Proof as an explanation of dynamic geometry generated conjectures – task design in a toolbox puzzle approach

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In the frame of a design-based research project, this paper presents analysis of Danish grade 8 students working together to prove conjectures, which they formulated based on guided explorations in a dynamic geometry environment. A systematic account of the rationale and hypothesis behind the task design is described in the form of objectives, hypotheses, and choices, which are then evaluated in light of the analysis of data. The case indicates that the designed task can bridge a connection between conjecturing activities in dynamic geometry environments and deductive reasoning. The students manage to explain theoretically, what is initially empirically evident for them in their exploration in the dynamic geometry environment. The proving activity seems to make sense for the students, as a way of explaining "why" the conjecture is true. The Theory of Semiotic Mediation frames the design of the study and the data analysis.

An ongoing issue in the mathematics education research field concerns the role of dynamic geometry environments (DGE hereinafter) in relation to proof. Several studies highlight the potentials of DGE in relation to development of mathematical reasoning, abilities in generalization and in conjecturing (e.g. Arzarello et al., 2002; Laborde, 2001; Leung, 2015; Baccaglini-Frank & Mariotti, 2010; Edwards et al., 2014). However, it is not clear whether such activities in DGE can support students' development of abilities in deductive argumentation. Some studies indicate that the empirical nature of the DGE investigations may impede the progression of deductive reasoning (e.g. Marrades & Gutiérrez, 2000; Connor et al., 2007). That is to say, once the students have explored a construction in the DGE and discovered some relationship, they may become so convinced by the empirical experience that it does not make sense

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for them to prove (again) what they "know". However, other researchers suggest that students' explorative work in DGE does not have to prevent development of deductive reasoning (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). Seemingly, the didactic design surrounding the work in the DGE and the role of the teacher is of utmost importance (e.g. Mariotti, 2012). De Villiers (2007) argues against a common method, which is for the teacher to devalue the result of the students' empirical investigation as a means of motivating students to undertake theoretical validation. Instead, he suggests highlighting the role of proof as an explanation. The teacher may turn the theoretical validation into a meaningful activity for the students as a challenge to explain "why" their DGE investigations are true (de Villiers, 2007). Trocki (2014) suggests that motivating the students to theoretically justify their empirical explorations may also be incorporated into the task design itself.

In light of the ongoing discussion in the field on the role of DGE in conjecturing and proof, the following research question arises:

How can students' conjecturing activities in DGE be combined with theoretical validation, to make theoretical validation a meaningful activity for the students?

The research question is investigated as a part of a larger design-based research project (Højsted, 2021), in which the overarching mathematical aim is to utilize potentials of DGE in order to support students' development of mathematical reasoning. The specific designed task that is reported upon in this paper aims at bridging a connection between students' conjecturing activities in the popular DGE software, GeoGebra, and the subsequent proving of those conjectures. However, diverging understandings exist regarding the meaning of the notion of proof in a teaching and learning context (Arzarello et al. 2007; Mariotti, 2012; Balacheff, 2008). Therefore, it is pertinent to address what is implied by the notion of proof in the context of school mathematics, both in the research field and in this paper.

Proof as a process

Mariotti (2012) elaborates on different understandings of proof in school context and unfolds two extremes; 1) proof as the product of theoretical validation of already stated theorems, and 2) proof as the product of a proving process, which includes exploration and conjecturing as well as proving conjectures. Sinclair and Robutti (2013) state that the view on proof in the context of school mathematics has largely shifted to comprise proof as a process, and that this may in part be attributed to the facilitation of experimentation provided by digital technologies. In alignment with this view on proof as a process, NCTM (2008) puts forward a broad meaning of reasoning and proof using a *reasoning and proof cycle*, which consists of *exploration* of a mathematical problem or context, making a *conjecture* about the problem/context, and finally putting forward *justification* for the conjecture.

In the next section, pertinent notions from the *Theory of semiotic mediation* are introduced, in which the study is theoretically anchored. Afterwards, the method and educational context of the study is explained, followed by an elaboration of the task design principles. Then empirical data is analysed closely, coming from two groups of students working together on the task. The results are discussed concerning the specific cases, but also referring to results coming from other groups and to research aims going forward. Finally, a conclusion is given.

