Preservice teachers negotiating roles and responsibilities in a proof-based teaching approach

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This study explores preservice teachers' learning processes as they engage in proof-based teaching with peers and collectively reflect pre- and post-teaching on their planned and enacted lessons. As a part of a didactics course on algebra and number theory, groups of preservice teachers embrace the role of 'teachers-of-the-week' and lead their peers in activities focused on explaining and justifying the reasons behind important mathematical results. Through a reflexive thematic analysis of data, three main themes in the preservice teachers' discourse on proof-based teaching were identified. The results suggest that engaging in proof-based teaching with peers and collective reflection on teaching provides preservice teachers opportunities to actively explore, negotiate and re-negotiate roles and responsibilities in such an approach.

Keywords: Preservice teachers, Proof-based teaching, Reflection, Teacher preparation.

Proofs and proving can play an important role in a mathematics classroom, as a vehicle for developing in depth understanding of mathematics (NCTM 2000; Ball & Bass, 2003; Hanna & de Villers, 2012; Stylianou et al., 2010; A. Stylianides, 2016). Hanna (1990, 2000) emphasised that proofs that explain, rather than proofs that only verify, are the types of proofs that are most useful in a school context, as they can help understand why a mathematical claim is true. Rather than merely verifying that a claim holds, explanatory proofs illuminate the underlying mathematical relationships and help learners develop a deeper understanding of the concepts involved. For example, an explanatory proof might use visual representations to reveal why combining two odd quantities necessarily produces an even quantity. Aligned with this perspective, Reid (2011) coined the term proof-based teaching to refer to an approach to teaching

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proving that emphasises explanatory proving. Because proof-based teaching is a central concept in this study, it is described in more detail in a later section.

The role of proof and proving in a Norwegian school context aligns with a proof-based teaching approach in several ways. In proof-based teaching, proving is considered a vehicle for learning rather than an object to be taught (Reid, 2011). Similarly, in the Norwegian curriculum, proving is not described in the curriculum goals as an object to be taught (Valenta & Enge, 2019). Rather, proving is visible in the core element *reasoning and argumentation*, serving as an overarching principle or approach to learning the mathematical content (Ministry of Education and Research, 2026). The core element 'reasoning and argumentation' states that pupils across all grades should understand that mathematical rules and results in mathematics are not random, but that they have clear reasons (Ministry of Education and Research, 2026). Furthermore, pupils should learn to give valid arguments for their approaches, reasoning and solutions, to prove that they are valid (Ministry of Education and Research, 2026).

Although the Norwegian curriculum expects pupils to engage in proving activities across all grades, it is not detailed or given much explicit attention in the concrete competence goals (Valenta & Enge, 2019), neither in the Norwegian textbooks (Pettersen, 2022; Gjerløw & Torsdatter, 2023). As teachers often rely on these sources, and since engaging pupils in proving activities in general is recognized as challenging (Reid & Knipping, 2010; Stylianides et al., 2013; Stylianides, 2016), it is crucial for preservice teachers to gain familiarity and experience with proving activities suitable for the grade levels they will teach.

The study

Although research has extensively documented the challenges pupils and teachers face with proof and proving, relatively few studies have investigated classroom-based interventions designed to address these challenges, particularly in teacher education contexts (Stylianides & Stylianides, 2017). This study addresses this gap by investigating preservice teachers' learning processes as they engage in a practice-based approach to learning proof-based teaching at the university (cf. Charalambous & Delaney, 2018).

Aligned with an international trend, practice-based approaches to teacher education have gained increasing attention in Scandinavia (Mosvold, 2018). The term 'practice-based' is an umbrella term that encompass a range of different approaches in teacher education (Char-

lamabous & Delaney, 2018; Forzani, 2014). Despite the differences in use, the term mostly emphasizes the importance of enacting practice as a means of learning to teach (Charlamabous & Delaney, 2018).

