Triangle and parallelogram area formulas: a critique of teacher education textbooks in Norway

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We survey Norwegian textbooks for preservice teachers, looking for attention to generality in their presentations of diagrammatic proofs for the area formulas for triangles and parallelograms. We find that most textbooks do little to bring issues concerning generality to the attention of the reader in the presented arguments, neither explicitly nor implicitly. We also find that the textbooks fail, as a rule, to present correct proofs of the area formulas for parallelograms and triangles, even given a liberal view of what constitutes a proof. The presented proofs are incorrect in that they are insufficiently general; they fail to cover crucial cases. This is unfortunate, as preservice teachers should be made aware of issues concerning generality in diagrammatic proofs.

Norwegian primary mathematics education guidelines promote an emphasis on understanding, argumentation, and proof, as opposed to rote learning, uncritical acceptance, and instrumental use of formulas (Kunnskapsdepartementet, 2019). One question arising in this context is how teachers are prepared for leading class discussions and activities concerning proof and argumentation by their preservice training. Not least as Norwegian primary and lower secondary school textbooks may leave much to the teacher and to the students themselves; arguments and proofs may be sketched or hinted at in discussion points or exercises for the students to explore, rather than written out in detail (see Oppgave 15.10, 15.12 Kongsnes & Wallace, 2020, for an example).

To be able to critically discuss mathematical arguments and separate valid arguments from non-valid ones is highly non-trivial and requires much training and experience. Diagrammatic arguments in school geometry can be seen as a form of proof by *generic example*, or generic argument, insofar as they involve ”making explicit the reasons for the truth

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of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class." (Balacheff, 1988, p. 219). A crucial aspect of such an argument is, of course, its generalizability or *generality*: the argument must not make use of any properties of the representative that are not shared by all members of the intended class; it should "establish the truth of [the] statement for all situations that satisfy the given conditions" (Knuth, 2002, p. 386). Thus it should be clear, or should be justified if needed, that the argument covers, or can easily be adjusted to cover, all cases for it to be accepted as a proof. This "awareness of generality" is, then, one of the central aspects of generic arguments that students need to be trained in (cf. e.g. Reid & Vargas, 2018).

While geometry can seem to be particularly well suited for learning to prove at a primary and lower secondary level because of the possibility to use visual representations and diagrammatic arguments, the issue of generality can be especially challenging when working with diagrams. To illustrate, if one's proof relies on the fact that a perpendicular from a point C to a line segment AB exists, then it is not enough that this perpendicular can be drawn on the specific diagram on which one is currently working; it must be possible to draw it on any appropriate diagram. That is to say, its existence must follow from the given conditions. To keep track of such issues and not be mislead by one's particular diagram is not always so easy, even for professional mathematicians. Thus the "modern view" in professional mathematics that "diagrams are at best a heuristic in aid of finding a real, formal proof of a theorem of geometry, and at worst a breeding ground for fallacious inferences" (Barwise & Etchemendy, 1996, p. 3). In school mathematics, finding a "real, formal" proof is not the aim, and arguments and diagrams are simpler. Yet the point remains that a central aspect of learning to assess and produce diagrammatic proofs is getting used to asking questions such as "Does this argument work also if we draw the triangle in a different way?".

This forms the background of our focus here, which is the proofs presented for the area formulas for triangles and parallelograms in textbooks used in Norwegian preservice teacher training. These area formulas, a part of the curriculum for Norwegian lower secondary school, are early examples of geometric theorems that can be justified by relatively accessible diagrammatic arguments. It is natural to treat them together, as one formula is often used in the justification of the other. As mentioned above, Norwegian school textbooks may leave much of the details concerning proofs and mathematical justifications to the teacher and the students themselves. This is also the case with the proofs for the area formulas for triangles and parallelograms, notably in the prominent school textbooks Kongsnes and Wallace (2020); Bråthe et al. (2021), and Hjardar and Pedersen (2020). In addition, all three of these present sketches of arguments for the area formulas which are not general (although the formulas themselves are clearly meant to be).[1] I.e. they do not work if we draw the triangle or parallelogram in a different way. Notably, these textbooks do not mention that the arguments do not generalize. If the students are not to uncritically accept the arguments as proofs of the general formulas, it is thus left to the teacher to point out, or lead the students to discover, the missing generality.

This can be a tall order if the teacher is not well prepared for it. Several studies have investigated teachers' or preservice teachers' conception of proof. Both e.g. Martin and Harel (1989), Healy and Hoyles (2000), and Knuth (2002) found that a substantial number accepted flawed arguments, of various kinds, as proofs. Knuth (2002), for instance, who concludes with a call to enhance teachers' conception of proof, writes that the teachers in his study "tended to be very proficient at recognizing proofs but had more difficulty recognizing nonproofs". One influencing factor (of several) in this context is the familiarity or truth of the statement to be proven. As highlighted e.g. by Reid and Knipping (2010, p. 67), students may be less critical of arguments for statements they already believe to be true. The arguments sketched in the aforementioned school textbooks are precisely nonproofs of statements that the teachers know to be true. Again, they are nonproofs in that they are insufficiently general; they do not establish the correctness of the formulas in all cases. As such they are not valid arguments for the statements they argue for, and thus fail to meet the second criterion in the often used conceptualization of proof by A. Stylianides (see Stylianides, 2007, p. 291). Apart from generality, however, the arguments are sound and accessible, and the fact that they are not fully general can (in our experience) be easy to miss. Thus teachers may easily fail to discover that there is a problem of generality to address, unless they have been trained to be aware of such issues.

