Proof-related competences in Nordic mathematics curricula: a cross-cultural comparison

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This study compares how national curricula in Denmark, Finland, Norway, and Sweden address proof-related competencies in school mathematics. Using a refined analytical framework based on previous research, the analysis reveals substantial variation in how and when competencies such as proof, argument, and structure are introduced and developed. Denmark and Finland present a more consistent and explicit integration of proof-related competencies throughout the curriculum, while Norway emphasizes reasoning but avoids the term "proof", and Sweden delays engagement with proof to upper secondary levels. These differences reflect varying curricular priorities and conceptions of mathematical thinking, with implications for international comparative research and curriculum development.

Keywords: mathematical proof; argumentation; school mathematics; cross-cultural comparisons

Proving is an essential part of doing mathematics (Schoenfeld, 2009, p. xii) and both researchers and policy documents recommend that proving be a part of school mathematics in all the years of schooling (e.g., Stylianou et al., 2009; NCTM 2000). However, much of the existing research in this field is limited to a single level of schooling (Stylianou et al., 2009), and a single national context (Reid et al., 2019). Hence, there is a need for more international comparative research on the role of proving across the years of schooling.

International comparative research allows researchers to examine what is assumed in each national context, which gives insight into a country's educational priorities. International comparisons may also

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point to new research directions or suggest new aspects for analysis (Pepin & Haggerty, 2001). Denmark, Finland, Norway and Sweden have all reformed their mathematics curricula in the past decade, and so this presents an opportunity to see how the international recommendation to include proving in school mathematics at all levels is reflected in these four national contexts.

One of the few comparative studies of proving in school curricula is Hemmi et al. (2013), who examined the development of proving and related competences in Estonian, Finnish and Swedish mathematics curricula. In this article I will expand and update their work, contributing not only to the field of comparative research on proof and proving in schools across cultures, but also refining the analytic frame used by Hemmi et al. (2013). I aim to answer the following questions:

- How are proof-related competences described in the national curriculum documents in Denmark, Finland, Norway and Sweden?
- How do these competencies develop through different school levels?
- What differences can be observed between the national curriculum documents?

Competencies

The four Nordic curricula considered here have been influenced by the Danish KOM (Kompetencer og Matematiklæring) project in the early 2000s (Niss & Højgaard, 2019; Niss & Jensen, 2002). This is explicit in the Danish curriculum and reflected in the terminology used in all the documents. In KOM framework (Niss & Jensen, 2002) a holistic mathematical competence is described through eight competencies. These competencies are intended to offer an alternative description of what it means to master mathematics.

> The competencies are closely related – they form a continuum of overlapping clusters – yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint. (Niss, 2003, p. 9)

The eight competencies are grouped into two sets of four:

- Competencies related to asking and answering questions about and by means of mathematics:
    - » Mathematical thinking,
    - » Problem handling,
    - » Modelling,
    - » Reasoning,
- Competencies related to mathematical language and tools:
    - » Representation,
    - » Symbols and formalism,
    - » Communication,
    - » Aids and tools.

Of these, the most obviously proof-related is Reasoning:

> This competency consists of, on the one hand, the ability to follow and assess mathematic reasoning, i.e. a chain of argument put forward by others, in writing or orally, in support of a claim. It is especially about knowing and understanding what a mathematical proof is and how this differs from other forms of mathematical reasoning, e.g. heuristics based on intuition or on special cases, and it is also about understanding how and when mathematical reasoning actually constitutes a proof, and when it does not. (Niss & Højgaard, 2011, p. 60).

The Thinking competency refers to conjectures and generalisations:

> ... this competency comprises being able to recognise, understand and deal with the scope of given mathematical concepts (as well as their limitations) and their roots in different domains; extend the scope of a concept by abstracting some of its properties; understand the implications of generalising results; and be able to generalise such results to larger classes of objects.
>
> This competency also includes being able to distinguish, both passively and actively, between different types of mathematical statements and assertions including ”conditional statements”,

> ”definitions”, ”theorems”, ”phenomenological statements” about single cases, and ”conjectures” based on intuition or experience with special cases. Of particular importance is an understanding of the role played by explicit or implicit ”quantifiers” in mathematical statements, not least when these are combined. (Niss & Højgaard, 2011, p. 53).