Prior research has explored learning through activities such as rehearsals, time-outs, and cycles of enactment and investigation (Fauskanger & Bjuland, 2019, 2021; Hovtun et al., 2021; Rø et al., 2019; Wæge & Fauskanger, 2021, 2023), highlighting their potential to support engagement with core teaching practices. Grossman et al., (2009) refer to such activities as approximations of practice, i.e. approaches providing safe opportunities to practice teaching, with support and feedback. My study builds on this line of work by examining a practice-based approach to learning proof-based teaching (cf. Grossman et al., 2009). Similar to other enactment-based approaches, this practice-based approach provides a structured setting in which preservice teachers can experiment with proof-based teaching and reflect collaboratively on their experiences (cf. Charalambous & Delaney, 2018).

This study aims to contribute to more insights into preservice teachers' learning processes in this specific context, by exploring a cohort of preservice teachers' collective pre- and post-teaching reflections. As peer exchange and collective negotiation of meaning is considered an important means for learning (Schön, 1983), and constituents of the learning process, the following research question is formulated to guide the study: "What characterises a cohort of preservice teachers' discourse on a proof-based teaching approach, when reflecting on their planned and enacted lessons?" The term 'discourse' is here used in a sociocultural sense to refer to the preservice teachers' active processes of negotiating meanings of proof-based teaching, through articulating and expressing their ideas together with their peers (Säljö, 2001). Through engaging in a discourse about proof-based teaching, the preservice teachers can explore what such an approach to teaching and learning mathematics can involve together with their peers.

Theoretical foundations

This study adopts an overarching sociocultural perspective on learning and development, considering preparation of preservice teachers as a process of *enculturation* (Bishop, 1988; Rogoff, 1990; Säljö, 2001, 2010). This means that the preservice teachers become socialised into a proof-based teaching approach in mathematics, gradually gaining familiarity with ideas, values, norms and practices aligned with such an approach. The concept of *appropriation* is a sociocultural metaphor for learning, that is here used in line with Rogoff (1990) and Säljö (2001, 2010), to refer

to preservice teachers' processes of becoming gradually more familiar with ideas, values, norms, and ways of describing and engaging in proof-based teaching.

Aligned with a sociocultural perspective on learning, Schön (1983) emphasises both practice and reflection in the process of professional learning. He points to how parts of our knowing is tacit and is inherent and implicit in our patterns of action, referring to it as knowing-in-action (Schön, 1983). Through reflection-on-action, implicit theories and assumptions existing within our routines and patterns of behaviour may become visible:

> Nevertheless, it is sometimes possible, by observing and reflecting on our actions, to make a description of the tacit knowing implicit in them. Our descriptions are of different kinds, depending on our purposes, and the language of description available to us. We may refer, for example, to the sequences of operations and procedures we execute; the clues we observe and the rules we follow; or the values, strategies and assumptions that make up our "theories" of action (Schön, 1983, p. 25).

In this study, collective reflection with peers is considered an important part of preservice teachers' learning processes, and it involves using appropriated cultural tools to explore and negotiate the meanings of their newly gained experiences (cf. Säljö, 2001). The preservice teachers are provided opportunities to reflect explicitly upon norms, values, routines, strategies and patterns of behaviour, making implicit theories, assumptions and ideas related to a proof-based teaching approach to mathematics visible in the social plane.

Proof-based teaching

Reid and Vallejo Vargas (2017) identified three key elements of proof-based teaching: (1) a toolbox of shared knowledge, (2) an expectation for explanation, and (3) deductive explaining. In the context of proof-based teaching, an expectation for explanation involves establishing a didactical contract (Brousseau, 1997), where students are accustomed to being asked to explore and justify mathematical claims and prove their validity through deductive forms of reasoning (Reid & Vallejo Vargas, 2017). A 'toolbox of shared knowledge' consists of mathematical concepts, definitions, theorems, axioms, or other accepted premises, that serve as reference points for the proving process (Reid & Vallejo Vargas, 2017). The toolbox thus founds the basis from which mathematical arguments can be made.

