Teachers' task-specific questions when monitoring students' combinatorial reasoning in mathematics

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The aim of this paper is to contribute to what kinds of task-specific questions teachers pose when students solve problems and reason mathematically in combinatorics. With departure in creative mathematical reasoning (CMR), which entails constructing and arguing for solutions on mathematical problems, two lessons with similar mathematical problems on combinatorics in grades 5 and 6 from two teachers participating in the same professional development program were analyzed and contrasted. The analysis was supported by the four principles for CMR-teaching: encouraging students to initiate, develop, justify, and verify their own reasoning. The results show that despite that both teachers have very high mathematical knowledge for teaching, they pose different kinds of task-specific questions – adhering to the four principles – in relation to combinatorial reasoning. The first teacher poses questions according to the first two principles of initiating and developing students' reasoning as well as funnels students towards answers and solutions, the latter which does not align with CMR-teaching. The second teacher poses questions according to all four principles for CMR-teaching. It is discussed how the differences in results between the two teachers' lessons regarding the four principles underline the importance of preparing task-specific questions. It is also discussed how the results may contribute to teacher education activities aimed at developing teacher's question-posing skills.

In the last decades, a large amount of research addresses problem-solving as a key to deeper mathematical knowledge (see e.g. Hiebert & Grouws, 2007; Lester & Cai, 2016). Teaching mathematics through problem solving means that both teachers and students should formulate and follow reasoning (Mueller et al., 2014). Lithner (2008) defines reasoning for problem solving as creative mathematical reasoning (CMR), which

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means that the solver constructs an origin solution method and formulates arguments for the solution founded in mathematics. In teaching, when monitoring students' different ideas and reasoning process during the explore phase, a challenge for the teacher is to maintain the cognitive demand, i.e., not telling students too much during their efforts to solve a problem (Stein et al., 1996). Teacher-student interactions during the explore phase are crucial for students' opportunities to reason and formulate mathematically founded arguments, which involves asking questions to students to support and challenge them as well as responding to students' questions without giving them solution methods. A particular challenge is to "release authority to the students to use themselves and one another as resources to decide upon correctness" (p. 70, Larsson, 2015). In this, students' justification of claims with mathematical arguments is central.

Supporting students to make details explicit in their ideas about a problem is integral to student learning (Franke et al., 2007). Wood (1998) suggests that teachers' questions can be funneling or focusing during teachers' monitoring of students' solutions. Funneling questions guide students towards a solution while focusing questions help students to focus on productive ideas in their own thinking. Franke et al. (2009) found that teachers often readily asked general questions to elicit students' reasoning but struggled with follow-up questions to help students to develop their mathematical ideas. Hence, if the purpose is to support students to develop CMR, a challenge for teachers is to pose fruitful questions that support students to develop their own reasoning, and not funneling them towards a specific solution or providing them with a ready-made solution strategy. In our own studies we have found that this challenge for teachers can be handled by posing general questions such as "Can you explain your thinking?" to elicit students' reasoning, and – even more importantly for students' CMR – asking task-specific questions in relation to both the mathematical task and to where in the reasoning process the students are (Olsson & D'Arcy, 2022; Olsson & Granberg, 2024). A limitation of these previous studies is that the task-specific questions were developed within the confines of our own projects. The teacher's awareness of the guiding framework therefore likely influenced the outcomes regarding which questions engage students' CMR. To further validate our findings, it is important to investigate task-specific questions in contexts that are not explicitly influenced by our framework, as in our current study reported in this paper.

A fundamental aspect of CMR involves the formulation of arguments grounded in mathematics, which are frequently evident in students' justifications (Olsson & D'Arcy, 2022). In their review, DeJarnette et al.

(2020) highlight a significant gap in the literature concerning the role of teacher questioning in facilitating students' mathematical justifications.

The aim of this paper is to contribute to what kinds of task-specific questions teachers pose when students solve problems and reason mathematically in combinatorics. In our study, the combinatorial problems involve selections of a number of objects from a finite set of objects (English, 2005).

The research question guiding this study is: To what extent and in what ways do the two teachers pose task-specific questions that encourage students to: (1) initiate an independent (line of) reasoning, (2) develop one's own reasoning, (3) justify one's own reasoning, and (4) verify the result of one's own reasoning, within the mathematical domain of combinatorics?

In CMR-teaching, posing task-specific questions is an essential part of the teacher-student interaction and focusing on these questions allows us to delve into important aspects of CMR-teaching that have previously been studied only to a limited extent (Olsson & Granberg, 2024).

This study also makes a methodological contribution by using design principles for teaching through CMR (Olsson & D'Arcy, 2022; Olsson & Granberg, 2024) as an analytical tool in a context that was not explicitly influenced by our framework. Further, in their review, Ratnayake et al. (2024) state that in the Nordic countries there is a common interest for implementing research findings in mathematics teaching. However, many small-scale research projects are too contextually limited and therefore not implementation-friendly (Ratnayake et al., 2024). This study is situated within a classroom teaching context that both teachers and researchers can relate to.