Theoretical framework

The *Theory of semiotic mediation* (TSM hereafter) considers students' formation of meanings as they use an artefact to solve a task, and on the important role of the teacher to support students' meaning formation in a way that is coherent with mathematical discourse (Bartolini-Bussi & Mariotti, 2008). According to the TSM, when students use an artefact to accomplish a given task, they develop initial *personal meanings*, which are evoked in relation to the artefact-based activity. In a school context, the initial personal meanings will typically differ (to a varying degree) to the *mathematical meanings* an expert mathematician (the mathematics teacher) would recognize. However, through her/his pivotal role in the classroom, the teacher can intervene and try to support students' evolution from personal meanings into mathematical meanings that are coherent with the discourse of mathematicians. Bartolini-Bussi and Mariotti (2008, p. 754) introduce the notion *semiotic potential* of an artefact to describe the dichotomy of personal meanings and mathematical meanings that an individual may generate as the artefact is used to solve particular tasks. If the teacher (or a researcher) is aware of this semiotic potential of an artefact, she/he can design didactic sequences involving tasks that exploit the semiotic potential of the artefact in order to promote specific mathematical learning.

When students work on the artefact-based tasks, *signs* will emerge in the form of students' verbal utterances, gestures (e.g. movement of hands, face or other body parts), written products and DGE interactions on the screen. The signs can be interpreted as the physical manifestations of students' meanings related to the artefact-based activity. As mentioned above, the meanings can, to a varying extent, be coherent with the mathematical meanings predetermined as the aim of the task. Taking into account the emerging signs, the expert teacher can act as a mediator trying to encourage the students' evolution of mathematical meanings. The mediation takes place in the teacher/student interaction for which the classroom discussions play a central role. If the teacher is conscious of their mediating role and intentionally employs the artifact to serve this purpose, then it becomes a *tool of semiotic mediation* (Bartolini-Bussi & Mariotti, 2008, p. 754). The TSM framework also models the teaching and learning process according to a *didactic cycle* comprising three types of activities.

1. Students working on artefact-based activities to solve tasks, which are designed to foster production of specific signs.

2. Students writing to reflect on the artefact-centred activity.

3. Classroom discussions in which the teacher mediation supports the evolution of meanings coherent with the aim of the didactic intervention. (Bartolini-Bussi & Mariotti, 2008; Mariotti & Maracci, 2011).

Originating from a Vygotskian (1978/1930) perspective, the TSM considers cognitive development as a process of internalization. The internalization process has two essential aspects "it is essentially social; it is directed by semiotic processes. In fact, as a consequence of its social nature, external process has a communication dimension involving production and interpretation of signs." (Bartolini-Bussi & Mariotti, 2008, p. 750). Therefore, analysing students' use of signs (gestures, verbal utterances etc.) in social activities (e.g. working in pairs on a task), can shed light on their internalization process and unveil the evolution of students' meaning. The evolution of meanings can be emphasized by recognizing specific semiotic chains, e.g. chains of relations of signification (Bartolini-Bussi & Mariotti, 2008).

From a research point of view, the analysis of students' internalization process can be used to investigate to what extent the designed tasks function as anticipated. That is, to see if the unfolding of the semiotic potential of the artefact corresponds to the a priori analysis embedded in the task design process (see the elaboration of design principles in *Task design principles*, and specific signs which will be recognized in the analysis to be coherent with the intended task design principles in *Signs, hypothesis and choice delimitation in the analysis*).

Method and educational context

The research project is anchored in the frame of design-based research methodology (Bakker & van Eerde, 2015), which is characterized by the dual objective of developing educational practice, as well as developing theory about the teaching/learning process of that practice, and therefore it is "claimed to have the potential to bridge the gap between educational practice and theory" (Bakker & van Eerde, 2015, p. 2). Based on analysis of DGE literature, a hypothetical learning trajectory was proposed (see more in Højsted, 2020a), leading to the development of a didactic sequence that included 15 tasks. This a-priori work is in the frame of TSM considered to be an analysis of the semiotic potential of DGE. The sequence design was also influenced by results from a survey (2020b). The didactic sequence was tested and redesigned in three design cycles in three different schools that each lasted approximately three weeks (14–16 lessons). The data presented in this paper is from the second and third design cycle. To investigate the research question in this paper, a "toolbox puzzle" task was designed with the aim of supporting the students to first formulate conjectures based on guided investigations in GeoGebra, and then to undertake theoretical validation of the conjectures (task design is further elaborated in *Task design principles*).

