Experiencing part-whole relations of numbers in a partitioning task

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The experience of part-whole relations has been identified as being of importance for the development of proficient counting strategies. In an intervention in first grade (seven-year-olds), one activity was about partitioning numbers in different ways using finger patterns. We analyzed the enactment of the activity using variation theory as a theoretical framework, and identified critical aspects for students' learning: to differentiate between finger pattern and finger number, to differentiate between parts and fingers, to make parts bigger than 1, to experience commutativity, and to see zero as one part. The findings provide new knowledge about what needs to be discerned by students in order to be able to partition numbers up to 10, using finger patterns for illustrating part-whole relations.

The focus of this paper is students' learning of part-whole relations of numbers in the first grade. In a part-whole relation, for example, 4 is the whole, and 1 and 3 are possible parts. Resnick (1983) argues that students' interpretation of numbers in terms of a part-whole relation is a major conceptual achievement for early school years since it makes it possible for students to think about numbers as a composition of other numbers. For a long time, students' learning of addition and subtraction has been assumed to develop through acquisition of basic counting strategies, e.g. count from the first addend, count from the biggest addend, etc., emphasizing counting of single units (Fuson, 1992). In recent years, a concern has been raised about students holding on to strategies involving counting that they have learned early on, and rejecting more advanced ones,

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Kullberg, A. & Nord, M. (2025). Experiencing part-whole relations of numbers in a partitioning task. *Nordic Studies in Mathematics Education*, 30 (2), 3–23.

such as using part-whole relations (e.g. Björklund et al., 2019; Cheng, 2012; Ellemor-Collins & Wright, 2009; Hopkins et al., 2022). Ellemor-Collins & Wright (2009, p. 52) state that "adding and subtracting without counting is a critical goal in achieving children's numeracy. Some students do not achieve this facility. Instead, they persist with strategies involving counting by ones for addition and subtraction in the range 1 to 20, and in turn use counting strategies in the higher decades". The early-learned counting strategies that involve counting single units (forwards and backwards) to solve addition and subtraction items become cumbersome for some students when dealing with numbers in a higher number range. However, alternative approaches for learning addition and subtraction have been studied to a lesser extent (see, for instance, Björklund, et al., 2021; Cheng, 2012; Ellemor-Collins & Wright, 2009; Schmittau, 2003; Venkat et al., 2021). Some studies suggest that part-whole relations can be used from the outset in students' learning of addition and subtraction, instead of a counting-based approach (e.g. Cheng, 2012; Hopkins et al., 2022; Kullberg et al., 2022). Neuman (2013) suggests that finger numbers[1] (showing the numbers starting with one on the little finger of the left hand) can help students to experience part-whole relations of numbers. However, we need to know more about what difficulties students encounter when learning about part-whole relations. The aim of this paper is to identify and discuss critical aspects of students' learning of part-whole relations via finger numbers and finger patterns. We use data from a teaching intervention in first grade (seven-year-olds) conducted in four classes where part-whole relations were emphasized from the outset. In this paper, one of the activities, about partitioning a number into two parts, is analyzed when enacted in teaching. The following research question is posed: What is critical for students to discern in order to be able to partition numbers below 10 in different ways using finger patterns?

Experiencing numbers as part-whole relations

Since the 1980s, it has been advocated that students should learn addition and subtraction skills through the acquisition of more and more advanced counting strategies (Baroody, 1987; Carpenter & Moser, 1984; Carpenter et al., 1982; Fuson, 1992). Later, the student develops strategies that don't involve counting, using part-whole relations. Because counting has been considered the developmental path for students' learning, it has also been used as a path for teaching addition and subtraction in lower grades. "Counting on" (from the first or the biggest number) and "counting back" are strategies taught by teachers. However, these counting strategies involving counting in single units *do not* entail that the student experiences the counted steps as a part of the counted number (Steffe & Cobb, 1988). Hence, the students only need to experience the number as single units (e.g. 7 + 6 = 7 + 1 + 1 + 1 + 1 + 1 + 1), as steps on a number line, and therefore do not need to see and create parts greater than one within the counted number (e.g. 7 + 6 = 7 + 3 + 3).