As regards diagrammatic proofs in geometry, the proofs for the area formulas for triangles and parallelograms are often some of the first examples that preservice teachers encounter in their curriculum for which the issue of generality naturally arises. In this paper we investigate in what way textbooks for preservice mathematics teachers in Norwegian present justifications for the area formulas of triangles and parallelograms and, in particular, how they address the issue of generality in this context.

Theoretical framework and research questions

In the conceptualization of mathematical proof in practice offered by Aberdein (2013), in his elaboration of Epstein (2013), ”proofs [...] are *arguments* by means of which mathematicians convince each other that the corresponding *inferences* are valid” (Aberdein, 2013, p. 362). This points to a duality, or two-layeredness, with regard to proofs in practice. There is, in the terminology of Aberdein (2013), an *inferential structure* consisting of a rigorous, deductive formal proof, never (usually) produced, or even in practice produceable. And there is the actually produced *argumentational structure* by which mathematicians convince each other, and themselves, that a valid inferential structure exists and can in principle be produced. As Reid and Vargas (2018) argue, generic arguments cannot be dismissed as nonproofs simply because they are not rigorous, formal deductions, as mathematical proofs in general are not such. Rather, to qualify as proofs, ”generic arguments must fulfill the function of argumentative structures, to *point* to the inferential structure” (p. 241). They propose two requirements for a generic argument to qualify as a proof, one of a psychological and one of a social nature. In their own summary (p. 250), ”Psychologically, for a generic argument to be a proof it must result in a general deductive reasoning process occurring in the mind of the reader that convinces the reader that there exists a fully deductive inference structure behind the argument. Socially, for a generic argument to be a proof it must conform to the social conventions of the context.” In the context of the classroom, in order to contribute towards a framework to aid teachers and students in deciding if a generic argument is a proof or not, they propose two criteria for the evidence that should be included in the students’ written work (p. 247):

Evidence of awareness of generality are explicit expressions to the effect that the student intends the argument to be, or is aware that the argument should be, general enough to be valid in all cases.

Mathematical evidence of reasoning are expressions that ”reveal the form of the reasoning behind the argument”, for instance, but not limited to, the kind of *narrative reasoning* that Kempen and Biehler (2015) require a generic argument to include in order for it to be considered a proof.

We adopt and adjust the framework proposed by Reid and Vargas (2018) to use as a lens through which to view the arguments for the area formulas in preservice teacher textbooks. In the context of a textbook for preservice teachers, the proofs should, according to the psychological requirement, be so as to evoke a general deductive reasoning process in the reader, so that the reader is stimulated or lead to see that as the (diagrammatic) example ”varies in some ways the theorem remains true” (Reid & Vargas, 2018, p. 246). Socially, textbook proofs should of course conform to the social conventions governing generic or diagrammatic proofs in this context, but they should also, by example or otherwise, to some extent introduce and convey those conventions.

In this paper we ask how the area formulas for triangles and parallelograms are justified in textbooks for Norwegian preservice teachers, whether they are justified by fully general arguments or not, and, in both cases, what attention to generality can be found in connection with those arguments. With regard to the latter, we look for two kinds of evidence:

O*vert attention to generality*, corresponding to Reid and Vargas' *evidence of awareness of generality*, are any expressions stating, reminding, or signaling to the reader that the argument is meant (or not meant) to cover all cases. This includes e.g. asking the reader to consider if the proof depends on the specific diagram or verify that it does not, directing the reader to the exercises for a different, or the general, case, etc.

Attention to generality in reasoning, corresponding to *mathematical evidence of reasoning*, are expressions in the argument that signal that the attention is on the general case, rather than the specific diagram. For instance, accompanying the drawing of a perpendicular from a point C to a line segment AB by saying ”the perpendicular exists because...”.

Our research questions are, then:

- What arguments are presented for the area formulas for triangles and parallelograms in textbooks used in Norwegian preservice teacher education?
- What attention to generality, overt and/or in reasoning, can be found in, or in connection with, those proofs?

Methodology and data selection

Norwegian compulsory school (Grunnskolen) lasts from age 6 to age 16 (grades 1–10). Before 2010, there existed a teacher education programme (allmennlærerutdanningen) preparing teachers for grades 1–10, while from 2010, this has been divided into two programmes, one for grades 1–7 and another for grades 5–10. In our study, we will include all these variants. There are national guidelines for the content of these teacher education programmes, but each university and university college design their own courses, including reading lists and exams. They are thus free to choose from Norwegian or foreign textbooks. The number of mathematics textbooks for teacher education predating 1998 is rather small, as the obligatory content of mathematics was increased from 15 to 30 ECTS as late as 1998.