In the experiment, the students were working in pairs using one computer, and handed a printed task booklet containing the designed tasks. The lessons were organized according to the didactic cycle in the TSM – i.e. several iterations of 1. Artefact-based activities, 2. Student production of written products, 3. Classroom discussions.

Data from each design cycle was acquired in the form of screencast recordings of the students' work in GeoGebra; external video of certain groups (chosen in collaboration with the teacher to comprise a spectrum of high-low achieving students); and written reports that were collected from the students.

In this paper, data is analysed from two pairs of students, in order to investigate to what extent the toolbox puzzle design supports them in proving their conjectures and if the activity seems meaningful to them. Some results coming from other groups is also mentioned in the conclusion.

Data is analysed by analysing the unfolding of the semiotic potential of the designed artefact-based activity. I.e., by interpreting signs (video, transcripts, screencast, written work) produced by the students as they work on the designed task. According to the TSM, these signs can shed light on the students' internalisation process, highlighting the evolution of meanings attached to the artefact-based activity. On the basis of this interpretation, an evaluation may be offered of the successfulness of the task design principles in promoting the mathematical meanings that are coherent with the educational aim of the task. The analytical focus in this paper is particularly on students' emerging signs indicating evidence of the effectiveness of the design in relation to supporting the transition from conjectures to proof (task design hypothesis 2/3 and choice 5 elaborated in the next section).

The educational context of the study is situated in two 8th grade (age 13–14) mathematics classrooms in Denmark. The students had some previous experience using the geometry part of GeoGebra, which is common in Denmark, since ability in relation to dynamic geometry programs are highlighted in the curriculum *mathematics common aims* already from grade 3 (BUVM, 2019). However, the students had *no* experience related to theoretical validation of conjectures or theorems, which is not surprising since it is almost non-existent in lower secondary school in Denmark, which is evident at curriculum level, in textbooks and in practice.

Task design principles

The task design principles are broken down into three interrelated levels; *objectives*, *hypotheses* and *choices*, which offers a systematic account of the design process that is coherent with the predictive and advisory nature of design-based research (Bakker & van Eerde, 2015) (see also Højsted & Mariotti, 2023). At the general level, there are objectives that describe the educational aim of the task design, i.e. what is the intended learning outcome for the students. Next, there are hypotheses concerning activities and types of tasks that may foster the specific student learning. Finally, choices are made at the micro level of design, which for example includes decisions concerning requests of specific student activity; order of requests; formulations in the task; and choice of figures. In order to assure alignment, the choices must be coherent with the hypothesis, and furthermore, the hypothesis must be coherent with the objective.

Objectives

The learning *objectives* concern developing the foundation of students' ability in relation to proof as a process, i.e. their ability in relation to exploration, conjecturing and deductive reasoning. More specifically, that they are able to investigate figures using a DGE in order to make conjectures about the figures, after which they are able to verify the conjecture by means of an inferential argument.

Hypotheses

Four *hypotheses* are submitted in relation to activities to foster the intended student learning. The hypotheses are related to the semiotic potential of a DGE to highlight dependencies between geometrical properties of a constructed figure (Højsted & Mariotti, 2023; Leung et al., 2013; Mariotti, 2014); to the educational value of approaching proof as an explanation (de Villiers, 2007); to the semiotic potential of a paper based "toolbox" containing theorems and diagrams to support students' deductive reasoning; and to the process of semiotic mediation.

Hypothesis (1) is partly described previously:

> Since any constructed figure behaves according to the geometrical relationships defined by its construction procedure, students acting on a figure produced by a construction command can observe the invariance of a property or the invariance of a relationship between properties (Mariotti, 2014), and the perceived invariants can be related to the construction process. (Højsted & Mariotti, 2023, p. 8)

The observed invariants can function as a starting point leading to students' formulation of conjectures about the construction. However, it seems necessary to develop specific prompts (e.g. questions, visual ques) that guide the students' attention onto the desired properties of the construction, as well as asking the students to explain unexpected observations, in order for active reflection to occur (Højsted & Mariotti, 2023).