Proof, Reasoning and Argumentation

The research literature uses a range of words to describe proof-related competencies, and words such as ”proving” and ”reasoning” do not always refer to the same competencies (Reid & Knipping, 2010; Reid, 2022). Therefore, it is important to anchor any framework to analyse proof-related competencies in established terminology used in the research literature.

Proof

Stylianides (2007) uses ”proving” to refer to ”to describe the activity associated with the search for a proof” (p. 290). He offers the following definition of ”proof”:

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
>
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
>
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community.
>
> (p. 291)

Reasoning and argumentation

As Hemmi et al. (2013) note ”The words (mathematical) reasoning and argumentation are often used in connection to proving, but views on the relationship between argumentation and proof vary in the field of mathematics education." (p. 358). In this section I review recent work clarifying the meanings of the words "reasoning" and "argumentation".

The word "reasoning" is used in a number of ways in the mathematics education literature (Jeannotte & Kieran, 2017, p. 2). Jeannotte and Kieran summarize the literature dealing with mathematical reasoning to clarify the resulting confusion. They divide mathematical reasoning into two aspects: a structural aspect and a process aspect (p. 6). The process aspect is closely related to proof-related competencies, and is divided into three types: processes related to the search for similarities and differences, and processes related to validating, and finally, exemplifying (which is not relevant here).

Processes related to the search for similarities and differences include (pp. 9–11):

- Generalizing: deriving general principles about one set of mathematical objects.
- Conjecturing: a mathematical reasoning process "that, by the search for similarities and differences, infers a narrative about some regularity with a likely or probable epistemic value" (p. 10).
- Identifying a pattern: recognizing repeating structures between mathematical objects.
- Comparing: looking for similarities and differences between mathematical objects. Comparing is a necessary part of identifying a pattern, as objects must be compared for repeated structures and common features to be observed (p. 11).
- Classifying: looking for common features between mathematical objects, and categorising them according to these common features.

Processes related to validating concern changing the epistemic value (e.g., likely, true, probable, false) of a statement (p. 11). They include four processes, each of which is a subset of the previous one:

- Validating: any process that aims to change the epistemic value.
- Justifying: aims to change the epistemic value by seeking grounds (data, warrants or backing in Toulmin's, 1958, terms).

- Proving is specifically focussed on changing the epistemic value from probable to true, and has three characteristics, strongly reminiscent of Stylianides' (2007): accepted statements, deduction, and appropriate realisations (Jeannotte & Kieran, 2017, pp. 12–13).
- Formal proving is restricted to statements drawn from a mathematical theory that would be accepted by an ”expert mathematician” (p. 13) and realisations accepted by ”mathematical communities” (p. 13).

The words ”argumentation” and ”argument” are used with a range of meanings in the mathematics education literature (Reid & Knipping, 2010). Here I will follow Stylianides (2007) in considering argumentation the process of producing arguments, and an argument as ”a connected sequence of assertions for or against a mathematical claim” (p. 291). This meaning of ”argumentation” is similar to Jeannotte and Kieran's (2017) category ”Justifying”.

Prior research

The only study of proving in of proving and related competences in Nordic school curricula is Hemmi et al. (2013)'s study of the Estonian, Finnish and Swedish mathematics curricula. Based on the literature available to them, they derive six categories (see table 1) for an analytical frame of developmental proof and use this frame to identify differences between the curricula. They found that, across all the years of schooling, only the Finnish curriculum addresses all the proof-related competences, while the Estonian curriculum addresses some of them, especially from the lower secondary level. The Estonian curriculum, however, has more explicit goals concerning proof and proving. They found that in the Swedish curriculum proof and related competences are addressed first at upper secondary level, and that prior to that only a few competences are addressed, weakly.