Later studies have shown that young children use a variety of methods for solving addition and subtraction tasks before entering formal school (e.g. Björklund et al., 2019; Clements & Sarama, 2007). For example, Björklund et al. (2019) showed that five-year-old children who structured part-whole relations on their fingers were more successful than children using fingers to count single units when solving a subtraction item (10 - 6 =). Experiencing numbers as part-whole relations is considered to be critical for development of arithmetic skills (Cheng, 2012; Resnick, 1983), as being aware of part-whole relations allows students to make use of powerful strategies such as decomposition ($c = a + b$), commutativity ($a + b = b + a$), and the complement principle (if $a + b = c$ then $c - a = b$) when solving addition and subtraction tasks.

The ways that students are initially taught for dealing with addition and subtraction have an impact on how students deal with operations later on (Hopkins et al., 2022). For example, in order to be able to solve addition and subtraction items bridging through ten, e.g. 15 - 7 =, in a proficient way (without having to count back in single units), students need be able to use part-whole relations, e.g. 15 - 5 - 2 =, where 5 and 2 are seen as parts of 7.

Finger patterns as a means of experiencing part-whole relations

Fingers can be used by students for ”keeping track” of counted units when solving an arithmetic task. For example, when solving 2 + 4 =, the student might say ”1, 2, 3 [one finger extended is one more], 4 [two fingers extended is two more], 5 [three fingers extended is three more], 6 [four fingers extended is four more] – 6” (Baroody, 1987, p. 134). When students count backwards or forwards in single units, they need to keep track of how many single units they have counted, and at the same time keep track of the number sequence in order to arrive at the answer. This ”double counting” on two parallel number sequences is a cumbersome strategy since it is difficult to keep track of both sequences simultaneously, and even more so in higher number ranges. It has been argued that using fingers to keep track of single units may not help students to develop an understanding of part-whole relations of numbers since parts larger than one are not seen within the number (Ekdahl, 2019; Runesson Kempe et al., 2022).

Several scholars suggest that finger patterns can be beneficial for students as a way to experience part-whole relations when solving addition and subtraction tasks (Ahlberg, 1997; Kullberg & Björklund, 2019; Kullberg et al., 2020; Neuman, 1987). Ahlberg (1997) argues that when students perceive part-whole relations by means of their fingers, they are ”grasping the numerosity of the numbers” (p. 70).

The use of finger patterns to solve arithmetic problems can promote students' use of parts larger than one. ”Seeing” the parts, or combinations of ”seeing parts” and counting single units when structuring a part-whole relation, have been found to be beneficial for solving an arithmetic task among four- to six-year-old children (Björklund et al., 2019). Several studies have shown relationships between young children's finger use and development of early arithmetic skills (e.g. Baroccas et al., 2020; Kullberg et al., 2020; Kullberg, 2022; Zhang et al., 2020).

As a part of the intervention, the students had been taught finger numbers (1 being the little finger on the left hand, 2 being the little finger and ring finger on the left hand, and so on) for the numbers 0 to 10 during their first month in first grade. Previous research has shown that some students may experience a finger number merely as a name with no cardinal meaning (also called word tagging, see Brissiaud, 1992). If a student experiences ”numbers as names” and not as groups of objects with cardinal meaning, this student will have difficulties showing the number with another finger pattern.

Theoretical framework

Variation theory (Marton, 2015) is the theoretical framework used in this study. The theory was developed in the late 1990s based on insights from phenomenographic research on learners' different conceptions of the same phenomenon (Marton, 1981; Marton & Booth, 1997). Learning is seen as the discernment of aspects (of an object of learning or a phenomenon) that have not been previously discerned. In the theory, the aspects the learner needs to discern in order to be able to learn the desired capability (the object of learning) are called critical aspects. In order to be able to add and subtract, young students need to experience, for example, cardinality, ordinality, part-whole relations, and modes of representation of number (Björklund et al., 2021). Students solve tasks according to the way they understand them, i.e., depending on which aspects they have already discerned. Hence, critical aspects can be observed when students solve tasks in qualitatively different ways. An aspect is not critical if the learner has already discerned the particular aspect. Variation theory states that it is primarily that which is varied that is likely to be noticed by the learner. To discern a new aspect of a phenomenon, for instance numbers, the learner needs to experience variation in regard to the critical aspects. To discern a critical aspect, it needs to be varied against a background of invariance. We suggest that in order to be able to partition number, students need to encounter variation in how the same number (invariant) can be partitioned in many different ways (varied). In this study, we focus on students' experience of partitioning numbers in an activity that intends to contribute to an understanding of number relations. Variation theory is used to analyze what aspects learners have or have not yet discerned. Identifying critical aspects can help teachers in becoming aware of what they need to vary in order to enhance students' learning of part-whole relations. In the research tradition of phenomenography and variation theory, several studies have been made regarding teaching and learning of addition and subtraction in the early years (e.g. Ahlberg, 1997; Björklund, et al., 2021; Kullberg et al., 2020; Neuman, 1987, 2013). This study adds to this research by analyzing students' partitioning of numbers in an activity used in teaching.