Hypothesis (2): Students may not see the value of theoretically validating what is already empirically evident for them in the DGE investigation (e.g. Marrades & Gutiérrez, 2000; Connor et al., 2007), however, in alignment with de Villiers (2007) suggestion, theoretical validation might be turned into a meaningful activity for the students, when it is presented as a challenge to explain why their conjecture is true.

Hypothesis (3) concerns the functioning of the toolbox. Since the students have no previous experience in developing an inferential argument, they are provided with a sheet of paper (the toolbox) that contains axioms and theorems to be used in their argumentation, as well as a support figure, which may highlight certain properties to guide the students in the argumentation. The hypothesis is that the toolbox activity will support the students to develop the idea that mathematical argumentation comprises developing chains of reasoning based on previously established truths. A learning trajectory can be envisaged that ultimately leads to an empty toolbox where the students can use any established truths that they find useable.

Hypothesis (4) concerns the process of semiotic mediation, which is related to the student activity described in hypothesis 1–3. As modelled in TSM, these specific semiotic activities (working on the task: observing invariants, formulating conjectures, writing, explaining and discussing what they observe, developing an inferential argument) are expected to foster the emergence of signs with underlying meanings (students' interpretation of the phenomena). Our fourth hypothesis submits that through the *mediation of the teacher*, particularly in classroom discussions, the students' interpretation of DGE phenomena can be linked to a geometrical interpretation; that theoretical validation can be interpreted as an explanation and therefore meaningful; and that mathematical argumentation is interpreted to consist of chains of reasoning based on established truths.

Choices

Five *choices* constitute our considerations at the micro level of task design. Four choices have previously been elaborated (see Højsted & Mariotti, 2023) and are reiterated in table 1.

Choices (3 and 4) regard a task heuristic known as Prediction-Observation-Explanation (White & Gunstone, 2014, p. 44–65), which concerns three types of requests for the students: The students are requested to consider an event or action, and to predict the result of that event/action as well as to justify their prediction. Afterwards, they are requested to observe what happens and to explain their observation, finally, they are requested to resolve any differences between what they predicted and what was actually observed.

Signs, hypothesis and choice delimitation in the analysis

The objectives, the four hypothesis and the five choices are integral aspects of the task design, and they are presented above to give the reader a full and coherent account of the task design principles. However, the analysis will primarily concern hypothesis 2 and 3, which are directly linked to the research problem that is investigated in this paper. Choices 1–5 will be addressed, however, choice 5 is the most relevant to be evaluated and referred to in the analysis because of the specific focus on explaining the conjecture using the proof sheet.

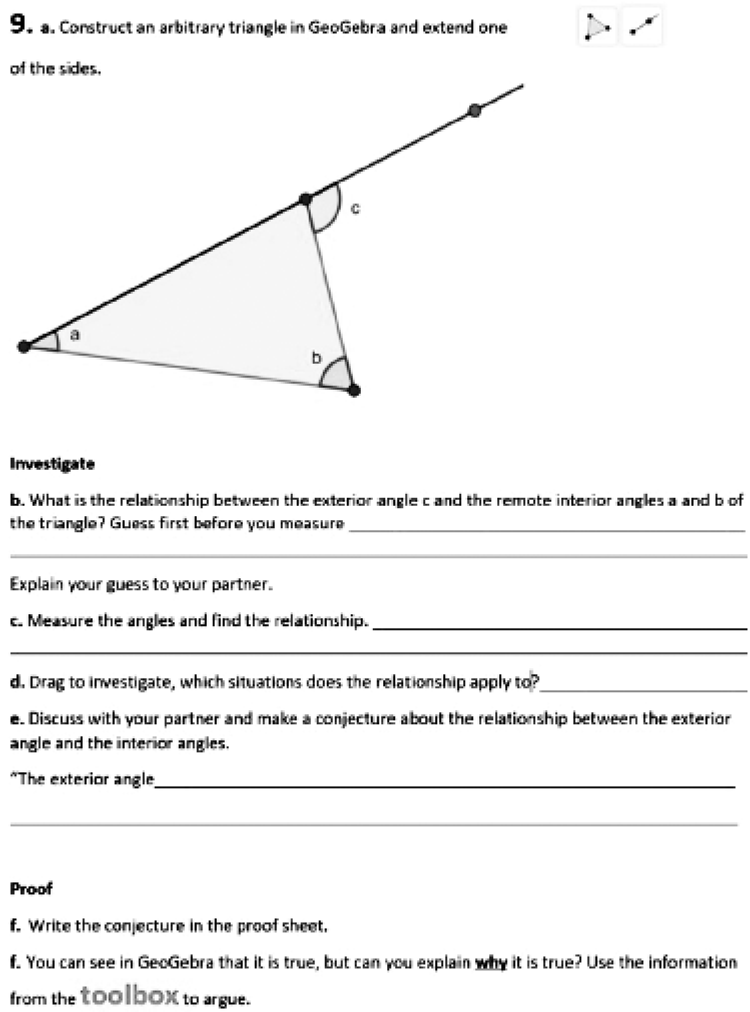
In the analysis, students' production of explanations in which they use inferential arguments based on the toolbox (solving the puzzle) are acknowledged as signs that the students find this theoretical validation meaningful.