To answer the research question, we decided to analyse all available mathematics textbooks used in Norwegian preservice teacher education that includes arguments for the formula of a parallelogram. The combined experience in teacher education of the authors goes back more than 25 years, which means that we were able to prepare a list of relevant books. We supplemented our experience-based list with searches in the database of the National Library of Norway, which is supposed to include all books ever published in Norway. Using the app Korpus from the National Library, we searched for books having ”matematikk” (mathematics) in the title and ”lærerutdanning” (teacher education) in the subject field. In an additional search, we searched for all books including the phrase ”areal av parallellogram” (area of parallelogram) anywhere in the book. These results were screened to include only books written for teacher education and including arguments for the areas of triangles and parallelograms. We then added a small number of foreign textbooks that we are aware have been used in Norwegian teacher education. There exists no collection of historical reading lists for all universities and university colleges in Norway. For this article, archival searches for such reading lists were considered unrealistic. Thus, there may exist texts used in some teacher education programme that has not been identified in our search. After identifying the arguments presented for area formulas for triangles and parallelograms in these textbooks, these proofs were analysed according to the questions laid out above: Are the arguments fully general or not, and (in both cases) are they supported by attention to generality, overtly and/or in reasoning. The results are presented in the section named *Generality and arguments for the area formulas in Norwegian preservice textbooks* starting on page 53. The section is structured by the two arguments that are by far the most prevalent in the textbooks. These arguments are presented on page 53–54 by an extensive textbook extract, which we use as a point of reference to summarize the situation in all included textbooks.

Table 1. *The full list of included textbooks.*

Reference	Title	Lan. (Orig.)
Beck et al. (1998)	Matematik i læreruddannelsen	Danish
Bjørnestad et al. (1998)	Matematikk 1 for allmennlærerutdanningen	Nor.
Bjørnestad et al. (2013)	Alfa	Nor.
Breiteig and Venheim (1998)	Matematikk for lærere 1	Nor.
Christensen et al. (1972)	Matematikk for lærerutdanning	Nor.
Hansen et al. (2007)	Ypsilon	Danish
Hinna et al. (2016)	QED 1–7	Nor.
Hinna et al. (2011)	QED 5–10	Nor.
Hole (2006)	Grunnleggende matematikk i skoleperspektiv	Nor.
Jensen (2004)	Grunnleggende matematikk for lærerutdanningen	Nor.
Kirfel et al. (1999)	Matematiske sammen-henger	Nor.
Pind (2011)	Håndbok i matematikkundervisning	Nor.(Dan.)
Solem et al. (2019)	Tall og Tanke 2	Nor.
Van de Walle et al. (2017)	Teaching Student-Centered Mathematics	English

The full list of included textbooks, 10 Norwegian, 3 Danish, and 1 American, is presented in Table 1. Of these, Hinna et al. (2016, 2011) and Solem et al. (2019) are, in our experience, currently the most widely used. Hinna et al. (2016) is for grades 1–7 and Hinna et al. (2011) for grades 5–10, but the relevant sections are for our purposes identical. Hinna et al. (2016) and Solem et al. (2019) have both recently appeared in new editions: Hinna et al. (2022) and Solem et al. (2023). A new edition of Hinna et al. (2011) is forthcoming. We focus on the earlier editions here, since the central issues of this paper were discussed with, or in the presence of, some of the authors of these books as the new editions were in preparation. We note the differences between Hinna et al. (2016) and Hinna et al. (2022) in Footnote 2. Solem et al. (2023) is excluded from consideration as the third author coauthored it while this article was in preparation.

All translations of excerpts from Norwegian and Danish textbooks into English are the authors' and all translation errors are ours.

Summary of findings

In the majority of the textbooks, we find very little in the way of attention to generality in the justifications of the area formulas for triangles and parallelograms. The arguments themselves are in many cases extremely tersely presented, to the point where diagrams are given as more or less standalone arguments, with few or no prompts that direct the attention of the reader to the general case. In several cases, this results in some uncertainty as to what exactly the authors are intending to prove, i.e. how general they intend their formula to be understood. Moreover, nearly all textbooks present a non-general argument for either the area formula for triangles or the formula for parallelograms, or both. Non-general arguments for the formula for triangles are often, but not always, explicitly or implicitly pointed out as being non-general. When it comes to parallelograms, however, this is not the case. With one exception, all textbooks that present a direct argument for the formula for parallelograms present a non-general argument for this formula. These textbooks either do not say or indicate whether the argument is meant to cover all cases or not, or they do so, but to the effect that the argument *is* meant to cover all cases.

Despite some differences in style, we find that the included textbooks are rather uniform in which arguments they choose to justify the area formulas with. In particular, the non-general diagrammatic argument for the area formula for parallelograms mentioned above is so prevalent in Norwegian preservice textbooks that it seems to have acquired the status of a standard argument for this formula. It is notable that this argument occurs not only in the Norwegian textbooks, but also in an American and a Danish textbook that we have included as they are used in Norwegian preservice training.

These findings are presented in the section named *Generality and arguments for the area formulas in Norwegian preservice textbooks*, starting on page 53.

Preliminaries

For clarity and disambiguation, we describe and explain some technical terminology and conventions used in the sequel.

A *base* of a triangle (parallelogram) is any chosen side of the triangle (parallelogram). We are not strict with distinguishing between base as a line segment and as a length of a line segment, if it is clear by context what is meant. Triangles (parallelograms) are understood to come equipped with a choice of base. Thus the phrase ”all triangles”, for instance, is elliptical for ”all triangles and all choices of base”. In a diagram, the base is understood to be the bottom side of the triangle (parallelogram), if not otherwise specified.

Let P be a triangle (parallelogram) with base b. Let ℓ_b be the line extending b, and let ℓ' be the line parallel to ℓ_b intersecting (the border of) P and so that P is included between ℓ_b and ℓ'. The *altitude* of P with respect to b is the distance between ℓ_b and ℓ'. We say that P has *internal altitude* (with respect to b) if there exists a perpendicular to ℓ_b that lies entirely within P between ℓ_b and ℓ'. If P does not have interior altitude, we say that it has *exterior altitude*.