Valenta and Enge (2020) did a similar analysis of the Norwegian curriculum using Hemmi et al.'s (2013) categories, within a theoretical framework based on Jeannotte and Kieran (2017) They found the categories Arg, Invest, Struct and Def at every level and Log in Year 5. However, they note that ”the formulations in the Defn category are quite informal and do not explicitly deal with the understanding of the role of definitions (but rather the use of mathematical concepts)” (p. 15, my translation)

They conclude that:

> the lack of emphasis on proof and proving, the limited focus on logic and the formal aspects of mathematics, and the lack of emphasis on the role of mathematical definitions suggest that the curriculum does not facilitate the development of argumentation into proof. (p. 15, my translation)

Analysing proof-related competences in curricula

Informed by the research literature, especially the work of Styliandes (2007) and Jeannotte & Kieran (2017), I have expanded the six categories in Hemmi et al.'s (2013) analytical frame to better describe the proof-based competencies in the curriculum documents. My revised categories are listed in table 1.

The main changes to Hemmi et al.'s (2013) frame are:

- the division of the Struct category into two parts according to whether the connection involved is related to Stylianides's (2007) "set of accepted statements" (p. 291) or to more general mathematical connections.
- the division of the Log category into references to logic related to the reasoning (mode of argumentation, Stylianides, 2007) involved, and references to precision and clarity, that is, to the way that reasoning is expressed (modes of argument representation, Stylianides, 2007).

Table 1. *Categories used in the analysis.*

Revised category	Hemmi et al. (2013) category	Definition	Reason for revision
Proof	Proof	Explicit mentions of "proof", "proving", or "deductive reasoning".	Not revised.
Argument	Arg	Statements that involve proof or deductive reasoning without mentioning those terms explicitly. Instead, terms such as "argumentation", "mathematical reasoning", "justification", "explanation", "assessment" or "evaluation" might be used.	Not revised.
Grounds	Struct	Statements that refer previously established knowledge that reasoning is based on.	Includes some aspects of Struct, to fit Stylianides (2007) "set of accepted statements"
Structure	Struct	References to understanding mathematical connections and structures.	More precise than Struct in that it refers only to connections and not to the grounds being connected to.
Investigation	Invest	Statements referring to finding patterns, dependencies or causalities, and making generalizations, conjectures or hypotheses.	Not revised.
Definition	Defn	References to understanding mathematical definitions and classifying	Not revised.
Logic	Log	Explicit mentions of "logic" or logic-related terms like "truth-value", "implication", etc.	Log has been split in keeping with Stylianides' (2007) distinction between forms of reasoning and forms of expression.
Language	Log	References to formality, precision and clarity of expression.	

Methods

The documents examined were the most extensive official documents in each country describing the mathematics curriculum. For Norway, this was the Curriculum for Mathematics year 1–10 (Utdanningsdirektoratet, 2019b) which includes only relatively brief descriptions of general competencies as well as goals for each grade. The document for Finland is Grunderna för läroplanen för den grundläggande utbildningen 2014 [Foundations of the basic education curriculum 2014] (Utbildningsstyrelsen, 2014), which includes descriptions of general competencies as well as goals for each grade for all subjects. It is published in Finnish and Swedish, and the two documents are prepared in parallel, so neither is a translation of the other. I have used the Swedish document (Utbildningsstyrelsen, 2014) for this analysis. Only the sections specific to mathematics were consulted. For Denmark, Matematik Faghæfte 2019 [Mathematics Curriculum Guide 2019] (Børne- og Undervisningsministeriet, 2019) was used, which is an extensive document that includes common goals, learning plans and teaching guidance.