Literature background

We first focus on combinatorial reasoning processes before we dig into research about teaching through problem solving and CMR.

Combinatorial reasoning

Combinatorics is a mathematical area that serves as a good ground for problem solving and exploration since it is a rich domain that suggests a variety of interpretations of and solutions to a problem (Batanero et al., 1997; English, 2005; Fransisco, 2013; Lockwood, 2013). Further, combinatorics allows for a wide range of representations and reasoning (see e.g. Ayalon & Rubel, 2022; English, 2005). English (2005) suggests that "[c]ombinatorics may be defined as a principle of calculation involving the selection and arrangement of objects in a finite set" (p. 121). Batanero et al. (1997) found that among 14-15-year-olds, the main difficulty prior to combinatorics instruction was to list objects systematically. At the same time, research has shown (e.g. English, 1991) that even children as young as 7-year-olds can combine items systematically, i.e. they are able to use the "'odometer strategy' holding one item constant while systematically varying each of the other items)" (p. 451, English, 1991), with meaningful problem contexts and suitable materials.

Research on students solving combinatorial problems accentuates the relationship between three aspects that represent important components of combinatorial thinking which are closely linked to each other: 1) counting processes, 2) sets of outcomes that are generated by counting processes, and 3) producing a formula (Lockwood, 2013). As Lockwood (2013) states, students who begin their process by listing elements from the set of outcomes are more successful in solving problems than those who use a formula immediately.

Hence, a critical aspect of middle school mathematics teaching in combinatorics is to provide opportunities for students to use combinatorial reasoning processes in line with Lockwood's (2013) three components. Hence, it is essential that teachers provide opportunities for students to generate sets of outcomes and organize these in a systematic manner so that students develop from unsystematic to systematic listing of sets of outcomes (e.g. English, 1991; 2005; Larsson et al., 2024a). In this study, a key feature of students' combinatorial reasoning is to organize sets of outcomes systematically. The aim of this paper is to contribute to what kinds of task-specific questions teachers pose when students solve problems and reason mathematically in combinatorics. An essential feature of teachers' task-specific questions is how the questions support students' systematic organization of outcomes in combinatorics.

Teaching through problem solving and creative mathematical reasoning

Schoenfeld (2020) proposes that a good mathematical problem opens for conjectures, making connections, abstractions, and generalizations. Moreover, in a classroom context, problem solving is a key to focus on students' thinking (Burkhardt & Schoenfeld, 2019). That is, in teaching based on problem solving students' thinking contributes to the conception of mathematics. To develop mathematical thinking, Goos and Kaya (2020) point at problem solving and mathematical reasoning as two key aspects. Lithner (2008) describes the process of constructing methods and mathematical founded arguments supporting that method as creative mathematical reasoning (CMR). The CMR approach contrasts with rote memorization or application of a pre-determined procedure believed to solve the problem. CMR is creative in the sense that the students use their mathematical knowledge to construct a novel solution method. A complete sequence of CMR also includes the construction of mathematical arguments to justify why the proposed solution method solves the problem (Lithner, 2008). To engage students in CMR, tasks should not include suggestions of solution methods but represent a reasonable creative challenge (Lithner, 2017).

A series of studies have shown that students who practice by creating methods to solve problems outperform, on post-tests, students who practice using provided methods (Jonsson et al., 2014; Norqvist, 2018, Norqvist et al., 2019). In those studies, the students solved tasks without support from a teacher. In a classroom context, to encourage CMR, it is important that the teacher avoids providing instructions on how to solve tasks but rather supports the student's thinking processes (Olsson & D'Arcy, 2022). We have for several years developed a design regarding how to support students' CMR (see e.g. Olsson & Granberg, 2024; Teledahl & Olsson, 2019) and have found that the phases of students' CMR can be categorized as initiating, developing, justifying, and verifying their own reasoning (ibid). These phases are easily disrupted if the teacher explains how to solve the current problem or funnels students towards a solution. This interruption can lead to the cessation of the students' CMR. Instead of explaining how to solve the problem or funnel students, we propose that teachers support these phases by posing prepared general and task-specific questions (Olsson & Granberg, 2024). As Boaler and Brodie (2004) emphasize, it is crucial to capture nuances in teachers' questioning that shape the cognitive opportunities given to students and to seize "the important subtleties of teaching that may make the difference between more or less productive learning environments" (p. 775). To facilitate the development of accurate and comprehensive explanations, probing questions that focus on specific aspects of students' work are shown to be effective (Franke et al., 2009). However, to elucidate mathematical concepts, teachers must pose questions that encourage justification, a practice that has been found challenging for many teachers (e.g. Drageset, 2015). Unlike probing questions, the role of questions that elucidate mathematical concepts has been less extensively studied (DeJarnette et al., 2020). In this study, we contribute to this by focusing on teachers' task-specific questions that encourage students to initiate, develop, justify, and verify their own reasoning in the context of combinatorics.