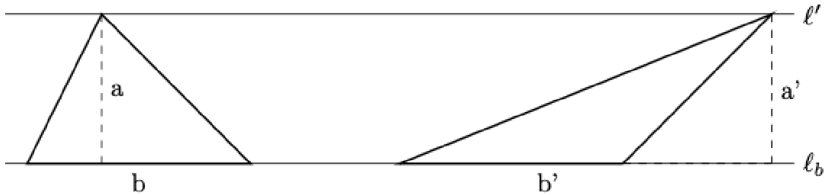

Figure 1. *Triangles with interior (left) and exterior (right) altitude*

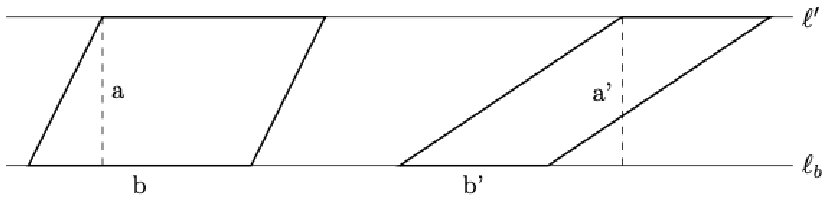

Figure 2. *Parallelograms with interior and exterior altitude*

Figures 1 and 2 display examples of triangles and parallelograms with internal and external altitude.

The area formulas for triangles and parallelograms are the following.

Theorem 1 (Formula for the area of a triangle). *A triangle with base b and altitude a has area A given by* $A = \frac{ab}{2}$

Theorem 2 (Formula for the area of a parallelogram). *A parallelogram with base b and altitude a has area A given by* $A = ab$.

Generality and arguments for the area formulas in Norwegian preservice textbooks

Introduction: three arguments

We structure this section by the two main direct arguments for the area formula for triangles and the area formula for parallelograms, respectively, found in the included textbooks. One, or both, of these arguments occur in all textbooks. Only one textbook adds a substantially different argument. We present this special case on page 62.

We refer to the direct argument for the area formula for triangles as the *Rectangle Argument (for the area formula for triangles)*. This is less commonly used in the textbooks; the preferred way to justify the area formula for triangles is to give a direct argument for the area formula for parallelograms and then derive the area formula for triangles from that, using an argument that we will refer to as the *Bridge Argument*. We refer to the direct argument for the area formula for parallelograms found in the textbooks as the *Standard Argument (for the area formula for parallelograms)*.

All three arguments, Rectangle, Standard, and Bridge, are presented together in the span of just over a page in Hinna et al. (2011). We quote this in full, in our own translation and with some minor layout differences, both to introduce the arguments and to serve as an example and a reference point. The figure numbers and captions are ours, for later reference. We have not included the original proposition numbers in the translation. We have also changed the variable for the base (grunnlinje) from g to b, and the variable for the altitude (høyde) from h to a.

> We now wish to find the area of other shapes than rectangles. Let us start with the area of a parallelogram. We start with the base and altitude of the parallelogram.

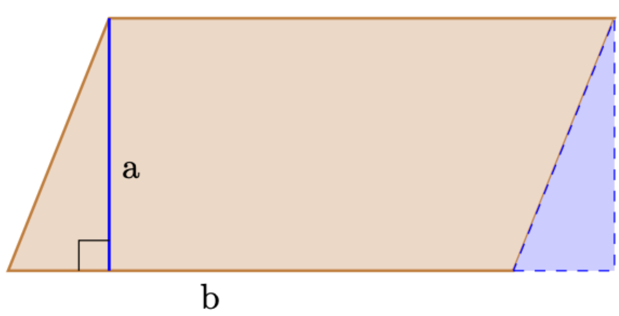

Figure 3. *The Standard Argument*

If we cut off a triangle and place it on the right side, the parallelogram is transformed to a rectangle with the same area. Therefore, we see that the area is given by $A = ba$.

Proposition (Area of a parallelogram). The area of a parallelogram with base b and altitude a is given by

$$A = ba$$

Let us now consider the area of a triangle with a base of length b and with altitude a. We can either inscribe the triangle in a rectangle, or we can add the same triangle to make a parallelogram.

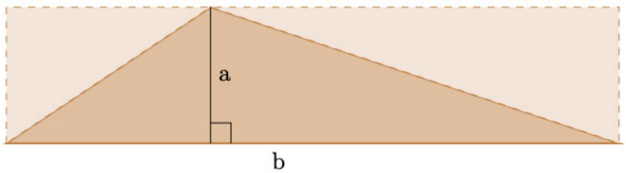

Figure 4. *The Rectangle Argument*

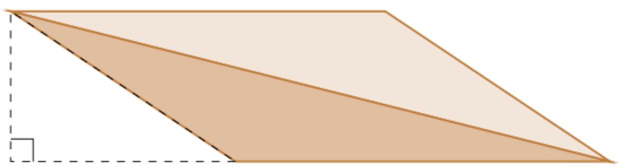

Figure 5. *The Bridge Argument*

In the first case we see that the area of the triangle is half the area of the rectangle, and in the second case we see that the area of the triangle is half the area of the parallelogram. Thus we have the following familiar formula for the area of a triangle.