The meaning of proof in a proof-based teaching approach aligns with Stylianides (2007) definition, and deductive forms of reasoning are thus the only acceptable mode of argumentation (Reid & Vallejo Vargas, 2019). Within this approach, particular emphasis is placed on explanatory proving, that is, proofs that help learners understand why a mathematical claim is true, rather than merely establishing that it is true (Hanna, 1990, 2000; Reid, 2011). Explanatory proving is therefore not viewed as an alternative to deductive proof, but rather as a specific orientation toward proof that prioritizes explanation and understanding. Deductive explanations using algebra tiles can, for example, be considered a form of action proof, where the validity of a mathematical statement is demonstrated through physical actions with tiles (cf. Semadeni, 1984). Such action proofs may serve an explanatory function when the actions make visible the mathematical relationships underlying a claim. To prove that multiplication is commutative ($a \cdot b = b \cdot a$), an action proof might involve demonstrating that the order of multiplication does not affect the product by arranging 3 rows of 5 counters each, then rearranging them into 5 rows of 3 counters each. In this way, the proof not only verifies the claim but also provides an explanation for why it holds.

Methodology

This study is grounded in a 'big Q' qualitative research paradigm (cf. Braun & Clarke, 2022), based on constructionist and interpretivist assumptions (Bryman, 2016). A constructionist orientation to reality involves considering reality as socially negotiated and created by the people involved (Bryman, 2016). From an interpretivist perspective, it is then the researcher's task to interpret the meanings that people give to their experiences and describe the world from their perspective (Bryman, 2016).

This research adopted an explorative case study approach (Bryman, 2016). The case study was conducted in the context of a didactics course of algebra and number theory. There were in total 32 preservice teachers in the course, all distributed into seven groups containing four to five members. Five groups were selected for this study, based on the content of the lessons they had chosen. The author participated as an external researcher and was not involved in the teaching of the course.

The primary data source in this study was generated through focus group interviews before and after the lessons in each of the five groups (Bryman, 2016). These interviews served as opportunities for collective reflection on action (cf. Schön, 1983). The focus group interviews were guided by the researcher based on an interview guide, consisting of

initiating prompts such as "tell me about this lesson you are going to teach" and "what is most important to you in this lesson" (Bryman, 2016). The goal was for the preservice teachers' interests to steer the conversation, so prompts merely served to encourage interaction and promote discussion, based on the preservice teachers' planned and enacted lessons. The pre-teaching reflection sessions took place the day before the lesson enactment, and the post-teaching reflection sessions were conducted immediately after the preservice teachers' enacted lessons. The interviews lasted between 20 and 30 minutes.

The research design draws on principles from ethnographic research, as the author spent time in the course setting to develop contextual understanding (cf. Bryman, 2016). Observations throughout the course, and specifically of preservice teachers' lesson enactment, served as an essential part of the research process. This prolonged involvement in the setting provided additional insights into the overall context and thus aided the data analysis. Videorecording of the enacted lessons further aided the analysis by allowing a closer examination of specific incidents referenced in the interviews.

In line with ethical guidelines (Bryman, 2016), informed consent was obtained from all participants, and data were anonymised, securely stored, and processed in accordance with Sikt regulations. Sikt is the Norwegian agency for shared services in education and research

Research context: "teachers-of-the-week"

The preservice teachers were introduced to proof-based teaching through two sample lessons modelled by their professor. In the roles as 'students', the preservice teachers were engaged in tasks and activities that involved exploring and justifying mathematical rules and results, using algebra tiles as concrete tools for argumentation. The sample lessons could be considered 'representations of practice' (cf. Grossman et al., 2009), making proof-based teaching visible and accessible to the preservice teachers. After the sample lessons, the preservice teachers took turns facilitating 90-minute proof-based teaching lessons with their peers. The mathematical content and specific material for each of the teachers-of-the-week lessons were carefully designed in advance by the professor to ensure a logical sequence, where each week's topic built on the previous lessons. The topics in the lessons aligns with the Norwegian grade 1–10 curriculum and were therefore already familiar to the preservice teachers from their own schooling. What was new to them in this context was engaging with the mathematical content through a proof-based teaching approach, focused on explanatory proving, as well as reflecting on the

content from the perspective of a teacher. Prior to enactment, the teachers-of-the-week met with their professor to go through the lesson material and tasks.