In the case of Sweden, the Curriculum for Compulsory School, Preschool Class and School-Age Educare – Lgr22 (Skolverket, 2024) was consulted. The mathematics sections of this document contain only a syllabus with a short introduction. The brevity of this text means that almost no useful data is available in it. For this reason, the Kommentarmaterial till kursplanen i matematik: Grundskolan [Commentary on the Mathematics Syllabus for Compulsory Education] (Skolverket, 2022b), which provides further elaboration of the curriculum, was also used as a source of data for Sweden. In cases where the documents are available in English as well as national languages, these versions were consulted in parallel. The quotations in this article are from the official English version of the documents, if available, or my translations.

Education systems across the Nordic countries have many similarities. Schooling is compulsory and non-selective for the first 9 or 10 years, usually organised into a six- or seven-year primary school (beginning at age 6 or 7) and a three-year lower secondary school. This is followed by upper secondary school in which students specialise in academic and vocational programmes. I have only examined curriculum documents for the primary and lower secondary schools, as the wide variety of different programmes offered in upper secondary schools make comparisons more difficult. The documents are usually arranged in three year-bands: 1–3, 4–6 and 7–9 in Denmark and Sweden, and 1–2, 3–6 and 7–9 in Finland. Norway does not use year-bands in official documents, but Years 8–10, lower secondary school, are often grouped together in practice.

I conducted a qualitative content analysis (Mayring, 2021) of the documents. The categories from Hemmi et al.'s (2013) analytical frame were used for a preliminary deductive analysis, with parallel revision of the categories responding to aspects of the documents that were not well captured by the existing categories. New categories were theoretically grounded in existing literature. I used the categories to code the sections of each country's curriculum documents that clearly addressed proof-related competencies. The unit of analysis was by grade or grade band, according to the organisation of the document. Each category was coded as present, possibly present, or absent, with no account taken for multiple mentions of a category within one unit of analysis (which happened very rarely in any case).

I worked with the original language of the documents, supplemented with English machine translations where no official English translation was available. In the case of the Finnish documents, I used the Swedish version primarily, but also compared this to the Finnish version. I found no examples where the texts were not parallel. After coding the documents, I prepared a summary which was checked by native speakers of each language (both Finnish and Swedish in the case of the Finnish documents).

Findings

Denmark

Table 2 summarises the categories of proof-based competencies most evident in the Danish mathematics handbook (Børne- og Undervisningsministeriet, 2019), for each of the three year-bands.

Table 2. *Categories observed in documents from Denmark.*

Year-band	**1–3**	**4–6**	**7–9**
Categories	Argument	Investigation Argument	Investigation Definition Grounds General Proof

The word ”explanation” occurs several times in the 1–3 year-band. Students should be able to ”give and follow informal mathematical explanations” (p. 11). These explanations seem to belong to the category Argument, as the learning plan also states, ”The teacher ensures that the explanations contain a justification and are not solely descriptions of procedures.” (p. 35).

In the 4–6 year-band the focus is on Investigation. A goal is that students ”can use reasoning to develop and test hypotheses” (p. 13). The learning plan elaborates that they should do this in ”investigative work” (p. 42). The learning plan also states that ”students give short and simple reasoning, e.g. in the form 'If I..., then it must happen that..., because...'” (p. 43) which seems to fall into the category Argument.

The Investigation category continues in the 7–9 year-band. The learning plan states that students should ”use reasoning to develop and evaluate hypotheses” (p. 50). The categories Definition and Grounds are also present, as students should ”distinguish between hypotheses, definitions and theorems” (p. 15) and use these as grounds for reasoning: ”The reasoning is increasingly based on the definitions and theorems that the students have already learned” (p. 50). Proofs are explicitly mentioned for the first time in this year-band, both in the goals, ”The student has knowledge of simple mathematical proofs” (p. 15) and in the learning plan, ”The step-by-step process includes exemplary examples of simple proofs in the teaching” (p. 50).

Finland

Table 3 summarises the categories of proof-based competencies most evident in the Finnish curriculum (Utbildningsstyrelsen, 2014), for each of the three year-bands.