Proposition (Area of a triangle). The area of a triangle with base b and altitude a is given by

$$A = \frac{ba}{2}$$

(Hinna et al., 2011, pp. 480–481)

We see that the three arguments are each given essentially by presenting a single diagram, with little or no text to support or direct the (general deductive) reasoning process that these diagrams are supposed to induce in the reader. Apart from some explanatory sentences—"cut off a triangle and place it on the right side" in figure 3, "add the same triangle" in figure 5—the reader is asked to "see" that the conclusion of the argument follows from the diagram. In the sequel we refer to such arguments as *standalone diagrammatic*.

We find no overt attention to generality in the quoted text (nor in the pages surrounding it). There are no statements informing or prompting the reader to consider whether either of the arguments depend in some aspect on properties that are specific to the displayed diagrams rather than general.

We interpret the inclusion of two cases of triangles as an instance of attention to generality in reasoning. That is, we take the argument to be saying that, for a general triangle, it may not be possible to enclose it in a rectangle with the same base and altitude. Hence the need for the second triangle diagram (figure 5).

Other than that, we find no instances of attention to generality in reasoning. To draw a parallel with Reid and Vargas (2018); if considered as a written submission by a student, we find no clear indication in the reasoning that the attention of the student is on the general case, rather than the particular diagram.

The most notable omission of attention to generality in the reasoning is the justification that you can cut a triangle off in figure 3 – i.e. in what we have named the Standard Argument. In general, if you cut vertically from the top left corner of a parallelogram (which is slanted to the right), you might get a triangle, but you might also get a trapezoid (as in figure 2). In the latter case, the argument does not work, and the area formula for parallelograms has not been justified. This propagates down to the argument for the area formula for triangles. Consider e.g. the skewed triangle BCA of figure 6.

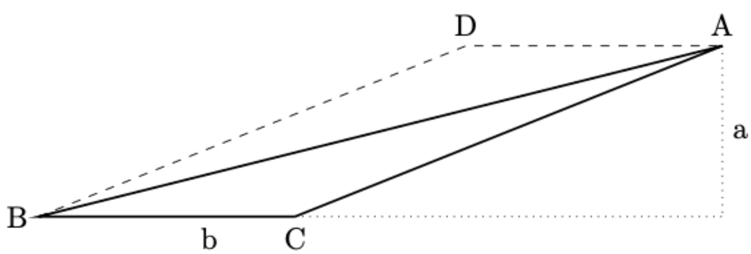

Figure 6. *A skewed triangle*

The Standard Argument does not work for the induced parallelogram BCAD, so although the area of the triangle is half of that of the parallelogram, we can not without further argument conclude that it is $\frac{1}{2}ba$.[2]

To summarize:

What we refer to as the *Standard Argument* for the area formula for parallelograms is the argument of figure 3. This argument establishes (only) that the area of a parallelogram with interior altitude is the base times the altitude.

What we refer to as the *Rectangle Argument* for the area formula for triangles is the argument of figure 4. This argument establishes (only) that the area of a triangle with interior altitude is half the base times the altitude.

We remark that the Standard Argument can not be saved by cutting off a different piece. It is known that there is in general no way to transform a parallelogram to a rectangle with the same base and altitude by cutting off a single polygonal piece and moving it. A parallelogram can always be transformed into a rectangle with the same base and altitude by cutting it into polygonal pieces, but there is in general no upper bound for the minimum number of pieces required to do this (Tarski, 2014).

We now present our findings for the remaining 13 textbooks, (overlappingly) grouped into those that present the Rectangle Argument for triangles and those that present the Standard Argument for parallelograms.

The Rectangle Argument

Of our 14 textbooks, 8 present the Rectangle Argument:

Table 2. *List of books that present the Rectangle Argument.*

Reference	Title	Page ref.
Beck et al. (1998)	Matematik i læreruddannelsen	114–115
Breiteig and Venheim (1998)	Matematikk for lærere 1	334–335
Hinna et al. (2016)	QED 1–7	457
Hinna et al. (2011)	QED 5–10	480–481
Hole (2006)	Grunnleggende matematikk i skoleperspektiv	126
Jensen (2004)	Grunnleggende matematikk for lærerutdanningen	12
Kirfel et al. (1999)	Matematiske sammenhenger	71–72
Pind (2011)	Håndbok i matematikkundervisning	230–231

With one exception, none of these present a direct, fully general proof for the area formula for triangles, i.e. for arbitrary triangles with an arbitrary choice of base; triangles with exterior altitudes are either excluded, not considered, or dealt with by the Bridge Argument and the formula for parallelograms. The exception is Pind (2011), which we return to on page 62. Hinna et al. (2016) is for our purposes similar to Hinna et al. (2011). In brief, the remaining textbooks are as follows.

Two textbooks, Hole (2006) and Jensen (2004), overtly alert the reader to the fact that the Rectangle Argument does not establish the formula in all cases. In Hole (2006),the argument is followed by an exercise asking the reader to prove the formula also for triangles with exterior altitude. The presentation of the argument itself is standalone diagrammatic. It is somewhat more elaborately presented in Jensen (2004), where the case of right triangles is done first. More notably, Jensen (2004), who instructs the reader to find the altitude of the triangle by drawing a line perpendicularly down to the opposite side, states after giving the formula that this formula holds also if the perpendicular should happen to not intersect the side, but only the line extending it.