Data analysis

A reflexive thematic analysis (Braun & Clarke, 2022), was employed to identify characteristics of preservice teachers' discourse on their planned and enacted proof-based teaching lessons. Such an approach involves exploring and interpreting reoccurring themes, patterns and key features within the data (Braun & Clarke, 2022). After an initial data familiarization and transcription, each group was systematically examined in turn. For each group, pre- and post-lesson reflections were analysed separately. The initial generation of codes involved systematically identifying meaningful data segments and assigning short phrases or keywords, that captured the essence of that part of the data. In the initial coding, the three key aspects of proof-based teaching identified by Reid and Vallejo Vargas (2017) served as sensitizing concepts, giving some general directions and guidance in approaching the data (Bryman, 2016). The three elements, 'a shared toolbox of knowledge', 'an expectation for explanation' and 'deductive explaining', were thus not used as rigid codes imposed on the data. Rather, they served as support to "uncover the variety of ways in which something can exist and be seen" (Bryman, 2016, p.16). Themes were thus not predetermined by Reid and Vallejo Vargas' (2019) framework but were generated from the data itself. Each interview was revisited multiple times, and codes were refined and adapted through several cycles. Related codes were then grouped into initial themes, which were then reviewed several times to ensure that they accurately captured significant characteristics of the preservice teachers' discourse. The second phase of the analysis focused on identifying similarities, differences, and nuances within and across the data (cf. Braun & Clarke, 2022). This involved systematically comparing pre- and post-lesson reflections, first within each group, and then across the different groups.

The results from this analysis are presented in the next section, and illustrative quotes and excerpts are presented and used to exemplify each theme. The five teachers-of-the-week groups are labelled A, B, C, D, and E, in the sequence the lessons were taught. Each preservice teacher is assigned a pseudonym beginning with the same letter as their group, to indicate which group they belonged to. Due to space restrictions only a subset of quotes can be included, and the extracts from the preservice teachers' reflections are chosen to exemplify the main themes. However,

they do not necessarily encompass the full range of perspectives and nuances emerging within each theme.

Results

The main results show that preservice teacher's discourse is characterised overall by an overarching negotiation of roles and responsibilities in a proof-based teaching approach. Through the thematic analysis, three themes emerged across pre- and post-teaching reflections: (1) priming students for proving - establishing grounds for explanatory proofs; (2) facilitating student-centred proving processes; and (3) orchestrating whole-class presentations of explanatory proofs. Each of these themes will be elaborated in the following sections.

Priming students for proving – establishing grounds for explanatory proofs

In their pre-teaching reflections, the preservice teachers discuss the importance of introducing and reviewing key ideas and important algebra tile concepts at the start of the lesson. In their post-teaching reflections, they re-negotiate the importance of being mindful and intentional and carefully consider when concepts and ideas are introduced to students.

Pre-teaching – recalling, repeating or reviewing prior ideas and concepts

Across different groups, the preservice teachers overall acknowledge the importance of ensuring that learners have the necessary 'things' that may be useful for proving. Some groups consider the opportunity to explicitly repeat what students have learned previously, and others highlight how their approach involves strategically sequencing tasks to prime students for the upcoming tasks:

Dylan I start with a bit of repetition from earlier to go over some useful things for the session—things they will need. So here, I start by checking if they remember what this is—that it's a multiplication frame, and what they can use it for. Then, a bit about the rules that will be useful. For example, showing that when these x tiles are multiplied together, they form a larger tile. Also, demonstrating what happens when you multiply a positive and a negative—that the result is negative. [group D, turn 30]

Elise And that's why we're starting with a review task from last time. To bring back [the concept of] completing the square. We don't think it's completely unmanageable. [group E, turn 95] You're putting them on the right track. [group E, turn 97]

Dylan expresses how their group plan to explicitly review 'some useful things' for the session, such as the multiplication frame and rules for multiplication. Elise, on the other hand, points to how their group will start with a review task. The purpose of the review task is to put their peers on the right track, by providing fresh and recent experiences with the idea of 'completing the square'. Other groups express how they will just rely on students' recalling from previous lessons, expecting prior exposure to be sufficient preparation:

Cameron I think it would have been difficult if we hadn't had the group that was there last time. Because they used this multiplication frame quite a lot. So, we're hoping that most of them remember it now. So that they get how it works. [group C, turn 12]

Caleb Yes, because we discussed whether we should do a review of it. But I think we decided that we – [group C, turn 13]