Table 3. *Categories observed in the documents from Finland.*

Year-band	**All**	**1–2**	**3–6**	**7–9**
Categories	Logic Language	Investigation Argument?	Argument Investigation	Language Logic Proof Investigation

In the introduction for each year-band, it is stated ”The mission of teaching mathematics is to develop logical, precise and creative mathematical thinking in students.” (p. 129/235/375). Hence, I consider Logic and Language to be overarching categories.

In the 1–2 year-band, the focus is on the Investigation category:

> Students are offered opportunities to find similarities, differences and patterns. They compare, classify and order, and observe connections between cause and effect. (p. 130).

There is a specific learning goal that refers to ”drawing conclusions” (p. 129), but the method of drawing conclusions is not specified. If it were deductive then this statement would belong to the category Argument, however the focus on patterns suggests that this refers to drawing conclusions inductively or abductively.

In the 3–6 year-band, the learning goals again mention drawing conclusions, but now they should have grounds: ”draw justified conclusions” (p. 236). This suggests inclusion in the Argument category. I have translated the Swedish word ”motiverade” as ”justified” here, though it could be referring to giving reasons in a broader sense. However, the Finnish version (Opetushallitus, 2014) uses the word ”perusteltuja” (p. 235), which comes from ”perustella” (to justify) and is directly linked to ”perusta” meaning a foundation or basis, hence my translation to ”justified”. The focus on investigation present in the 1–2 year-band continues, and is described using similar language (Utbildningsstyrelsen, 2014, p. 236).

In the 7–9 year-band, the overarching categories Logic and Language are first related to specific learning goals. Students should, ”solves tasks that require logical and creative thinking and develop the skills needed for this” (p. 375) and ”express themselves precisely and mathematically” (p. 375). Logical and precise thinking are also mentioned in the opening sentence of the 7–9 year-band section (p. 375).

In the elaboration both Logic and Proof are explicitly mentioned:

> Students practice activities that require logical thinking, such as finding rules and dependencies and presenting them in an accurate manner. ... They strengthen their ability to justify and draw conclusions. ... Students gain insight into the basics of proof and practice determining the truth value of propositions. (p. 376).

Investigation is also suggested by the process of ”finding rules”.

Norway

The Norwegian mathematics describes six core elements. The first five are process competencies and the last includes content such as number, algebra, geometry and probability. There are no year-bands, but competence goals and formative assessment guidance are listed for each year from 2 to 10. Overall, the competencies referred to in the Norwegian curriculum are consistently Argument, Investigation and Structure in every year.

The core element that is most directly proof-related is Reasoning and Argumentation:

> Reasoning in mathematics means the ability to follow, assess and understand mathematical chains of thought. It means that the pupils shall understand that mathematical rules and results are not random, but have clear and logical grounds. The pupils shall formulate their own reasoning to understand and to solve problems. Argumentation in mathematics means that the pupils give grounds for their methods, reasoning and solutions, and prove that these are valid. (Utdanningsdirektoratet, 2019b, p. 3)

The reasoning referred to in the *Reasoning and Argumentation* core element fits very well in the Argument category. It seems to be deductive, as it can show that mathematical rules have ”logical grounds” (p. 3). This phrase is a translation of the Nynorsk ”grunngivingar” (Utdanningsdirektoratet, 2019a, p. 3), literally ”to give grounds”, suggesting deductive justification.

I interpret the word ”prove” (”beviser” in Norwegian) in the phrase ”prove that these are valid” in the everyday sense of confirming or documenting something. I do not believe ”beviser” here means ”provide a mathematical proof”. If this is what is meant here then ”prove that reasoning is valid” seems circular. Hence I do not code this in the category ”Proof”. This is the only use of ”prove” or ”proof” in the curriculum.

Other proof-related competencies can be found in other core elements. The category of Investigation is central to the core element Exploration and Problem solving: ”Exploration in mathematics means searching for patterns, finding relationships and discussing one’s way to a shared understanding.” (Utdanningsdirektoratet, 2019b, p. 2).