A similarly more elaborate presentation of the Rectangle Argument is found in Beck et al. (1998), in which the special case of right triangles is also done first. Here, we find overt attention to generality in accompanying statements, e.g.: ”*In general*, the area of *any* right triangle is half that of a rectangle which has the two short sides of the triangle as its length and width”; followed by ”*any* triangle can be divided into two right triangles” (pp.114–115, our emphasis). Beck et al. (1998) also present two diagrams of triangles with different orientation. Finally, Beck et al. (1998) state that, ”according to tradition” (”Traditionen har givet [...]”), the altitude of the triangle is the line segment that divides the triangle into two right triangles, and the base is the side that is perpendicular to it. (Our formulation, they use a diagram). With this, we take it that Beck et al. (1998) exclude triangles with exterior altitudes. That is, the statement of the area formula for triangles in Beck et al. (1998) is with respect to a restricted notion of base. Whether that is fortunate or not, the Rectangle Argument is a valid argument for this more restricted proposition.

In contrast, the presentation in Kirfel et al. (1999) is entirely standalone diagrammatic, with no attention to generality either overtly or in reasoning. Unlike in Beck et al. (1998), there is no discussion of the notion of base or internal or external altitude in Kirfel et al. (1999). It is therefore unclear if the formula is meant to cover all choices of base, or only meant for triangles with internal altitude. Looking at its ensuing treatment of parallelograms, however, it seems that the formula is meant to apply also to triangles with external altitude, see the next section.

The presentation in Breiteig and Venheim (1998) is similar to that of Kirfel et al. (1999), also with no mention of the possibility that triangles might have external altitude. They do say that if the area formula for parallelograms is established first, then the formula for triangles can be derived. This is shown using the Bridge Argument, but unlike Hinna et al. (2011), the triangle in the diagram used for this argument has internal altitude.

The Standard Argument and the Bridge Argument

11 out of 14 textbooks present the Standard Argument for the area formula for parallelograms. None of these, arguably except Jensen (2004), note that the argument does not work for all parallelograms, i.e. for all parallelograms and all choices of base.
Again, we deal with Pind (2011) separately on page 62.

Table 3. *List of books that present the Standard Argument.*

Reference	Title	Page ref.
Bjørnestad et al. (1998)	Matematikk 1 for allmennlærerutdanningen	69
Bjørnestad et al. (2013)	Alfa	414–415
Breiteig and Venheim (1998)	Matematikk for lærere 1	335
Hansen et al. (2007)	Ypsilon	78–80
Hinna et al. (2016)	QED 1–7	456–457
Hinna et al. (2011)	QED 5–10	480–481
Jensen (2004)	Grunnleggende matematikk for lærerutdanningen	13
Kirfel et al. (1999)	Matematiske sammen-henger	72
Pind (2011)	Håndbok i matematikkundervisning	230–231
Solem et al. (2019)	Tall og Tanke 2	208–209
Van de Walle et al. (2017)	Teaching Student-Centered Mathematics	357–358

Hole (2006) is not included in this list, as he only mentions the area formula for parallelograms in an exercise (Exc.10, p. 129). The exercise is to ”derive a formula for the area A of a parallelogram with ’base’ *g* and 'altitude' *h*, as shown on the figure". The figure in the exercise is essentially identical to figure 3 above.

The presentation of the Standard Argument in Jensen (2004) is standalone diagrammatic. However, together with the statement of the formula, Jensen (2004) states that any side of the parallelogram can be chosen as the base, and he includes a diagram with a (seemingly) external altitude. He does not comment on the justification of the formula in this case. Recall from page 57 that Jensen (2004) accompanied the Rectangle Proof with a statement that the formula for triangles holds also if the altitude is external. It is therefore possible to, but difficult to conclusively, interpret Jensen (2004) as indicating that the argument is only meant as a partial justification.

In addition to Hinna et al. (2011), the textbooks Van de Walle et al. (2017), Hansen et al. (2007), Bjørnestad et al. (2013), Bjørnestad et al. (1998), Hinna et al. (2016), and Solem et al. (2019) use the Standard Argument and the Bridge Argument as the principal means to derive the area formula for triangles.

Van de Walle et al. (2017) contains several instances of overt attention to generality, both in its preceding discussion of the notions of base and height (altitude)—where it is emphasized that any side of a figure can be called its base—and in direct connection with the arguments. E.g. in the teaching tip

> "Be sure to vary the shapes so that the height falls inside some of the shapes and outside others." (p. 356);

in the conclusion of the Standard Argument

> "Parallelograms can always be transformed into rectangles that have the same base and height." (p. 357);

and the conclusion of the Bridge Argument

> "Two copies of any triangle will always form a parallelogram with the same base and height." (p. 358).

As in Hinna et al. (2011), however, the arguments themselves are (essentially) standalone diagrammatic, with no apparent attention to generality in reasoning. And as in Hinna et al. (2011), the diagram presenting the Bridge Argument shows a triangle with external altitude that induces a parallelogram with internal altitude.