Method

During the intervention, which lasted two semesters, four teachers and the research team planned tasks jointly and the teachers enacted them in their own first-grade classes (seven-year-old students). The teachers worked in three compulsory schools located in a suburban area of a large Swedish city. All teachers were qualified teachers with more than five years of teaching experience and participated in the study voluntarily. In this paper, video data from the enactment of one activity is analyzed. During the activity, the students were sitting on the floor in a small group together with the teacher. In this paper, data from five lessons in three of four participating classes was analyzed. The rationale for this selection was that in these five lessons, the camera was positioned in a way that captured the students' fingers. The selection of students (N = 26) was based on the choice of lessons. The lessons were video recorded by the teachers. The students' legal guardians had given written permission to allow participation in the study.

The task

The analyzed task is about partitioning one number into two parts on two hands, e.g. showing 4 as 1 finger on one hand and 3 on the other hand (see figure 2). The task is a version of the "statement game" developed by French researchers (Sensevy et al., 2015). Later in the lessons, the ways of partitioning were connected to written numbers in triads. The teacher conducted the lessons in small groups, each consisting of approximately six students. A die was a thrown, and the students were expected to show the number on the dice with finger patterns using both hands. The students were told to show the number in a different way from the other students in the group. The purpose of the task was to make it possible for the students to experience that the same number can be partitioned in different ways and with different finger patterns. In this sense (i.e., the number was invariant while the partitioning varied), variation theory was used when planning and enacting the tasks in the lessons. The teachers pointed out what was the whole (e.g. 6) and the different parts (e.g. 3 on one hand and 3 on the other hand, or 4 + 2, or 1 + 5) in the part-whole relation. The finger patterns were used as a means for students to be able to instantly "see" (perceive) the numbers in a part-whole relation. In this paper, in order to identify what may be critical for learning, we direct our attention to what parts the students chose, and what fingers the students used to show these parts.

Process of analysis

In the process of analysis, recordings of the lessons were watched repeatedly. The second author analyzed students' finger patterns, speech, and movements during the activity. The authors then jointly analyzed, based on the data, what critical aspects students had or had not yet discerned. The unit of analysis was instances in the lessons in which students used their fingers to represent numbers. The way the students partitioned the numbers and which fingers were shown by the students were coded. For example, when a student partitioned 5 into 2 + 3, using the middle and index finger on the left hand, and the thumb, index, and middle finger on the right hand, this was coded as ○○●●○ ●●●○○. Students' different finger patterns, as well as speech and visual signs (pointing, folding, watching others), were considered in the analysis. How the student represented the number the first time, and whether and how the student changed fingers and/or parts, were analyzed. The finger patterns that the students showed to the teacher are given in bold in the tables, e.g. when the dice showed 6 (see table 4), the student showed 2 + 4 to the teacher. When the student tried by him/herself, this is shown as 5 + 1. By identifying differences between what parts and fingers the students showed, it was possible to identify differences among the students and hence see aspects critical for being able to partition numbers below 10 with finger patterns. We illustrate these critical aspects with examples from individual students' enactments.

Results

In order to identify what is critical for learning, we searched for qualitative differences in how the students partitioned numbers into two parts with the help of finger patterns. From this analysis, we found the following aspects to be critical: to differentiate between finger pattern and finger number, to differentiate between parts and fingers, to make parts bigger than 1, to experience commutativity, and to see zero as one part.

To differentiate between finger pattern and finger number

Being able to represent a number from the dice on two hands e.g. show 7 as 4 fingers on one hand and 3 on the other, and thereby show the finger pattern in another way than the finger number (where the finger number would be all fingers on the left hand and the thumb and index finger on the right hand, 5 + 2) was an important part of the activity. In our study, there was only one student (Elsa) who had difficulties with showing finger patterns other than the finger number for the numbers discussed in the task. During two analyzed lessons, Elsa managed twice to show another finger pattern than the finger number.