Cameron We're counting on them remembering it. [group C, turn 14]

Caleb Yeah, that they remember it, really. [group C, turn 15]

Although the different groups suggest varying approaches, the preservice teachers generally acknowledge the importance of establishing the groundwork necessary for engaging in explanatory proving, to support students' engagement with the tasks. The preservice teachers suggest how priming students for proving may involve either repeating or reminding them of algebra tile-specific concepts, such as zero pairs and multiplication frames, as well as ensuring familiarity with key mathematical ideas, rules and operations, such as the idea of multiplication as repeated addition, negative times positive is negative, and the fact that $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ etc. In this sense, the preservice teachers begin to recognise the importance of making relevant mathematical tools available to students as a foundation for proving.

Post-teaching – being mindful, intentional and careful about when concepts and ideas are introduced

Through enacting their proof-based teaching lessons, the preservice teachers indicate gaining a renewed appreciation of being mindful, intentional and even careful when preparing their peers for what they will work on during the lesson. Some explicitly emphasise the pitfall of introducing ideas that are not yet relevant for the following tasks as students may take this as a "hint", potentially putting them on the wrong path and making them narrowminded. Others highlight the need to carefully consider what concepts and ideas should be reinforced and what

aspects should be left for students to productively struggle with. These two quotes from group B and E illustrate this issue:

Belinda I was thinking a bit about what we just talked about—that thing about handing out a multiplication frame at the beginning. Maybe that's something we could have held back on. (...) Maybe then they wouldn't have immediately linked it to that. [group B, turn 151]

Elise I think it's a bit—when you give an exploratory task, like we're doing now—it's important to be very conscious of what you say before presenting the task. What hints we decide to give should be well thought out and adapted to the group that will be exploring. If there are things you expect them to already know, you don't mention those. But if there are things you know they will get stuck on no matter what, then it might be useful to address them. For example, we made a choice to state that one-half times one-half is one-quarter. We knew they would likely struggle with that, and it wasn't really the focus of what we wanted them to work on. So, being intentional about how we introduce tasks is important. [group E, turn 70]

Although the quotes presented here reflect some of the preservice teachers' explicit growing attention to the importance of being mindful and intentional about what to say, or not say, before students engage in proving tasks, not all groups engaged in such direct meta-reflection. Others instead refer to moments during the lesson when their peers became stuck, and they had to highlight key ideas later in the lesson. These reflections suggest a developing sensitivity to the consequences of instructional choices, particularly how premature or poorly timed input may constrain students' opportunities to engage in explanatory proving. This points to an emerging understanding of the teacher's role in carefully orchestrating when and how ideas are introduced to the students during the proving process.

Facilitating student-centered proving processes

In their pre-teaching reflections, the preservice teachers discuss their envisioned role of providing their peers with hints and tips. In the post-teaching reflections, they discuss specific strategies for supporting students, such as providing directive suggestions, prompts and questions.

Pre-teaching – providing hints and tips

The preservice teachers' pre-teaching reflections illustrate how they envision themselves as facilitators rather than direct instructors, emphasizing student-centred learning where their peers take an active

role in justifying and explaining their solutions. The preservice teachers articulate this approach with varying emphasis, envisioning how they will circulate among the groups, observing their peers as they work on tasks, listening in and gaining overview of students' solutions:

> Daisy I think a lot of what we do is walk around and see what they are working on and try to guide them a bit. (...) And sort of being able to give them some tips to help them move forward. (...) So that they do as much as possible on their own, without everything just happening because someone shows it in front. But rather that they figure it out themselves first. [group D, turn 66]

The preservice teachers suggest that a key responsibility for them as teachers is to provide their peers with supportive hints and tips to help them on their way:

> Clay So what we haven't nailed down yet is how to give tips. Because it is quite important here that we do not give different tips, or too much or too little to certain groups. [group C, turn 7]
>
> Eva We have to give some guidance if we see that they are stuck. For example, the hint that they can cut up to make it a square. Like, try to give some hints, without saying too much. [group E, turn 82]

While Clay emphasises the importance of maintaining consistency in their guidance, being mindful not to offer uneven or varying amounts of assistance, Eva points to how the guidance comes in when their peers get stuck, and they might provide some suggestions for small actions, such as cutting the tiles up to make a ¼-unit tile. These reflections indicate that the preservice teachers are beginning to appropriate a facilitative role in proof-based teaching, where supporting students' independent reasoning is prioritised over direct instruction. At the same time, their uncertainty about how to provide appropriate guidance highlights the complexity of balancing support and autonomy in students' proving processes.