The categories Investigation, Structure, and Language are involved in the core element *Abstraction and Generalisation*:

> Generalisation in mathematics refers to the pupils finding relationships and structures, without being presented a finished solution. This means that the pupils can explore numbers, calculations and figures to find relationships, and then formalise by using algebra and well-reasoned representations. (Utdanningsdirektoratet, 2019b, p. 3).

The term "well-reasoned" in this core element does not refer to anything proof-related. The Nynorsk original is "formålstenlege" which would be better translated as "appropriate for the goal".

The development of Argument over the years of schooling is visible in the formative assessment guidelines for each year. In every year there is sentence stating that students should demonstrate their competence by arguing for their own solutions. In Year 2, this sentence is "The pupils also demonstrate and develop competence in mathematics by having a sense of wonder, asking mathematical questions and explaining and arguing for their own solutions." (Utdanningsdirektoratet, 2019b, p. 6). Similar sentences occur in Years 3 and 4. In Year 5 the corresponding sentence is "They also demonstrate and develop their competence in mathematics when they reason about and argue for solutions and mathematical relationships." (Utdanningsdirektoratet, 2019b, p. 10). The word "explaining" has been replaced with "reasoning about". This change persists in Years 6–10. I interpret this change as the use of increasingly technical terms, "reason" versus "explain", to describe argumentation, rather than a shift in focus.

Structure is an important category of proof-related competencies in the Norwegian curriculum, at all year levels. "Relationships" are mentioned in the goals and guidelines for every year from Year 3 up, and "structures" and "patterns" are also mentioned several times.

The other categories are present rarely or not at all. There is one occurrence of the phrase "logical reasoning", in a goal for Year 5, and one explicit mention of minimum definitions in a goal for Year 6.

Sweden

Table 4 summarises the categories of proof-based competencies most evident in the Swedish commentary on the curriculum (Skolverket, 2022b), for each of the three year-bands.

Table 4. *Categories observed in the documents from Sweden.*

Year-band	**All**	**1–3**	**4–6**	**7–9**
Categories	Logic Language	Argument?	Grounds Structure Argument Logic?	Argument Structure Grounds

In the introduction to the curriculum, an aim is that ”The teaching shall help pupils to develop the ability to argue logically and make mathematical inferences” (Skolverket, 2024, p. 56). The phrase ”make mathematical inferences” is the official translation of ”föra matematiska resonemang” (Skolverket, 2022a, p. 54), which could be read as more broadly than making inferences. The commentary elaborates, under the heading Föra resonemang (conduct reasoning):

> Teaching should contribute to the students developing the ability to argue logically and conduct mathematical reasoning. Carrying out mathematical reasoning includes, among other things, reasoning to solutions using both informal and, with increasing age, increasingly formal arguments. It can, for example, be about justifying the choice of calculation method or explaining why a certain formula describes a given relationship. (Skolverket, 2022b, p.7)

This refers to Logic (”argue logically”). Language is also important here (”using both informal and, with increasing age, increasingly formal arguments”).

There is only one proof-related concept in the core content listed for the year-bands in the curriculum, under Geometry for Years 7–9: ”Geometric theorems and formulas, and arguments for their validity.” (Skolverket, 2024, p. 61).

The commentary (Skolverket, 2022b) elaborates:

> Geometric theorems and formulas, as well as argumentation for their validity, are content areas that are only explicitly stated for grades 7–9. This may, for example, involve encountering proof-like arguments for the sum of the angles of a triangle, or making the formula for the area of a circle seem plausible. The content means that students are given the opportunity to argue for the validity of formulas, demonstrate relationships between fundamental geometric concepts, and see how properties of geometric objects can be derived from knowledge of other concepts. Students should also be given the opportunity to reason about how, in mathematics,

> one determines whether something is true or not. In this way, the curriculum lays the foundation for students' understanding of the meaning of the concepts of theorem and proof in future studies. (Skolverket, 2022b, p. 22)

Here Proof is explicitly mentioned, but ”proof-like arguments” are better classified as Argument (as they are only proof-like) and proof is a topic for ”future studies”. Structure and Grounds are also present here.