Table 1. *Choices of task design, partly adapted from Højsted and Mariotti (2023)*

Choice	Rationale
(1) A construction is proposed that contains certain dependencies, leading to a clearly recognizable invariant.	This choice is related to our aim of exploiting the semiotic potential of dragging in DGE to reveal invariance of properties.
(2) The students are requested to produce the construction with guidance.	The choice reflects that the goal is to foster awareness of properties in the construction. Therefore, the students perform the commands themselves and in so doing are expected to reflect on which properties they are supposed to induce in the construction. In addition, they may interpret the behaviour of the construction after dragging, as a consequence of their construction method, and are therefore expected to reflect on the possible consequences of the construction steps. Some guidance is given in the form of accompanying pictures of commands, which may be useful to complete the construction, as well as a picture of the required construction (see example in figure 1).
(3) The students are encouraged to predict, before they drag objects, what will happen on the screen when they drag certain points, and to justify their prediction to the co-student they are working with.	Asking the students to predict the properties of the diagram before they drag, directs their reflections towards properties of the construction (the general objective) and may give rise to conflict, if what they observe does not coincide with their prediction. The conflict can provoke intellectual curiosity (Laborde, 2003). Encouraging students to justify their prediction serves two aims; firstly, it supports the development of a mathematical attitude to look for a reason – to justify the conjecture. Secondly, it supports the production of reasons that can develop into mathematical reasons. Both these types of support concur with the aim of the students becoming able to justify mathematical claims to others, which is a characteristic of mathematical reasoning.
(4) The students are encouraged to drag certain points and to describe what happens.	This choice is added so that the students can confirm the expected outcome or wonder why it did not go as expected and try to figure out why. Again, with the goal of students becoming aware of the relationship between the properties induced by the construction and the properties that appear invariant by dragging."
(5) The students are requested to write their conjecture in a proof sheet and to explain why the conjecture is true using the statements, axioms and support figure provided in the toolbox.	This choice comprises a semiotic dimension, explicitly requesting that the students produce signs in the form of a written product as they explain why the conjecture is true (de Villiers, 2007). The toolbox serves the purpose of supporting the students in developing an inferential argument, since they have no previous experience with this. Looking for the explanation becomes solving the puzzle, using the pieces that are in the toolbox. The idea is then, that after several such tasks, the toolbox can be empty.

The tasks in the didactic sequence

The initial tasks in the sequence were designed to highlight the theoretical properties of figures, and how they are mediated by DGE in the form of invariants, for example by requesting the students to construct robust figures in "construction tasks" (Mariotti, 2012). These initial tasks adhered to task design choices 1–4. In subsequent tasks, the students were engaged in constructing and investigating the constructions in order to make conjectures and then they were asked to explain why their conjectures were true adhering to task design choices 1–5. An example of such task is task no.9 in the didactic sequence, which is reported on in this paper (figure 1). It consisted of an initial construction part, followed by questions (predict-observe-explain) to guide the students to discover and



9. **a.** Construct an arbitrary triangle in GeoGebra and extend one of the sides.

Investigate

b. What is the relationship between the exterior angle c and the remote interior angles a and b of the triangle? Guess first before you measure ______

Explain your guess to your partner.

c. Measure the angles and find the relationship. ______

d. Drag to investigate, which situations does the relationship apply to? ______

e. Discuss with your partner and make a conjecture about the relationship between the exterior angle and the interior angles.