Post-teaching – providing directive suggestions, prompts and questions

When reflecting on their enacted lessons, the preservice teachers revisit their attempts to support their peers' engagement with the tasks. Although many admit having failed, by providing too many tips, or too little hints, the preservice teachers also report on successful attempts, replicating and referring to the types of prompts or suggestions they provided and the strategies they employed to support students' proving processes:

Barbara So, then we ended up saying a bit more, or like, explaining the idea of repeated addition and repeated subtraction a bit more explicitly than we had originally planned. [group B, turn 80]

Cameron (...) when we gave most of the groups the tip to use multiple frames, it seemed like a lightbulb went off for most of them. [group C, turn 21]

Barbara points to how, although they originally wanted their peers to realize how they could use the idea of repeated subtraction themselves, they ended up explaining it to them. Cameron refers to how their tip about using the multiplication frames helped their peers in the proving process. Several preservice teachers describe how they adopted different types of strategic questioning approaches, such as eliciting explanations for clarification or providing directive prompts to direct students' attention to specific aspects of the task or features of the tiles:

Caleb Yes, I tried, like, if they had found a number to start with, I tried to get them to think: "How should the number below in the table look then? Where is the difference here?" And then, "Yes, there you are multiplying by one more, like that one factor." "How would you express that within the [multiplication] frame?" [group C, turn 73]

Alice But regarding communication with the groups, I had a small situation with one of the groups where I asked, "Yes, is it possible to..." or, I think I said, "Is it possible to do this with zero pairs?" And then I noticed that the group got a little frustrated because they thought I was asking them to show a specific way of doing it, when I was really just asking, like, is it possible to do it with these. [group A, turn 102]

The preservice teachers here describe somewhat different questioning approaches. Caleb describes how he attempted to scaffold students' thinking through incremental prompts to help them recognize patterns and relationships. Alice, however, refers to an encounter where her questioning approach was intended to encourage further reasoning, suggesting "zero pairs" as a tool in their reasoning, and indicates how her approach resulted in her peer's hesitation and visible frustration. These reflections illustrate how the preservice teachers are experimenting with different strategies for supporting students' reasoning, while also becoming aware of how their interventions can shape or constrain students' proving processes.

Orchestrating whole-class presentations of explanatory proofs

In their pre-teaching reflections, the preservice teachers point to their envisioned responsibility of managing student presentations and providing teacher explanations. In their post-teaching reflections,

they discuss the need to balance student-led explanations and teacher explanations during whole-class presentations.

Pre-teaching – managing student presentations and providing teacher explanations

When reflecting on their planned lesson, the preservice teachers consider different responsibilities related to orchestrating the presentation of solutions. They point to how they could manage the presentations, by deliberately choosing and inviting specific groups to share their solutions, to ensure that the presented explanations include key ideas or desired solutions they want to be brought forward:

Annie Walk around and observe. Possibly go into groups to help—hear what certain groups are doing. Like, Mr. Proof [the professor] is very good at that, and if he thinks that maybe the one— the ones who show something first—if he thinks it's missing a bit, or if he knows that there is someone who has done it in another way, he makes them come up too. And we also have the opportunity to do that. [group A, turn 24]

Belle (...) if you know that there is someone you have gone around to, whom you know has a good answer, and you have spoken to them, like, "do you also want to go up afterwards?", and if they are willing to do so, you can let them. Especially if they have come up with a very good example, which we would like to have for the rest of the class. Then you can keep it a bit like that in the back of your mind. [group B, turn 36]

Here, both Annie and Belle point to how they as teachers can manage the student presentations by inviting specific groups to present their solutions, based on an overview gained during group work. They point to how this can ensure that particularly good explanations, or explanations adding more details or filling in on previous explanations can be brought to the attention of the whole class.