Analytical framework

The analytical framework stems from Lithner's (2008) definition of CMR associated with problem solving. To be categorized as CMR, the expressed reasoning must include an (for the reasoner) origin solution method and arguments for the solution method as well as the solution being anchored in mathematics (Lithner, 2008). Based on Olsson and Granberg (2024), an overarching design principle for teaching through CMR has been formulated: If the goal is that students will learn mathematics by CMR, teacher-student interactions should guide students into initiating, developing, justifying, and verifying their reasoning when constructing and formulating arguments for the solution. Further, Olsson and Granberg (2024) used the four phases (initiating, developing, justifying, and verifying) to operationalize CMR-teaching, which is to guide teacher-student interactions in ways that promote CMR. If teacher support promotes students to engage in all four phases, the teacher-student interaction is considered as supporting students' CMR. A fundamental idea behind the principle for CMR-teaching is to avoid funneling students towards a specific solution. In this study, the four phases are used as an analytical tool to investigate teachers' support to students' combinatorial reasoning. A particular interest is how the teacher's task-specific questions (i.e. questions based on the demands of the task) support these four phases, which we henceforth call the four principles for CMR-teaching.

Method

The context of the study and the selection of cases are first outlined. The two combinatorial problems used by the teachers in the study are then presented. Finally, the analytical method is described.

Context of the study

Teaching for mathematical proficiency is internationally recognized (e.g. Kilpatrick et al., 2001; Niss & Jensen, 2002) and lays the foundation for the nation-wide Swedish PD 'Boost for Mathematics' (BfM) from which the analyzed lessons stem. Moreover, focusing on teacher-student interaction around content based on students' mathematical thinking is foundational for BfM (see Boesen et al., 2015).

In BfM, teachers plan, enact, and discuss lessons in collaboration supported by a coach. BfM is hosted by The Swedish national agency for education, available on a web portal[1]. Teachers participate in modules with eight two-weeks cycles of: a) preparing by reading texts and watching videos, b) discussing the texts and videos and planning lessons together,

c) enacting the lessons, and d) discussing different aspects of the lessons. During the academic year 2015/2016 a large data set was constructed[2]. 174 lessons from BfM were video-recorded and two of them constitute data for this study (see Selection of cases). To gather comprehensive background data, video recordings of lessons were supplemented with interviews with the teachers. A segment of the interview analysis focused on assessing each teacher's Mathematical Knowledge for Teaching (MKT) (cf. Ball et al., 2008), including questions on different student strategies and common student errors. See Larsson et al. (2024b) for more information of the analysis procedure for teachers' MKT and the interview questions. The concept of MKT is instrumental in elucidating variations in teaching effectiveness (Ball et al., 2008). The fact that both teachers in this study achieved very high MKT scores is used as background data in this study, indicating that they possess comparable prerequisites for teaching as measured by MKT indicators.

The two teachers in this study worked with the module Problem solving for grades 4-6 in BfM. Before the current cycle (which was named Adaptations of problems) the teachers conducted a cycle where the focus was on communication with certain goals[3]. The texts introduced the notions of funneling versus scaffolding. Funneling is framed as the teacher putting the words in the mouth for the students and that the students do not have to do their own reasoning to solve the problem. It is emphasized that – in contrast to funneling students towards an answer – a way of scaffolding students is that the teacher poses relevant, carefully selected questions to the students for them to extend and deepen their mathematical reasoning and be able to solve the problem based on their own reasoning, which is in line with CMR.

Selection of cases

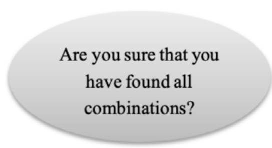
The two lessons chosen for analysis were taught by two different teachers, in grade 5 and 6 respectively. The two lessons are both part of the BfM module Problem solving for grade 4-6. The selection of these two particular lessons from the 174 lessons in total was based upon that several features of the two lessons and the two teachers were similar but still, the nature of teacher-student interaction seemed to be different. This makes the two lessons interesting to analyze and contrast regarding the different task-specific questions that the two teachers pose to their students in combinatorics. The similarities consist of that a) the two combinatorial problems are in essence the same mathematical problem (see Combinatorial problems), b) the two teachers teaching the lessons both have very high mathematical knowledge for teaching (MKT) according to the MKT measures taken in the larger study (see Larsson et al., 2024b), c) the two teachers have similar teaching experience (14 and 12 years, respectively), d) both teachers participate in the same PD module in BfM, e) the grade levels are similar, and f) the two lessons are of similar length and number of students[4]. In both lessons the problem was first introduced by the teacher on the board, whereupon the students worked on the problem.

Combinatorial problems

The teacher of lesson 2 used the original combinatorial selection problem in BfM, Glass beads, while the teacher of lesson 1 adapted the problem to a mathematically similar combinatorial problem with Swedish candy cars. As English (2005) emphasizes, this type of combinatorial selection problems is crucial for early development of understanding in probability. The problems were considered to represent a reasonable creative challenge regarding the students' ability to solve problems (cf. Lithner, 2017).