Table 1. *Elsa's ways of partitioning numbers in Lesson 1a.*

Number	Elsa's ways of partitioning		
4	4 + 0	●●●●○ ○○○○○	finger number (See Figure 1a)
	4 + 4	●●●●○ ○●●●●	after the teacher clarifies "Show four on two hands"
	2 + 2	○○●●○ ○●●○○	shows another finger pattern than the finger number by imitating another student
6	**5 + 1**	●●●●● ●○○○○	
5	5 + 0	●●●●● ○○○○○	finger number
	5 + 5	●●●●● ●●●●●	after the teacher clarifies "Show five on two hands"
	5 + 0	●●●●● ○○○○○	helped by the teacher, who folded the right hand, 5+0
6	**5 + 1**	●●●●● ●○○○○	finger number
2	**2 + 0**	○●●○○ ○○○○○	shows another finger pattern than the finger number

Elsa did not (without imitating other students in the group) manage to partition any number smaller than 6 on two hands when asked to do so. In the first lesson (with six students), Elsa quickly presented the finger number for 4 on her left hand (figure 1a) when the dice showed 4.

When the teacher emphasized that the number should be shown ”on two hands”, Elsa showed 4 fingers on *each* hand (figure 1b).

a.  b. 

Figure 1. *Elsa showing the finger pattern for 4 (a), and later 4 on each hand (b).*

When the teacher discussed each student’s way of partitioning 4, it seemed as though Elsa eventually imitated her classmates. When a student showed 4 and 0, Elsa also closed her left hand but did not make four with the other hand. After having observed all the other students’ ways, Elsa finally showed 2 and 2. The excerpt below shows Elsa’s actions when listening to two students sitting next to her, who both partitioned the number by showing 2 fingers on each hand. She succeeded in showing the finger pattern 2 and 2 for 4, most likely by imitating Ester sitting just opposite (figure 2). She was able to articulate that the number in the second part was ”two”, even though 2 was shown with the index and the middle finger, two consecutive fingers.

Teacher:	Ester also has two, one group with two and one group with two [like the student before] is four. Harry [turning to the next student] has two and two [Elsa looks at Ester and quickly makes 2 and 2 with the same fingers as Ester], and the whole is four.
Harry:	[Harry quickly changes to 3 and 1] Three and one.
Teacher:	And you can have one and three [points to the one finger on his left hand and the three fingers on his right hand], it is also four. (...) You had three and one, the whole is four.
Teacher:	And you, Elsa, have one group with two [pointing to the fingers on the left hand], and there is one group with....
Elsa:	Two.
Teachers:	And the whole is four.

When partitioning numbers from 6 to 10, the finger numbers include both hands, and thus allowed Elsa to partition 6 by showing the finger number she had learned (all the fingers on her left hand and her right thumb). The second time the dice showed 6, the teacher told the students to "show it in another way than the last time". Elsa then showed 6 in the same way as she did before, i.e., the finger number, saying: "I can't do it in any other way". Although Elsa had the opportunity to see how other students partitioned 6 in different ways, this did not seem to be sufficient for Elsa to change her way of partitioning the number. The first time Elsa did not use the finger number was when the dice showed 2. With the help of her right hand, she made the finger pattern for two with her ring finger and middle finger (see figure 3a and 3b).