While emphasising that students should take the lead in explaining their solutions and learning from each other, the preservice teachers also anticipate the need to step in and clarify concepts when necessary. They highlighted how teacher explanations might be warranted if students struggled to express or understand key ideas:

Dan (...) they should at least get some time to think on their own, try to explore a bit with the algebra tiles, and then see if anyone wants to share. I've also prepared my own tiles, just in case I need to use them. (...) [group D, turn 40] Say, if someone shares, and it's crystal clear, then that's fine—I can just skip it. But if not, then it might be useful to clarify it a bit. [group D, turn 42]

Caleb When Mr. Proof [the professor] showed us up there in the room [planning meeting], it was just like — when he sort of put together those squares, then it was like, "yes" — it all clicked, in a way. It all made sense. So, I think it's important that—if they can't figure it out—we show them, using the tiles. [group C, turn 83]

Both Dan and Caleb here indicate that if their peers struggle to articulate key ideas, or cannot figure out the solutions on their own, they see it as their responsibility to provide explanations. In different ways, the preservice teachers' reflections illustrate that although they envision their peers to be the ones explaining, they also acknowledge the value of good teacher explanations. The preservice teachers' reflections indicate a sense of feeling responsible for their peers' learning and a desire to ensure understanding, which is the overall goal of proof-based teaching according to Reid and Vallejo Vargas (2017, 2019).

Post-teaching - eliciting explanations vs. clarifying key ideas

Based on a variety of experiences, the preservice teachers' discourse reveals some tensions and considerations during whole-class presentations. They reflect on the challenge of navigating the balance between letting students lead the explanation and stepping in as teachers to clarify when needed. The preservice teachers continually emphasize that their goal was for their peers to be the ones explaining and learning from each other. However, they also acknowledge that some student-led explanations did not fully bring out the key ideas. As a result, they felt the need to supplement the students' explanations with additional input to emphasise key ideas:

Barbara So, it was actually during the break that we were discussing whether one of us should just say it. That, ok, we start with zero pairs because we need them to be able to subtract, to make repeated subtraction possible. But then the point about them figuring it out on their own kind of disappears. And that they shouldn't worry, if it takes a long time. But when should you give them the hint, and when do you just explain it? [group B, turn 12]

> Annie But I think a big part of the point here [in the teachers-of-the-week lessons] was that it's the students who should be explaining, not us. So, it's about finding that balance, where like, you're not supposed to take over what each student is showing. [turn 44]. But we saw that—something was missing. (...) So, it was kind of to show that [method] as well—that this is another good way to do it. [group A, turn 48]

These reflections indicate that the preservice teachers are navigating the complexity of balancing student-led explanations and teacher explanations during whole-class presentations. Although being committed to a student-centred approach, the preservice teachers recognize the need for teacher intervention to ensure clarity. In many ways, their reflections indicate a strong commitment to an overall goal of understanding.

Discussion

Results of this study suggest that learning proof-based teaching through a practice-based approach (cf. Grossman et al., 2009) offers valuable opportunities for preservice teachers to grapple with core ideas and challenges in such an approach. The preservice teachers' reflections reveal how the collective reflection sessions provided space for refining their ideas related to roles and responsibilities in a proof-based teaching approach to mathematics. As the preservice teachers reflect and negotiate their developing practice, their discourse reflects an evolving sensitivity to complexities in enacting proof-based teaching, and their ongoing processes of appropriating key ideas, norms, practices and values in line with a proof-based teaching approach (cf. Rogoff, 1990; Säljö, 2001).

The research question guiding this investigation was: What characterises preservice teachers' discourse on a proof-based teaching approach, when reflecting on their planned and enacted lessons? The results show that, across pre- and post-teaching reflections, the preservice teachers consistently acknowledge the importance of priming students for proving tasks by reviewing key concepts and ideas likely to support their work during the lesson. They also emphasize the teacher's responsibility for offering timely hints and prompts to support students' proving processes. These reflections suggest that the preservice teachers recognize a central role for the teacher in managing students' "toolbox of shared knowledge," one of the key elements of proof-based teaching highlighted by Reid and Vallejo Vargas (2017, 2019). Although Reid and Vallejo Vargas do not specifically address preservice teachers' learning of proof-based teaching, the present findings reinforce their claim that such a toolbox is essential to proof-based teaching. At the same time, the study extends this idea by specifying more precisely how teachers may manage this toolbox in practice.