> *Ahlgren's cars – Lesson 1 (grade 5)*
> In a candy bag there is mixed colors of Ahlgren's cars, they are green, white, pink, brown and black. Alexandra picks two cars without looking.
> Which colors can the cars have (all combinations) if there are:
>
> a) 1 car of each color?
>
> b) 2 cars of each color?
>
> c) 1 brown, 1 black, 2 green, 2 white, 2 pink cars?

> *Glass beads – Lesson 2 (grade 6)*
> In a bag there is red, green, yellow, and blue glass beads. Ville picks two beads from the bag without looking. Which colors can Ville's two beads have if there are:
>
> a) 1 bead of each color?
>
> b) 2 beads of each color?
>
> c) 1 red, 1 blue, 2 yellow and 2 green beads?



Two additional sub problems were also posed to students who had reasoned about sub problems a, b, and c in lesson 2 on Glass beads:

> d) Which colors can Ville's two beads have if there is 1 bead of each color, and the number of colors increases with one all the time? That is, if there are 5 colors, 6 colors, 7 colors etc.
>
> e) Which colors can Ville's two beads have if there are 2 beads of each color, and the number of colors increases with one all the time?

Analytical method

The two selected lessons were analyzed regarding the teachers' task-specific questions according to the four principles for CMR-teaching that were developed through previous studies (Olsson et al., 2024; Olsson & D'Arcy, 2022; Olsson & Granberg, 2024; Teledahl & Olsson, 2019). The video-taped lessons were watched several times by the first author and notes were taken. Relevant parts were transcribed by the first author, i.e., the parts of the lessons in which the teacher interacts with students during their pair/small-group work in the explore phase. The teacher-student interaction was divided into interactional sequences that begin with either a student utterance or an utterance from the teacher when the teacher has stopped at a student pair/small group to listen to and monitor their reasoning. Each interactional sequence ends when the teacher moves to another student pair/small group. Each sequence was analyzed by the two authors in accordance with the four principles for CMR-teaching, i.e., more precisely whether the teacher encouraged the students to (1) initiate an independent (line of) reasoning, (2) develop one's own reasoning, (3) justify one's own reasoning, or (4) verify the result of one's own reasoning. These four principles for CMR-teaching, developed through previous studies (Olsson et al., 2024; Olsson & D'Arcy, 2022; Olsson & Granberg, 2024; Teledahl & Olsson, 2019), hence constituted the analytical tool in our study reported in this paper. In addition to these four principles for CMR-teaching that constitute analytical categories, a fifth category, funneling students' reasoning, emerged during the analysis and was used in the analysis as an additional analytical category. It is well-documented that it is common to funnel students' thinking (Wood, 1998).

To identify the four principles in the analysis, the following procedure has been used. For an interactional sequence to be categorized according to principle 1, i.e. that the teacher encouraged the students to initiate an independent (line of) reasoning, the students have not initiated a line of reasoning when the teacher enters the discussion. Typically, the students have very little written on their papers. Either the student(s) approach the teacher's attention by typically asking "What should we do/write?" or stating "I/we don't get it!", or the teacher notices that the students are stuck and starts a dialogue to engage the students in understanding what the task asks for, without funneling them (see further about funneling below).

For an interactional sequence to be categorized according to principle 2-4 (i.e., encouraging students to develop, justify or verify their own reasoning), the students have already – as opposed to principle 1 – initiated an independent line of reasoning when the teacher enters the conversation. What differs between principles 2-4 is what unfolds next. For an interactional sequence to be categorized according to principle 2, i.e. the teacher encouraging the students to develop their own reasoning, the teacher typically challenges the students to see patterns and generalize their initiated solution or extend their solution to higher numbers. For an interactional sequence to be categorized according to principle 3, i.e. the teacher encouraging the students to justify their own reasoning, the teacher typically asks students questions that requests them to delve deeper into why their solution (system/pattern) works. For an interactional sequence to be categorized according to principle 4, i.e. the teacher encouraging the students to verify the result of their own reasoning, the teacher typically asks students to test whether their answer is correct, e.g., to test whether their pattern or system works.

Funneling is identified by the teacher's use of guiding questions towards an answer or a specific solution on occasions when the teacher notices that the students either are stuck or have difficulties with the task (visible either in the students' notes or in their oral reasoning about the task). The students merely contribute with short answers on the path of questions that the teacher lays out for the students to reach an answer or a specific solution.

Next, according to the four principles for CMR-teaching and funneling, an overview of the results is first provided of the number of interactional sequences in the two lessons, followed by representative excerpts from the two lessons regarding the teachers' task-specific questions.

Results

The results reveal that despite the similarities between the two mathematical problems, the two teachers' mathematical knowledge for teaching, the two teachers' years of teaching experience, and the students' age, the two lessons are shown to have fundamental differences regarding the distribution of teacher questions with respect to the four principles. Only principles 1 and 2 are present in lesson 1 (Ahlgren's cars), while in lesson 2 (Glass beads) the teacher poses task-specific questions in line with all the four principles. Moreover, funneling is only present in lesson 1. Table 1 below provides an overview of the number of interactional sequences in the two lessons, according to the four principles for CMR-teaching and funneling.