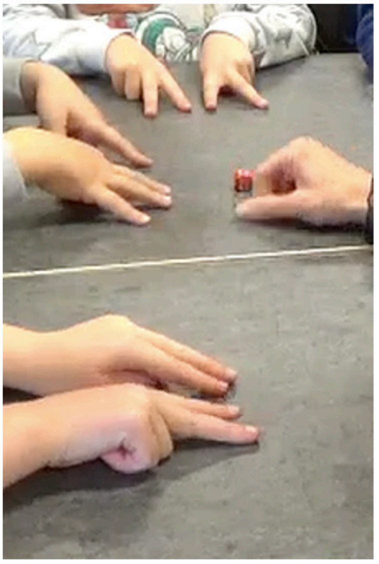

Figure 2. *Elsa (in the front left corner) showing 4 as 2 and 2.*

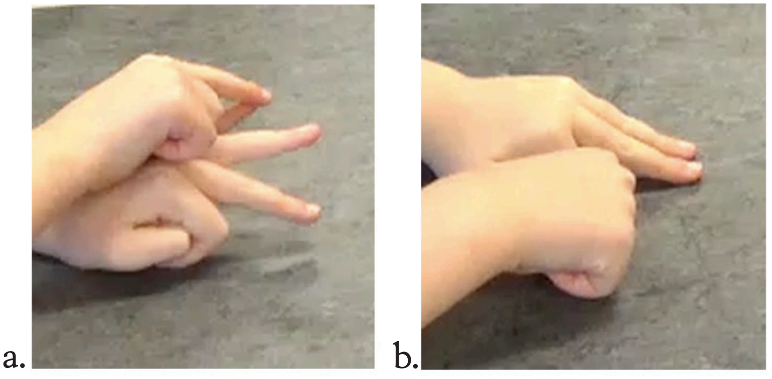

Figure 3. *Elsa folding her little finger (a) in order to show 2 with her ring and middle fingers (b).*

Elsa seemed to know the finger numbers up to 6, but she was not able to partition these numbers. Although Elsa had the opportunity to experience a variation in partitioning the numbers during the lesson, this was not sufficient during this sequence for her to be able to partition the number in other ways herself.

To differentiate between parts and fingers

The instruction "show it in another way" was interpreted by some students in a way that we did not foresee. The intention was to enhance the variation of the parts within numbers, not only the fingers used. We identified that many students, when first asked to show the number on the dice with two hands, and next to show it in another way, often showed the same parts but changed the fingers used to show the parts. These students interpreted the instruction "can you show it in another way" as meaning to show the same parts with other fingers, rather than showing different parts.

Eric is an example of a student who varied the fingers used, instead of the parts. Table 2 shows his different ways of showing parts for the different numbers discussed. For example, when the numbers 3 and 4 were discussed, he showed the same parts, varying order (commutativity) and fingers.

Table 2. *Eric's ways of partitioning numbers in Lesson 1b.*

Number	Eric's ways of partitioning
3	3 + 0 ●●●○○ ○○○○○ 1 + 2 ●○○○○ ○●●○○ **1 + 2** ●○○○○ ●●○○○ **2 + 1** ●●○○○ ○○○○● 2 + 1 ●○○○● ●○○○○
4	Says "zero and four" 3 + 1 ●●●○○ ●○○○○ 3 + 1 ●●●○○ ○○○○●
5	**1 + 4** ●○○○○ ○●●●●
6	6 + 0 ●●●●● ●○○○○ **5 + 1** ●●●●● ○●○○○
8	8 + 0 ●●●●● ●●●○○ **4 + 4** ●●●●○ ○●●●●

When the dice showed 3, Eric first showed the finger number (figure 4a). He then quickly changed finger pattern to showing one with his left little finger and two with his right index and middle fingers (figure 4b). Then he changed the fingers on his right hand to the thumb and index finger to illustrate 2 (figure 4c).

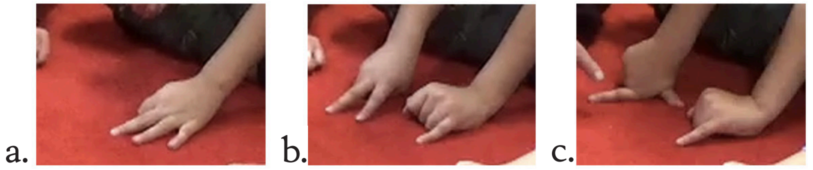

Figure 4. *Eric showing the finger number (a), and two ways of partitioning 3 into 1 + 2 but using different fingers (b and c).*

Having discussed the different ways of showing partitioning 3 on two hands with the group, the teacher said: ”I said two and one for everyone didn't I .... You have shown with different fingers. Can one partition it [the number 3] in another way than two and one” Eric quickly said: ”I know”, and showed 1 and 2 with other fingers (showing awareness of commutativity) (figure 5a), and a little later he said ”Look” and switched the fingers used on the left and right hand (figure 5b). Hence, in this part of the lesson, he was varying the finger patterns used for the same parts, 2 and 1.