It is difficult to analyse how the proof-related competencies in the Swedish curriculum change through the three year-bands. In the assessment criteria for Year 3, the criterion related to proof is ”The pupil makes and follows the reasoning behind mathematical inferences by asking and answering questions generally related to the topic.” (Skolverket, 2024, p. 63). The commentary notes that ”The wording [in the assessment criteria] for Year 3 has less explicit formulations about mathematical argumentation compared to later grades” (Skolverket, 2022b, p. 36), which may indicate that it is less important. For Years 6 and 9 the commentary indicates that assessment should be based on ”how well-founded, defensible and sufficient the arguments are” (p. 36) as well as ”how logically coherently the student justifies positions and conclusions and how well the student legitimizes solutions to problems through reasoning” (p. 36). This indicates the importance of Grounds, as well as Structure, Argument and possibly Logic.

Comparison

Table 4 summarises the main differences in the categories of proof-based competencies in the documents from Denmark, Finland, Norway and Sweden. When a category is mentioned in relation to a specific grade or grade band that is indicated. When it occurs in a general text applying to all grades, ”All” is indicated, although perhaps only some grades are intended.

Table 4. *Differences observed in the documents.*

	Denmark	**Finland**	**Norway**	**Sweden**
Proof	7-9	7-9		
Logic		All, 7-9	5?	All
Argument	1-3, 4-6	1-2? 3-6	All	7-9
Grounds	7-9			7-9, 4-6?
Structure		3-6	All	7-9, 4-6

In category Proof, the curricula of Denmark and Finland explicitly refer to ”proof”, but only in the 7–9 year-band. In the case of Sweden, ”proof” is used only in the context of Geometry and the reference is to proof-like argumentation and the use of proof in future years, not to proof in that year-band. The curriculum of Norway uses the word ”prove” but in the everyday sense of ”verify”, not to refer to mathematical proof.

Logic is mentioned in the curricula of Finland, Norway and Sweden. In Finland, developing logical and precise thinking are listed as the first task of mathematics teaching, in a specific learning goal for the 7–9 year-band, and in the description of Mathematical Thinking at that level. The Norwegian and Swedish references to logic are much weaker. In Sweden, ”logic” is mentioned in the introduction to the curriculum: ”The teaching shall help pupils to develop the ability to argue logically” (Skolverket, 2024, p. 56), but then only once more, in the assessment criteria for Years 7–9, where the teacher should assess how ”logically coherent” the students' arguments are. The Norwegian reference is limited to one mention of solving equations using ”logical reasoning” in Year 5.

The category Argument is present in the curricula of all four countries. In Denmark it occurs in the 1–3 and 4–6 year-bands, although not clearly so. In Finland it is also present in the first two year-bands. Note that these two countries clearly refer to Proof in the year 7–9 year-band. This suggests a shift in describing proving from implicit descriptions in early years to explicit descriptions in later years. In Norway, Argument occurs in the core element Reasoning and Argumentation and so should apply to all school levels. In Sweden, Argument occurs only in the 7–9 year-band, especially in connection with geometry.

Recall that Stylianides's (2007) definition of ”proof” refers to use of a ”set of accepted statements” (p. 291). This relates to the categories of Grounds and Structure. The Denmark and Finland documents have few examples of these categories. Grounds occurs only in the 7–9 year-band in Denmark. Structure is referenced only in the 3–6 year-band in Finland. In Norway, Structure is connected to the core element Abstraction and generalisation, and also in the goals and guidelines for every year from Year 3 up, which refer to ”relationships”, ”structures” and ”patterns”. In Sweden, Grounds and Structure occur in the commentary for the 4–6 and 7–9 year-bands, especially in connection with geometry, and the assessment of arguments and reasoning, which should be ”well-founded” (Skolverket, 2022b, p. 36).

Discussion

This discussion focusses on the third research question:

- What differences can be observed between the national curriculum documents?