Table 1. *Overview of the number of interactional sequences in the two lessons, according to the four principles for CMR-teaching and funneling.*

	Number of interactional sequences (with %) in lesson 1 Ahlgren's cars	Number of interactional sequences (with %) in lesson 2 Glass beads
Principle 1 – Initiate an independent (line of) reasoning	11 (44 %)	4 (26.7 %)
Principle 2 – Develop one's own reasoning	5 (20 %)	3 (20 %)
Principle 3 – Justify one's own reasoning	0 (0 %)	6 (40 %)
Principle 4 – Verify the result of one's own reasoning	0 (0 %)	2 (13.3 %)
Funneling – Guiding questions towards an answer/solution	9 (36 %)	0 (0 %)
Total	25	15

As visible in table 1, the mostly used principle in lesson 1 is encouraging students to initiate an independent (line of) reasoning (principle 1), followed by funneling[5]. One fifth of the interactional sequences are constituted by encouraging students to develop their own reasoning (principle 2). However, in lesson 1 the teacher neither encourages students to justify their own reasoning (principle 3) nor to verify the result of their own reasoning (principle 4). In lesson 2 – in contrast to lesson 1 – all four principles are present in the interactional sequences while funneling is not present at all (see table 1). Encouraging students to initiate an independent (line of) reasoning is mostly seen in the beginning of lesson 2 when students are trying to grasp what the problem asks them to do[6].

Principles 2, 3, and 4 are then used in a well-balanced way in the remaining interactions.

Now we turn to excerpts[7] from the two lessons that are representative for the kinds of task-specific questions that the two teachers pose during the lessons as a whole.

Lesson 1 – Ahlgren's cars

In lesson 1, the first two principles are present in the interaction between the teacher and the students during the monitoring in the explore phase.

Principle 1 – Initiating an independent line of reasoning

In the beginning of students' work, the following conversation unfolds on sub task a when a student begins by asking "What should we write?", illustrating how the teacher encourages students to initiate an independent line of reasoning:

Student:	What should we write?
Teacher:	All combinations if you pick two [cars] [shows a movement with the hand picking up something].
Student:	It says so, one of each.
Teacher:	That's what's in the bag.
Student:	Yes.
Teacher:	If you close your eyes and pick, in how many ways can you get them?
Student:	Brown-black, pink-brown, lots of colors.
Teacher:	Sort of. But there will be different combinations.
Student:	Yes.
Teacher:	So, if you start to write them you will probably see-
Student:	All combinations, should I write them?
Teacher:	That you can think of, yes, all of them you just said now.

In the conversation above, the teacher tries to initiate an independent line of reasoning from the students. The students have not really grasped what the problem asks them to do, and the main work done by the teacher here is to make clear what the problem says (that there is one candy of each color in the candy bag) and what the problem asks them to do (find all the different combinations of two candies). The students start to list sets of outcomes (brown-black, pink-brown) and the teacher encourages the students to list all combinations. The same lesson, the following conversation unfolds with other students on sub task a.

Principle 2 – Developing one's own reasoning

Student 1: But how many are there actually?

Teacher: How many combinations?

Student 1: Yes.

Teacher: You want me to say the answer to how many there will be.

Student 1: Yes [smiles].

Teacher: Hehehe. I will not do that.

Student 1: Ah.

Teacher: But do you see- do you see a pattern or something?

Student 1: No, I don't know. Maybe.

Teacher: So how many did you get now?

Student 1: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Teacher: Ten.

Student 2: I have ten, too

The teacher asks if student 1 sees a pattern but student 1 has not organized her combinations and seems unsure. Both students have ten combinations in total on their papers but with a varied level of systematicity. After this excerpt, the teacher emphasizes that student 2 has organized her sets of outcomes in a systematic manner and asks them to compare their solutions.

Funneling

The teacher funnels students' thinking on several occasions during the lesson. For example, when one student has difficulties with sub task b, the teacher shows with colored pencils until the student states that you can get two purple, two black, two white, two green, and two pink candies. Then, referring to the students' prior work on sub task a, the teacher also adds "Perhaps you also need to think that some of the old combinations would work too".

Lesson 2 – Glass beads

In lesson 2, all four principles are present in the interaction between the teacher and the students. Each of the principles is illustrated below with excerpts from the teacher-student interaction during the teacher's monitoring in the explore phase.

Principle 1 – Initiating an independent line of reasoning

In the excerpt below, that illustrates how the teacher encourages students to initiate an independent line of reasoning, a student begins by stating that (s)he does not understand what to do when (s)he is about to begin on sub task b:

Student:	I don't get it, it can come equally many of each color, it's not like the odds are bigger that it becomes one of these colors
Teacher:	If you look at the question, what is it that you are supposed to find out now?
Student:	Which colors Ville's two beads can have if there is
Teacher:	Exactly. So, it's which colors you are supposed to find out, not the chance or the odds.
Student:	No, I know. So
Teacher:	But which combinations could he get
Student:	All
Teacher:	Tell me a combination.
Student:	One red and one yellow.
Teacher:	There you have one combination. Are there more combinations that he could get?
Student:	Yes, all. Yellow and blue.
Teacher:	There you have another one. Can you try to find all the combinations that he could get?
Student:	Okay.