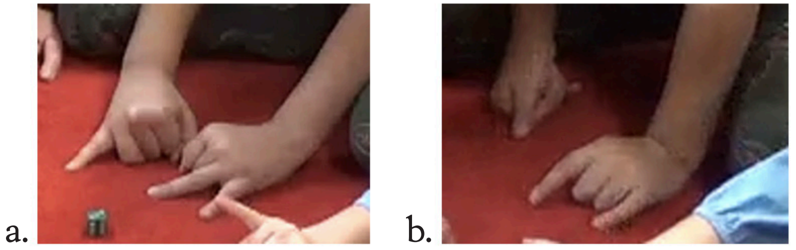

Figure 5. *Eric showing 2 + 1 with different fingers (a and b).*

The teacher continued by asking ”Is there another way?” and another student showed 0 and 3. The teacher also showed one hand with 3 and one hand with all fingers folded to represent 3 and 0, and thereby introduced a variation in the way of partitioning the number besides 2 and 1. When the number on the dice changed to 4, Eric first said ”zero and four”, most likely inspired by the previous example, and then showed 3 and 1, using the little finger, ring finger, and middle finger on his left hand, and the index finger (he later changed to his little finger) on his right hand. When prompted to show the same number (4) in another way, he showed the same partitioning, but changed one finger (See table 2).

To create parts that are bigger than one

Although many ways of partitioning the numbers were present on a group level in the teacher-led discussions, individual students differed in the number of combinations they showed. For example, Bill showed 6 in five different ways, while Eric showed one way.

When the number 5 appeared on the dice, Eric showed 1 and 4 (figure 6a). When the dice showed 6, he showed the finger number for 6 first (figure 6b), and later made a small change, from showing one with his thumb to showing one with his index finger (figure 6c).

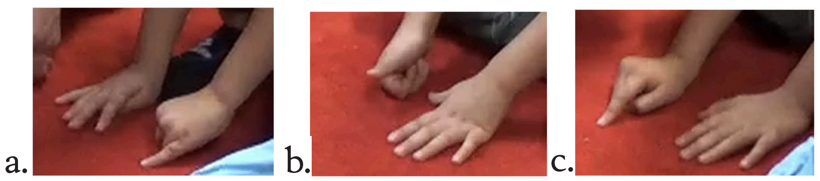

Figure 6. *Eric partitions 5 into 1 + 4 (a) and 6 into 5 + 1 (b and c).*

For the next number, 8, he similarly showed the finger number (5 + 3) but then folded his thumbs and made 4 on each hand. However, Eric did not show any of the other ways of partitioning the different numbers discussed, apart from moving one finger to the other hand.

Another similar example is Ester (table 3), who showed doubles when asked to partition an even number (e.g. 2 + 2, 3 + 3), or only changed the number of fingers by one when asked to partition in another way than earlier.

Table 3. *Ester's ways of partitioning numbers in Lesson 1a.*

Number	Ester's ways of partitioning
4	**2 + 2** ○○●●○ ○●●○○
6	3 + 3 ○●●●○ ○●●●○
5	4 + 1 ●●●●○ ●○○○○
6	4 + 2 ●●●●○ ●●○○○
2	1 + 1 ○○○●○ ○●○○○
5	4 + 1 ●●●●○ ●○○○○ 2 + 3 ○○●●○ ●●●○○ struggles

When Ester was asked to partition 5 in another way than 4 + 1, she seemed to struggle with the task, taking a long time to think, and repeatedly saying ”I don't know”. Ester most likely identified the parts 2 + 3 by looking at the whiteboard where the teacher had written the different ways (1 + 4, 4 + 1, 5 + 0, 3 + 2) that had already been discussed.

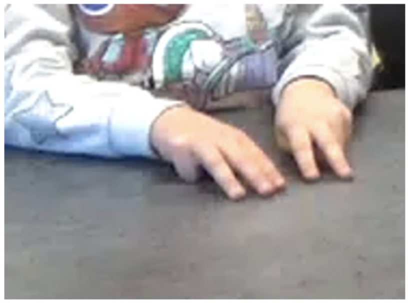

Figure 7. *Ester's way of partitioning 5 into 2 and 3.*

Ester:	I don't know.
Teacher:	Try. [Ester looks at her fingers]
Walter:	I know, I know.
Teacher:	Soon.
Ester:	I don't know.
Teacher:	We'll listen to the last students. Please think some more Ester.
Ester:	[Ester looks at her hands and then at the whiteboard where the teacher had written different ways of partitioning with numbers in triads (e.g. 3 + 2), makes 2 on her left hand, and then 3 with her right hand].