Hansen et al. (2007) stands out among the textbooks we have considered by taking an axiomatic and relatively rigorous approach to geometry. The proofs in Hansen et al. (2007) are also significantly more formal than in our other textbooks, with abundant instances of attention to generality, both overt and in reasoning. This is also the case with regard to its presentation of the Standard Argument. The choice of the bottom line of the presented parallelogram, for instance, is made with the formulation "Let us now choose AD as the base", and it is argued for that the two triangles (left and right in figure 3) are congruent. In the end, however, the claim that the perpendicular from the top left corner to the extension of the base intersects the base is not justified. Intriguingly, in an exercise immediately after the argument, Hansen et al. (2007) asks the reader to repeat the argument with one of the shorter sides of the parallelogram as the choice of base (another instance of overt attention to generality). However, the displayed parallelogram is so that the altitude is internal also in this case.

Although less formal than Hansen et al. (2007), the arguments in Bjørnestad et al. (2013) are not standalone diagrammatic, but accompanied with a supporting narrative, in which, for instance, the reader is asked to verify that the relevant triangles are congruent. In addition to such instances of attention to generality in reasoning, we find in Bjørnestad et al. (2013) an instance of overt attention to generality which is unusual in our selection of textbooks with regard to its explicitness and emphasis:

> "The parallelogram ABCD [in the diagram] is an arbitrary parallelogram. *Nothing that is particular to the shape of the figure is used in the reasoning concerning the area*." (p. 415, our emphasis)

(Again, however, this is incorrect; the existence of an internal altitude is particular to the shape of the figure.)

With regard to attention to generality and the way the arguments are presented, the remaining textbooks are essentially similar to Hinna et al. (2011), except in the following.

Bjørnestad et al. (1998) state that any side of a triangle or parallelogram can be chosen as the base.

Kirfel et al. (1999) does not state the area formula directly. The reader is asked to "complete the argument to show that the area of the parallelogram is correct" (p. 73). The intention is probably that the reader should find the formula as an exercise, given that the reader is asked to find the formula for trapezoids as an exercise later on the same page. There is no discussion of the notions of base and altitude. It is therefore not clear if Kirfel et al. (1999) intend the reader to deduce the general area formula for parallelograms or not. Kirfel et al. (1999) give three possible standalone diagrammatic arguments for the reader to complete in order to establish the area of a parallelogram: the first is the Standard Argument; the second divides the parallelogram into a rectangle and two triangles[3]; and the third is the Bridge Argument as an argument from the formula for triangles to the formula for parallelograms. Recall from page 57 that Kirfel et al. (1999) only present the Rectangle Argument for triangles. So the latter will not work for a parallelogram that only divides into two copies of a triangle with external altitudes. Interestingly, in the diagram in Kirfel et al. (1999), the parallelogram is divided into two copies of a triangle with external altitude. If one takes this to indicate that the formula for triangles is intended to be general, one may suspect that the formula for parallelograms is also meant to be general, although none of the argument sketches in Kirfel et al. (1999) will justify the general formula.

Finally, after presenting the Standard Argument (essentially as in Hinna et al., 2011), Solem et al. (2019) remark:

> ”Notice by the way that the parallelogram here is in the standard position with the broadest side placed horizontally. This saves space in a textbook, but such a placement can be a contributing factor to the prototype phenomenon explained in the geometry chapter. Therefore I will try to place the next figures [triangles and trapezoids] less conventionally.” (p. 209)

We interpret this, if not confidently, as a statement to the effect that the argument does not depend on this ”standard” placement of the parallelogram.

An algebraic approach

Similar to several of the other textbooks in using standalone diagrammatic arguments and containing little apparent attention to generality, the approach in Christensen et al. (1972) is slightly different. The Rectangle Argument is only used for the special case of right triangles. The formula for a triangle with internal altitude is then derived algebraically, by dividing it into two right triangles and computing the sum of the areas. That this presupposes the altitude to be internal is not remarked upon, and the similar proof for triangles with external altitude that computes their areas as a difference between that of two right triangles is not mentioned. The area formula is introduced as holding ”for all triangles” (p. 185), but we cannot determine if this is meant in the sense of holding for all triangles and all choices of base. The formula for parallelograms is then derived from that of triangles by using the Bridge Argument, where the (standalone) diagram shows a parallelogram divided into two copies of a triangle with internal altitude.

A fourth argument: Cavalieri's principle

The presentation in Pind (2011) differs from those of the other textbooks in central aspects; specifically in what constitutes the principal argument, what kind of argument this is, and how it is referred to in the text. Pind (2011) begins with the Rectangle Argument, with a supporting narrative and starting with the case of right triangles. Noting that triangles might have external altitude, Pind (2011) proceeds with the illustration recreated in figure 7 and the following explanation.

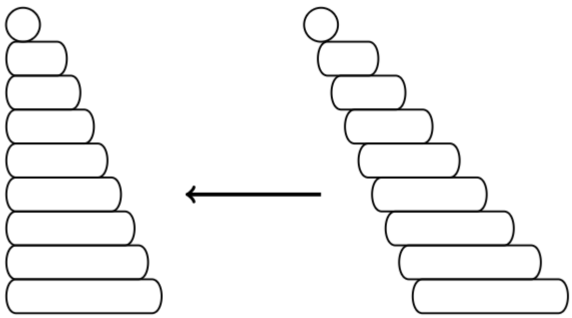

Figure 7. *Stack of books*

> ”The area formula for triangles can be illustrated as shown above, by having the students imagining that the triangle is a stack of books (of different sizes) seen from the side. The stack is about to fall, and to prevent this one slides the books against the wall. Now the triangle is a right triangle, and it is easy to compute the area as one half of a rectangle. The area of the books seen from the side is not changed by sliding the books, therefore the area of the two triangles is the same.” (p. 231)

A similar picture is used to ”explain” the formula for parallelograms.

