show that such collaboration between different teachers also helps to reflect several aspects of SoMM. Activities that reflect many of the aspects in SoMM do impact on the traditional norms in the classroom at least compared to traditional teaching using the textbooks (Sundtjønn, 2021). The sawing pattern activity shows one example where the students are working practically outdoors (taking measures, sawing, etc.) and theoretically indoor (calculating, reporting, etc.) and do not use any textbooks.

The activities do not reflect all aspects of the SoMM, and there are several opportunities to develop them further and prepare students more effectively for their future careers. For example, the relevance, design and possibly consulting aspects would have been reflected in the hairdressing salon if students had applied their solutions to dyed dolls' hair or invited real clients. If the students had developed a workplace identity and had bleached and coloured real clients in the hairdressing salon, I would assume that they would finish their work even if the bell rang, potentially reflecting the aspect of flow. It is not easy to develop a commitment such as flow, but if many of the other aspects of SoMM are reflected, it may be easier than if none are reflected. To develop the sawing pattern activity further, the students could visit a large sawmill and discuss with the workers how they or the computer decide on sawing patterns, etc. The aspect of justice, ethical considerations of mathematical modelling, was not reflected in the two activities. It's possible that if the prices for bleaching someone's hair were set based on gender, such a discussion might develop, and in the sawing pattern activity, the ethical discussion might be about the quality of the sawing timer versus getting the most timber out of the log, which relates to high yield.

The outcome of the analysis is based on the selected method. To apply theoretical principles for analysing classroom observations is non-trivial. There can be a gap between theory and practice, where theoretical principles may not readily translate into insights of teachers' actions or may not align with students' practical experiences of learning (Robson, 2002). In addition, classrooms are dynamic environments where context matters a great deal and theoretical principles may not account for the unique cultural, social, and individual factors that influence each classroom setting. However, the aspects of SoMM are developed as a tool intended for applicability in analysing 'all' classroom environments. Yet, an inherent consequence of such broad applicability is that the aspects are delineated in a general way. Teaching is a complex activity that involves multiple, simultaneous actions and decisions and since the theoretical principles of SoMM are generally defined, they may oversimplify this complexity, making it challenging to apply them accurately to analyse real-life classroom situation. For example, when teachers in one part of the lesson use much direct instructions focusing on traditional teaching and later give students more autonomy and working more collaborative with problem solving with an aim of fostering, makes it difficult for the researcher to classify if the activity is about fostering or not. As the single researcher conducting the observations I may have unconscious biases that affect my interpretation of the events (Robson, 2002), which can lead to a misalignment between the principles developed and the actual observations. On the one hand, more robust and clear definitions of the different aspects of SoMM could be useful for interpretation of observations, but on the other hand, that would affect the generalization and applicability.

Nonetheless, this constituted an initial attempt to apply the aspects of SoMM to the analysis of classroom observations within the context of mathematical modelling in vocational education. While numerous aspects were discerned, not all were identified, suggesting that there exists room for further enhancement of the activities. The analysis was concentrated at the Meso and Macro levels; it is conceivable that this study may catalyse additional research inquiries that explore the aspects of SoMM at the Micro level. Furthermore, there appears to be potential for employing the Spirit of Mathematical Modelling framework within design research to develop activities predicated on these theoretical tenets.

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