In particular, the preservice teachers emphasize the importance of making essential mathematical tools visible and accessible, by revisiting them during instruction, including concepts such as "zero pairs," "multiplication frames," $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and multiplication as repeated addition. Their reflections indicate that they see it as the teacher's responsibility to remind students of relevant tools and to suggest how these tools can be used in reasoning and argumentation. Overall, their discourse highlights the teacher's role in ensuring that students have access to and can actively draw on this shared toolbox during proving processes.

The preservice teachers suggest how they as teachers have the opportunity to gain an overview and deliberately choose and invite specific groups to present their explanations. A strategic and deliberate use of student responses aligns with practices described by Stein et al. (2008) as monitoring student responses" and "purposefully selecting student responses for public display" (p. 326–327). Although the preservice teachers consistently emphasize, both before and after teaching, that students should take the lead in explaining their reasoning, their reflections also reveal an awareness of the teacher's role in supporting and reinforcing ideas when needed. In particular, they recognize that when student explanations lack clarity or miss key concepts, teacher contributions may be necessary to ensure that all students grasp the underlying reasoning.

The preservice teachers' reflections indicate that, in this peer-teaching activity, they engage with key aspects of teaching proving that have been identified in the literature as particularly challenging. These include interpreting and supporting students' reasoning during group work and orchestrating productive whole-class discussions (Crespo, 2000; Stein et al., 2008; Stylianides et al., 2013; Stylianides & Stylianides, 2017). At the same time, the data suggest that some challenges described in the literature are less visible in the peer-teaching setting. In a secondary school classroom, pupils may for instance not be accustomed to justifying and arguing for why their solutions are valid. Consequently, issues related to having 'an expectation for explanation' (Reid & Vallejo Vargas, 2017) and establishing a didactical contract (Brousseau, 1997), might thus be more prominent with pupils in school than with peers at the university.

Although not explicitly evident in the data presented in this article, the preservice teachers' reflections nevertheless suggest an emerging curiosity about how a proof-based teaching approach might be implemented with school pupils. This highlights both the affordances and limitations of peer-teaching environments for learning proof-based teaching, and points to a need for further research in school classroom settings with pupils.

In their collective reflection on action (cf. Schön, 1983), the preservice teachers are reflecting both on a meta level and more implicitly on their envisioned roles and responsibilities of the teacher in a proof-based teaching approach. In their reflections, they envision roles, responsibilities and considerations in different parts of the lesson. Post-teaching, the preservice teachers narrate experiences, describing emerging roles, responsibilities and challenges they grappled with. The preservice teachers often re-visit some of the same ideas in their post-lesson reflections, continuing the negotiation of the roles and responsibilities in a proof-based teaching approach based on their experience from teaching. Their discourse indicates a growing sensitivity to tensions and a need for balancing and considering different values and responsibilities. The preservice teachers' nuanced and more concrete reflections post-teaching demonstrate their ongoing processes of appropriation of ideas, values, norms and practices aligned with a proof-based teaching approach (cf. Rogoff, 1990).

Concluding remarks

This research highlights the potential of peer teaching as an approximation of practice (cf. Grossman et al., 2009) and indicates preservice teachers' enculturation into proof-based teaching through joint engagement in peer-teaching and collective reflection on action (cf. Schön, 1983). Through reflecting on their teaching experiences in a university course context, the preservice teachers are developing into reflective practitioners of proof-based teaching, gradually gaining familiarity with key ideas, values, norms and practices in line with this approach (cf. Schön. 1983).

Future studies should explore the preservice teachers' enactment of proof-based teaching with peers, to gain more insight into their developing practices and how this relates to their emerging discourse on proof-based teaching. This can provide further insight into the potential for learning proof-based teaching through peer teaching and give more insight into preservice teachers' opportunities to engage with concrete and practical challenges and demands involved in a proof-based teaching approach. Further research should also explore the preservice teachers' continued learning processes as they transition into classrooms with pupils, as the didactical contract and students' prior experiences with justification may present additional challenges.

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