> ”The area of a parallelogram is also explained well with the book metaphor” (p.231)

Finally, she adds that the formula ”can also be understood by moving one end of the parallelogram to the other” (p. 231), at which point the Standard Argument is presented in a standalone diagrammatic way.

Note that Pind (2011) does not refer to the stack of books figure as an ”argument”, ”proof”, or ”justification”, but as an ”illustration” and a ”metaphor”. This is in line with our theoretical framework: recall that an argument should, in order to be such as to be considered as a proof, ”fulfill the function of argumentative structures, to *point* to the inferential structure”, and be such as to ”result in a general deductive reasoning process occurring in the mind of the reader that convinces the reader that there exists a full deductive inference structure behind the argument”. The mathematical reasoning that the stack of books illustration alludes to, viz. the statement and justification of Cavalieri's Principle, is technically and conceptually of a nature that we would not normally consider as known to, or within the conceptual reach of, the community of preservice mathematics teachers or lower secondary school students (cf. Stylianides, 2007, p. 291). Thus we regard Pind (2011) as not aiming to present an argument in the sense of a proof or a justification, but more in the sense of a heuristic argument; that is, a loose or informal argument, or conceptual picture, that is meant to convince the reader that the result is reasonable without getting into the machinery or technicalities on which the actual proof depends. That being said, it is clear from the presentation that the attention of Pind (2011) is on the general case. The Rectangle Argument is presented only as a way of finding the area of a triangle (for which one may choose a suitable base). This is followed by the general formula, with respect to which it is pointed out that any side can be chosen as a base. The stack of books illustration is then, as it seems, given to cover the cases where the altitude is external.

Discussion

Part of the background of this survey is the fact that Norwegian school textbooks often leave much concerning justification and proof to the students to explore and fill in, and the question of how teachers are prepared for leading such activities by preservice teacher education. Here we have restricted attention to the way textbooks for preservice teachers present justifications of the area formulas for triangles and parallelograms. In this particular case, the preparation that preservice teachers receive from their textbooks is limited in important respects. The textbooks usually restrict themselves to a brief presentation of a suggested argument, often presented in terms of a single diagram from which the reader is encouraged to see that the result follows. The issues concerning generality that have been the main focus here receive little attention. Briefly, there is little emphasis on not just finding the right diagram (and the argument to go with it), but also verifying that the diagram really represents (and the argument applies to) all relevant cases.

There are some notable (at least partial) exceptions to this. Some textbooks, such as Pind (2011) and (especially) Van de Walle et al. (2017), bring attention to the general case in the related discussion concerning the notion of altitude. Bjørnestad et al. (2013) and (seemingly) Solem et al. (2019) underscore that the argument they present is meant to apply to all cases, and not just the displayed diagram. Bjørnestad et al. (2013), together with Hansen et al. (2007), is also exceptional in signaling to the reader, as the argument goes along, that the object of their argument is an arbitrary shape of the appropriate kind, while the diagram is just an example.

The latter kind of attention to generality in reasoning is rare. Most of the textbooks do nothing to show that the constructions they perform on the diagram can be performed on any shape of the intended kind. The importance of such justifications, and the ease with which one can otherwise be mislead by one's diagram, are perhaps exemplified by the fact that most of the textbooks (including Bjørnestad et al. (2013)) present a particular argument for the area formula for parallelograms which does not work for all cases. The ubiquity of this non-general argument, seemingly or explicitly presented as fully general, was unexpected. Future work includes a systematic search to determine how widespread this argument is outside Norway.

Another unexpected finding was the extent to which we sometimes had problems determining what exactly the textbook intended to show. Concerning the area for triangles, for instance, the Rectangle Argument (figure 4) can be introduced, not as a justification of the general formula, but just as a way of finding the area of a triangle: choose the longest side of the triangle and enclose the triangle in a rectangle which has that side as its length. This is done in e.g. Pind (2011). It can be introduced as a partial argument; justifying the formula in some cases but not all. This is done in e.g. Hole (2006). It could also be introduced as an argument for the general formula, in which case it would be invalid. In several cases, it took some interpretation and careful reading to find out what was intended, and in a few, we simply could find no cues. Although somewhat on the side of (if not unrelated to) our research focus here, we find it notable that textbooks for training mathematics teachers do not put a higher emphasis on clearly and unambiguously stating what the claim is and what the argument shows. Especially as they, as textbooks, ought not only to conform to the norms governing mathematical practice, but also teach those norms.

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Notes

1 In the terminology introduced on page 53, Kongsnes and Wallace (2020) and Hjardar and Pedersen (2020) sketch the Standard Argument for the area formula for parallelograms, while Bråthe et al. (2021) sketch the Rectangle Argument for the area formula for triangles.

2 The new edition, Hinna et al. (2022), is mostly unchanged in the quoted section. To the Standard Argument is added the remark ”The formula holds also for other parallelograms”. To the argument for the area formula for triangles is added the remark that the argument gives the students a ”good understanding” for why the formula holds, even if it is not a ”general proof”. The section is also slightly more explanatory, and has an added section on the altitude of parallelograms. See Hinna et al. (2022, pp. 344–346).

3 We do not go any deeper into this argument as the division anyway presupposes that the parallelogram has internal altitude.

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