In the excerpt above, the teacher first directs the student to formulate what is actually asked for in the problem. The teacher then supports the student in starting to generate sets of outcomes as a way for the student to start initiating an independent line of reasoning.

Principle 2 – Developing one's own reasoning

To encourage students to develop one's own reasoning, the following excerpt illustrates how the teacher poses questions challenging the students to find the numbers of combinations when the number of colors increases (sub task d):

Student 1: Eeh, then we take (.)

Student 2: It's another one then.

Student 1: 1, 2, 3, 4, 5, 6, 7 [counts the number of combinations for 8 colors] it becomes 28.

Student 2: Yes.

Student 1: [looks at the teacher] How high were we supposed to go?

Teacher: I think it might be enough to 8. But here I want you to think further girls. Can you see a pattern in this? [There are only answers on the students' paper for 5, 6, 7 and 8 colors]

Student 1: Yes, well for each (.) if- if those colors were from the beginning [marks the first four colors in her figure] well red yellow blue and green so for this bead [points at the fifth bead] so it should be paired with those four and then it becomes plus four

Teacher: Yes.

Student 1: and then for this [points at the sixth bead] then it becomes plus five 'cause that should be paired with all those five

Teacher: Can you write it somewhere so you can see it? And can you try to figure out, without trying things out, how many the ninth will be?

Students: Mm. Yes.

Teacher: Work on that. And write clearly so that we can use it later. [The teacher leaves.]

When the teacher enters the conversation in the above excerpt, the teacher stops the student from moving on to higher numbers of beads by saying, "I think it might be enough to 8". Instead, the teacher asks if they can discern a pattern based on their answers and thereby develop their own reasoning by figuring out the number of combinations for nine colors instead of trying things out.

On another occasion, a student asks the teacher if their solution works, and the following conversation unfolds:

Student 1: So, does it work, that when you add a new one you get one for each?

Teacher: It seems to work pretty well, huh? Can you see a pattern in this?

Student 1: Yes, it increases with 1 here [points].

Teacher: Then perhaps you can figure out how many the next will be.

Student 1: Seven plus. Or there it was five plus, there it was six plus, it should be seven plus on that [points at 8 colors]. The first one was four plus.

Teacher: That's right. Is Student 2 also following you on that reasoning? [The teacher leaves.]

When the teacher leaves, Student 1 starts to explain his reasoning to Student 2.

Principle 3 – Justifying one's own reasoning

In the following excerpt, a student starts the conversation by stating that they think that they have figured out a general system how to calculate the number of combinations (sub task d) whereupon the teacher responds by posing the question "Alright, which system do you have?" to encourage students to justify their own reasoning:

Student: We think we've figured out a system.

Teacher: Alright, which system do you have?

Student: Like this, that (.) eehm when we added a bead then it could make combinations with all the other beads

Teacher: Mm

Student: So, then we added four [6+4=10 is written on the paper for five beads]

Teacher: Yes.

Student: Then we've got (.) we added five here 'cause then it could be added to the other five so then we took ten added with five is fifteen [10+5=15 is written on the paper for six beads]

Teacher: Yes.

Student: Then we did the same here, so we have six beads there, so then (.) we added [points at 15+7=22 on the paper for seven beads] no it was seven beads, so it's just to continue that way

Teacher: Let me take a look here

Student: No wait! It went wrong now, it should be six there [points at 15+7]

Teacher: Correct it if you think that you've made an error.

Student: [Corrects 15+7=22 to 15+6=21] Then we just continued.

Teacher: Okay, can you try to figure out the ninth too before you move on?

Student: Yes. [The teacher leaves.]

The student seems puzzled over the incorrect statement 15+7=22 on their paper but first tries to justify it (although not convincingly). Not until the teacher says, "Let me take a look here", the student realizes what went wrong and corrects it, after which the teacher challenges the student pair further in their reasoning.

Principle 4 – Verifying the result of one's own reasoning

The following excerpt, where the students have concluded the number of combinations of four colors to 9 (sub task b), illustrates how the teacher encourages students to verify the result of their own reasoning:

Student 1: How many did you get?

Student 2: I got 1 2 3 [...] 1 2 3 4 5 6 7 8 9. Then I've missed one.

Student 1: Mm.

Teacher: Can you go through them systematically, so that you see which one you've missed?

Student 2 herself realizes that (s)he has missed one combination as can be seen in the above excerpt. After the teacher's request to go through the combinations systematically by posing the question "Can you go through them systematically, so that you see which one you've missed?" (see the above excerpt), student 2 compares her combinations with student 1. Student 2 then exclaims: "I counted incorrectly".

Task-specific questions according to the four principles for CMR-teaching

Representative questions for each of the four principles have emerged from the analysis of all excerpts and our categorization of these, according to which principle for CMR-teaching they adhere to. In table 2, the two teachers' task-specific representative questions are summarized in an overview.