Commutativity

We found that some students showed an understanding of commutativity during the task whereas other students did not change the order of the parts when dealing with numbers higher than 3. When the students were asked to show different ways of partitioning a number, the teacher saw 4 + 1 and 1 + 4 as different ways. For example, in the discussion (Lesson 1b), one student showed 1 and 4 as parts of 5, while another student (Sara) crossed her hands and thus got 4 and 1, showing commutativity. Other students, for instance Bill, did not show awareness of commutativity when showing different parts for the same number when the number was bigger than 3.

Table 4. *Bills's ways of partitioning numbers in Lesson 1b.*

Number	Bill's ways of partitioning
3	1 + 2 ○○○●○ ○●●○○ **2 + 1** ○○●●○ ○●○○○ 0 + 3 ○○○○○ ○●●●○ quickly changes to below **1 + 2** ○○○●○ ○●●○○
4	**2 + 2** ○○●●○ ○●●○○ **3 + 1** ○○●●● ○●○○○
5	5 + 0 ●●●●● ○○○○○ **4 + 1** ●●●●○ ○●○○○
6	5 + 1 ●●●●● ●○○○○ 5 + 4 ●●●●● ○●●●● quickly changes to below 5 + 1 ●●●●● ○●○○○ 5 + 1 ●●●●● ●○○○○ 5 + 1 ●●●●● ○●○○○ 1 + 4 ○○○○● ○●●●● 3 + 4 ○○●●● ○●●●● **2 + 4** ○○○●● ○●●●● **3 + 3** ○○●●● ○●●●○
8	3 + 5 ○○●●● ●●●●●

For example, in the excerpt below, he didn't change the order of the parts from the previous way shown (2 + 4) when it was his turn to change his way of partitioning the number. Instead, he chose "Three and three. Anna-Lena displayed an awareness of commutativity when showing 4 and 2, pointing out that other students showed 2 and 4, saying "No, they have two and four".

Teacher: Viktoria had two and four. And you have the whole [6]... and you also have two and four. How can you change?... If you cannot have two and four. Can you change so you get something else? [Bill shows three fingers on each hand.] Now you changed and then you got...

Bill: Three and three

Teacher: Oh it was, after all, it wasn't like Viktoria's, because she had two and four.

Rosie: It's like mine.

Teacher: And it's like Fredrika's but you [to Bill] changed so you changed from Viktoria's [way of partitioning]. Now let's see what you have over here [turns to Anna-Lena]. Look here, Anna-Lena, you came up with something. How did you get there? You have six [points to the whole, both of the student's hands, and then the student's left hand].

Anna-Lena: Four
Teacher: Four ?the teacher points to the student's right hand?.
Anna-Lena: And two.
Teacher: And two. Oh, no one else had that.
Anna-Lena: No, they have two and four.
Teacher: Right. They have two and four.

Bill may be aware of commutativity, since he changed the order of the parts with numbers lower than 3; however, he did not use commutativity during this session to show different parts of bigger numbers.

To see zero as one part

For several students, it was not obvious to see zero as being one part when partitioning a number. The following excerpt shows how one student, Bill, did not see zero as one part shown by one folded hand. When the teacher asked if there were more ways to partition 3 than 2 and 1, Eve showed 0 and 3.

Teacher: Is there another way [than 2 and 1]?
Eve: Yeah, zero and three [shows middle, index, and thumb on the left hand and folds the right hand].
Teacher: Look, now you are showing both your hands. Then we have three as the whole, and you have partitioned it into three and zero. This is a possible way actually, isn't it?

Bill argued that showing 0 with one hand is not using two hands. He said "That you should show three with both your hands [taps his hands showing one and two fingers]" and continued "That is no hand" when Eve showed 0 with one folded hand.

Bill: [looks at Eve's hands] Two hands, two hands [quiet].
Teacher: (...) This was a really good question from Bill. Did you hear what Bill said? He said that you should use two hands. Tell me what you were thinking.
Bill: That you should show three with both your hands [taps his hands showing one and two fingers].
Teacher: Yes, and what did Eve do? Show what she did. Please show [to Eve].
Eve: This is this one [puts her left hand showing three fingers on the carpet].
Teacher: There is one hand.