Proof-related competences are included in the national curriculum documents of all four countries. Overall, there are more proof-related competences addressed in the curricula of Denmark and Finland than in Norway and especially Sweden. The curricula of Denmark and Finland are more explicit in presenting proof-related competencies throughout the curriculum, while Norway emphasizes Argument, but avoids the term ”proof”. Sweden has the least explicit discussion of proof-related competencies.

These differences can also be seen in the development of proof-related competences through the different school levels. In Denmark and Finland, Argument is included in the first six years, and ”proof” is explicitly referred to in the 7–9 year-band. In Norway, Argument is included in all years, but ”proof” is not explicitly referred to. In Sweden, Argument begins in the 7–9 year-band and the curriculum explicitly says that ”proof” is a topic for later years. Overall, Proof comes later than Argument, but Sweden delays both the longest, and Norway, while introducing Argument early, delays Proof.

It is possible to compare my results for Finland and Sweden to those of Hemmi et al. (2013). Both countries have made curriculum changes, but the place of proof-based competences remains the same. In Finland proof-based competences are addressed from the beginning of schooling, and proof and logic are explicitly included, at least in Years 7–9. In Sweden fewer proof-based competences are addressed, and only Investigation seems to be expected in Years 1–3. Proof is explicitly excluded, as appropriate only for later years.

One difference arising from my revision of Hemmi et al.'s (2013) framework can be seen by comparing my results to those of Valenta and Enge (2020) who analysed the current Norwegian curriculum using Hemmi et al.'s (2013) categories. The main difference is the category Def/Definition. Valenta and Enge observe Def at every level, while I see Definition only in Year 6. Valenta and Enge ”considered that explaining a concept may involve providing a (possibly informal) definition, and we have therefore included such formulations in the Defn category” (p. 10, my translation). I used a stricter interpretation of Definition, which I believe is more in line with Hemmi et al. (2013).

Limitations

The descriptions of the ”Reasoning” and ”Mathematical Thinking” competencies in Niss and Højgaard (2011), and the ”Processes related to the search for similarities and differences” described by Jeannotte and Kieran (2017) suggest finer divisions of the categories used here. Proof and Argument could be divided to capture the distinction between understanding a proof or argument, producing a proof or argument and understanding what constitutes a mathematical proof. These distinctions have been important in mathematics education research (Stylianides et al., 2024). Investigation could be divided to reflect five related processes described by Jeannotte and Kieran (2017): Generalising, Conjecturing, Identifying a pattern, Comparing and Classifying. Generalising and Conjecturing are also distinguished in Niss and Højgaard's (2011) ”Mathematical Thinking” competency. These finer distinctions were not useful in the present analysis, but they might be relevant, for example, for textbook analysis, where a larger data set is involved.

It should be noted that the quotations from the documents have been presented here in English translation, and that not all relevant passages have been included due to space limitations. This limits readers' opportunity to judge the interpretation for themselves No translation can exactly capture the original, though I have done my best to represent the original meaning and have checked my interpretations with native speakers.

Conclusion

This study provides evidence that the framework created by Hemmi et al. (2013) allows one to make useful distinctions between school curricula. My incorporation of additional theoretical elements expands what can be observed. Revealing the differences between school curricula can reveal different educational priorities, and allow policy makers to make more conscious and explicit choices about what aspects of mathematics to emphasise. The same framework could be used for textbook analysis, to reveal different priorities of authors and publishers, and it could also be adapted to comparison of classroom lessons, to give insight into teachers' priorities. Such research would also allow analysis of the ways in which the intended curriculum is reflected in classroom practices.

This study also shows that the differences between Finland and Sweden that Hemmi et al. (2013) observed have persisted in spite of curriculum changes in both countries, that took place in the context of international recommendations that proving be a part of school mathematics in all the years of schooling. The Nordic countries share many educational aims and draw on many of the same traditions, so it is surprising that there seem to be stable differences in national curricula that resist conformity to neighbours' priorities.

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