Table 2. *Overview of teachers' task-specific representative questions in relation to students' combinatorial reasoning, according to the four principles for CMR-teaching.*

	Task-specific questions lesson 1 Ahlgren's cars	Task-specific questions lesson 2 Glass beads
Principle 1 – Initiate an independent (line of) reasoning	- All combinations if you pick two [...] in how many ways can you get it? [...] So, if you start to write them you will probably see.	- Tell me a combination [...] Are there more combinations? [...] Can you try to find all the combinations that he could get?
Principle 2 – Develop one's own reasoning	- Do you see a pattern or something? Follow up: - The teacher asks a student with an unsystematic solution to compare the combinations with the other student in the dyad who has a systematic solution.	- Can you see a pattern in this? Follow up: - Then perhaps you can figure out how many the next will be (after student(s) suggest(s) a pattern).
Principle 3 – Justify one's own reasoning		- Alright, which system do you have? (after student(s) suggest that they have a system/a systematic solution)
Principle 4 – Verify the result of one's own reasoning		- Can you go through them systematically, so you see which one you've missed? (when a student realizes that (s)he has missed some combination(s))

Discussion

A well-documented challenge for the teacher when students solve problems and reason mathematically is to interact in productive ways to support students to develop their own reasoning (e.g. Larsson, 2015; Stein et al., 2008). The results show that the teachers in this study provide opportunities for the students to generate and organize sets of outcomes, which is emphasized by Lockwood (2013) as important for middle school students' development in mathematics. Both teachers encourage their students to try to discern patterns and to develop their reasoning from unsystematic to systematic listing of outcomes, to have students develop from a random listing of sets of outcomes to a more systematic listing (cf. English, 1991; 2005). However, there is a difference in the use of the principles for CMR-teaching between the teachers' lessons. That is, teacher 2 uses all four principles for CMR-teaching while teacher 1 uses the first two principles only (see table 1 and table 2). Hence, it is only teacher 2 who encourages students to justify their solutions and verify[8] the result of their reasoning. These results resonate with Franke et al.'s (2009) findings that teachers often are prepared to ask questions to elicit students' reasoning but struggle with following up students' mathematical ideas.

Looking closer into both teachers' actions reveals that teacher 1 seems to be prepared to encourage the students to develop their own line of reasoning. For example, the teacher comments that (s)he will not tell the answer and asks questions to support students to initiate and develop their own reasoning. However, instead of asking challenging questions with the purpose of encouraging students to justify and verify their reasoning, teacher 1 often funnels students towards the solution. In contrast, teacher 2 not only poses questions that promote students to initiate and develop their reasoning but continues by challenging students to justify and verify their solution methods. It appears that teacher 2 has prepared these more challenging questions in advance which signals that the teacher's expectations are high on students' ability to reason mathematically. For example, teacher 2 has prepared questions on what happens if the number of colors is increased from four to five, six, and seven colors (see section Combinatorial problems). Students who have developed their reasoning for the originally posed questions were asked these more challenging questions. Teacher 2 has also added the question "Are you sure that you have found all combinations?" to the problem for all students. Furthermore, a difference between the two teachers' interactions with students is that teacher 2 stays shorter with the students and often leaves them after posing a question. This indicates that the students are familiar to teaching in which they are supposed to construct and discuss solutions together in problem solving. This also gives teacher 2 time to listen to every group during the lesson; even if students do not call for the teacher's attention, teacher 2 listens to their reasoning and sometimes does not say anything at all, sometimes asks the students how it is going[9].

The results indicate that the difference between the two teachers' interactions regarding challenging students to justify and verify their reasoning may be due to teacher 2 having prepared task-specific questions in advance, which is in line with Olsson and Granberg's study (2024). Even though none of the teachers had prepared their lessons with the four principles from Olsson and Granberg (2024) in mind to guide teacher-student interactions in ways that promote CMR, the differences in the results underline the importance of preparing task-specific questions, at least if the purpose is that students will solve problems through reasoning.

In this study, teacher 2 apparently had anticipated possible difficulties and need for further challenges in students' problem-solving and based on combinatorial reasoning processes had prepared task-specific questions to challenge students' reasoning. Given that it is routine for a teacher to prepare task-specific questions, it is reasonable to assume that such preparations contribute to development of teaching skills. By recognizing the four principles for CMR-teaching in students' reasoning when they explore and solve mathematical tasks, teachers can modify the task-specific questions they pose and thereby create opportunities for development of students' CMR about key mathematical ideas over time. The development of teaching skills can be further strengthened by systematic planning and evaluation of lessons in collegial collaboration. Furthermore, planning and rehearsing (Lampert et al., 2013) lessons with the four principles for initiating, developing, justifying, and verifying students' mathematical reasoning as guiding principles can be a central element in teacher education activities. Translated into a practice-based teacher education in which pedagogies such as representing, decomposing, and approximating practice (e.g. Grossman et al., 2009) are central elements, students can analyze their own and others' teaching with the four principles for CMR-teaching as guiding principles for decomposing teaching. In this, teacher education can focus on teachers' questioning practices in relation to specific mathematical areas such as combinatorics.