Eve: And this is two hands.

Bill: That is no hand.

Eve: Yes, this is one hand [lifts her right folded hand], this is one hand.

Teacher: Keep it there, and how many finger is she showing on that hand?

Bill: Zero.

Teacher: Yes, but she is using that hand. It would be different if she had done this [puts the hand behind her back] three [only shows one hand], but you [Eve] have shown like this [shows 3 on one hand and 0 with the other hand]. The whole three is three and zero.

At the end of the excerpt above, the teacher made a contrast between showing zero as one part, (holding the hand folded beside the other hand) and not showing zero as a part (holding the hand behind her back), in order to point out the difference. Even though the teachers in the analyzed lessons pointed out that zero can be one part, there were students who were still hesitant to use zero as a part when partitioning numbers up to 5.

Discussion and conclusions

Piaget (1941/1965) has argued that the ability to experience part-whole relations of numbers does not appear until students are six or seven years old, while other studies show even younger students are able to do so (e.g. Cheng, 2012; Hunting, 2003). We show that this development is not always easy, even for seven-years-olds. Our study adds to previous knowledge on students' learning about number, number relations, and decomposition of numbers by identifying aspects that may be critical for students' learning of partitioning numbers into part-whole relations using finger patterns (e.g. Sensevy et al., 2015; Hopkins et al., 2022). Based on our findings, we suggest it is critical for student learning to: a) differentiate between finger pattern and finger number, b) differentiate between parts and fingers, c) create parts bigger than 1, d) experience commutativity, and e) see zero as one part. We suggest, as do Sensevy et al. (2015), that it is essential that students become aware that the same number can be represented (with finger patterns) in different ways. For example, students need to be able to vary both the parts and fingers they use (b, e), and vary the number of fingers used in each part (a). Previous research points out the importance of being able to create parts bigger than 1 (c) as fundamental for learning addition and subtraction (Hopkins et al., 2022). Students also need to discern the change (variation) in the order of the parts, while keeping the parts invariant, in order to experience commutativity (d). Zhou and Peverly (2005) argue similarly that young students need to encounter mathematical principles, e.g. commutativity, early in their education.

However, there may be additional critical aspects for other groups of students that we have not identified in our data. It is also important to stress that the identified aspects were not critical for all students in our sample since many of them had most likely already discerned them. Our findings show, as in previous studies, that finger numbers can help students to experience part-whole relations of numbers (Neuman, 1987, 2013; Brissiaud, 1992; Hopkins et al., 2022; Kullberg et al., 2020).

We also identified two students who used a way of partitioning that was not anticipated by the teachers, and which does not fit into the categories presented in the results section above. Although the task involved partitioning a number on two hands, these students used two hands to show one part. For example, Harry partitioned 6 into 4 and 2, by showing 4 with four fingers on his left hand and 2 with his left and right thumbs, saying "Four and two (putting his thumbs together)". We suggest this is vital for students to discern when using fingers to partition numbers. Kullberg and Björklund (2020) showed that when preschool children were solving a missing addend problem (3 + _ = 8), the children who were able to see the missing addend on two hands (2 + 3) were more successful in solving similar tasks one year later than children who solved the task by structuring one part on each hand (3 + 5 = 8).

There are certain limitations to this study that need to be considered. The number of participating students and teachers are rather few. A larger number of participants would have provided a larger data set and a more stable interpretation. It is also possible that analyzing several activities would have contributed to a fuller picture of critical aspects.

In sum, there are several critical aspects that students need to become aware of while learning about part-whole relations of numbers in first grade. To identify critical aspects of experiencing number relations, we looked closely at how students use fingers to show number parts. We advise teachers to notice what fingers their students use when partitioning numbers. We argue that our findings are useful for teachers, since by understanding what is critical for students to experience when learning to partition numbers, teachers know better what to teach and what needs to be brought to their students' attention.

Acknowledgements

This study was funded by the Swedish Institute for Educational Research [Grant number 2018-00038].

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Note

1 In this paper, we differentiate between finger number (e.g. showing 6 as 5 fingers on the left hand and the thumb on the right hand (5 and 1)) and finger patterns (e.g. showing 6 as 3 and 3 or 4 and 2 etc.).

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