Conclusions

We here conclude by summarizing the results of the work, by answering the research question guiding this study: To what extent and in what ways do the two teachers pose task-specific questions that encourage students to: (1) initiate an independent (line of) reasoning, (2) develop one's own reasoning, (3) justify one's own reasoning, and (4) verify the result of one's own reasoning, within the mathematical domain of combinatorics?

In sum, the kinds of task-specific questions adhering to the four CMR principles posed by the two teachers in our study in relation to combinatorial reasoning are:

> **Principle 1**: To initiate an independent (line of) reasoning, students are encouraged to start to list sets of outcomes and to find more (all) possible combinations.
>
> **Principle 2**: To develop one's own reasoning, students are encouraged to organize sets of outcomes going from unsystematic towards more systematic listing and to try to see patterns and find a rule for the number of combinations.
>
> **Principle 3**: To justify one's own reasoning, students are encouraged to justify their systematic solution and the patterns/rule they discern by mathematical arguments.
>
> **Principle 4**: To verify the result of one's own reasoning, students are encouraged to go over the combinations systematically or compare their own combinations with others'.

Regarding to what extent the two teachers pose the different kinds of task-specific questions, the results showed (see table 1) that teacher 1 only posed task-specific questions that encouraged students to initiate and develop their own reasoning and in addition funneled students, the latter which does not align with CMR-teaching. In contrast, teacher 2 did not funnel students and posed task-specific questions adhering to all four principles of CMR-teaching, i.e. teacher 2 encouraged students to initiate, develop, justify, and verify their own reasoning.

The findings have implications for teachers who teach mathematics through problem solving and would like to have access to useful tools and frameworks for supporting their students' CMR, which is crucial since problem solving and reasoning are found to be essential aspects of doing mathematics to develop students' mathematical thinking (e.g. Goos & Kaya, 2020; Lester & Cai, 2016). In this interactional work of teaching, it is important that teachers not only pose general questions on how students are thinking but also have access to the tools to pose task-specific questions, adhering to the principles for CMR-teaching (Olsson et al., 2024; Olsson & Granberg, 2024). Since it is known to be a challenge for teachers to pose follow-up questions that help students to develop their mathematical ideas (Franke et al., 2009), task-specific questions within different mathematical domains can be powerful tools for teaching (see e.g. Olsson & Granberg, 2024). Thus, to develop students' combinatorial reasoning in the interactional work of teaching, well-crafted, prepared task-specific questions within the domain of combinatorics can facilitate for teachers to respond productively in the moment to students' ideas, to mitigate the well-known challenge of in-the-moment decision-making (e.g. Stein et al., 2008).

The lessons from BfM were planned with a problem-solving approach to mathematics. Reasoning is undoubtedly an integral part of the problem-solving process (Lithner, 2008). Nevertheless, the fact that the analyzed lessons were not designed with the principles for CMR-teaching could be seen as a limitation. However, this study shows that the principles for CMR-teaching are possible to use as analytical framework for lessons that are not planned with the CMR-principles in mind. That is, the principles for CMR-teaching seem to work not only as a planning tool when preparing for lessons in which development of students' reasoning is integral, but also as an analytical tool and therefore this study contributes to methodological development.

The conclusions drawn regarding task-specific questions in relation to the principles of CMR-teaching closely align with instructional practices in the domain of combinatorics. This alignment appears to be a defining feature of task-specific questions. However, in order to formulate more generalizable guidelines for educators on how to design such questions, further research across diverse mathematical domains is needed. A robust empirical foundation is essential for the future development of reliable and transferable principles for constructing task-specific questions within the CMR-teaching framework.

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Notes

1 https://larportalen.skolverket.se

2 The data set was constructed within the research project Theorizing teacher use of curriculum material within mathematics classroom practice, funded by the Swedish Research Council.

3 The goals with this module part were for the teachers to: (i) deepen your knowledge about the significance of mathematical communication, (ii) develop your ability to identify funneling, scaffolding and mathematical ideas when communicating in problem solving, and (iii) develop your ability to plan problem solving with communication, with the aim of making explicit a certain mathematical content.

4 Lesson 1 in grade 5 was 42 minutes with 20 students. Lesson 2 in grade 6 was 46 minutes with 18 students.

5 Principle 1 and funneling together constitute 4/5 of the interactional sequences.

6 3 of the 4 interactional sequences with principle 1 in lesson 2 take place during the first three interactions.

7 The excerpts were translated from Swedish to English with emphasis on spoken language rather than on correct grammar.

8 English (1991) showed that 4-9-year-old children who hold one item constant and systematically varies the other item when they combine outfits with colored tops and pants know when all possible combinations have been listed, i.e. they can verify their own solution.

9 If the students' answer signals uncertainty or frustration, teacher 2 poses follow-up questions such as "Where did you get stuck?" and lets the students explain their reasoning.

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