"The exterior angle ______

Proof

f. Write the conjecture in the proof sheet.

f. You can see in GeoGebra that it is true, but can you explain **why** it is true? Use the information from the toolbox to argue.

Figure 1. *English translation of task no. 9 – exterior angle and remote interior angles of a triangle*

make a conjecture about the relationship of an exterior angle of a triangle with its remote interior angles.

Afterwards, the students were encouraged to explain/prove the conjecture in a proof sheet (on the left in figure 2), using a toolbox (on the right in figure 2), which contains a support figure as well as information (angle over a line is 180°, and the angle sum of a triangle is 180°) to be used in the argumentation.

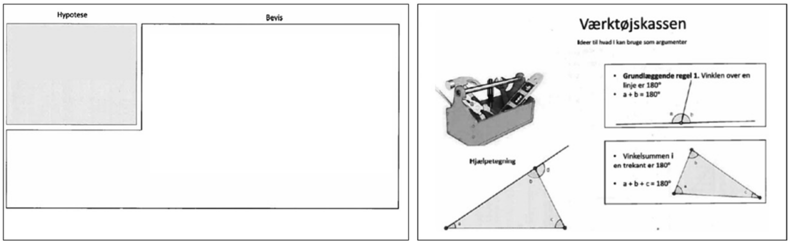


Figure 2. *The proof sheet and toolbox*

The unfolding of the semiotic potential and ensuing analysis

In this section, students' emerging signs are interpreted as two pairs of students work on task no. 9. The data presented from the first group concerns the whole task solution (i.e. 9a–f), in order to address and analyse each choice (1–5) from the task design principles. This is followed by a shorter presentation and analysis from the second group focusing only on the theoretical validation of their conjecture (9.f), which is the main focus in this paper. In the presentation of data, brackets are used when information is added and *italics* are used when non-verbal signs (actions/gestures) are described.

The case of Ole *and* Lis

The teacher describes Lis as a high achiever in mathematics, while Ole is described as a medium achiever in mathematics.

We enter their conversation just as Ole has finished constructing the triangle (task 9.a) and measured the angles marked on the task figure using GeoGebra's angle measuring command.

Lis predicts that the exterior angle equals the sum of the remote interior angles (task 9.b). To check if it is correct, they take out their calculator.

a258 Lis And there's a little calculator here. So... I then had the idea that a + b is equal to c. So it is F + H = I

(angles a, b and c from the figure in the task booklet correspond to angles at points F, H and I in the students GeoGebra figure)

[...]

a268 Ole 64.62 plus 57.45 [O*le reads the measures found with* G*eo*G*ebra*]

a269 Lis It makes 107, yes, so our thing (conjecture) was true.

a270 Ole Absolutely

While Lis considers how to proceed with the proof, Ole's is preoccupied with the GeoGebra construction still, and is dragging the point at angle b at first in a random manner, seemingly without any clear purpose, however he then proceeds to drag in order for angle a to become a right angle.

a290 Ole What, its lying! That is not true. 90 plus 27.57, it is impossible to add up to 117.58.

a291 Lis No, but it gives almost the same thing.

a292 Ole 0.1 off. They are lying. (He does not say 0.01 off, which is a common mistake)

[...]

a295 Lis It may be rounded.

a296 Lis The exterior angle is equal the sum of the (remote) interior angles [*reading the conjecture*]. We have to prove it now – I dare bet, via this [*flips the page and looks at the toolbox from the previous task*].

As Lis looks at the toolbox from the previous task, she seems confused as how to proceed. Ole is gradually zoning out from the work and frequently talks to another student about out of school issues.

a302 Lis But these arguments, we are supposed to... maybe you can use them, right?

[...]

a306 Lis But can one use these arguments from toolbox to prove it? Or what? Maybe you can do that, right? I do not know! Should ask Ulf (the teacher).

The teacher explains to Lis that she is using the wrong toolbox.

a315 Lis [*Flips to the correct page*] Well, that's the one we need. I thought we were ... Ahh! Well, now we have some nice rules here. Then it's much better all together. The sum of the angles in a triangle is 180 [*reading the information from the toolbox*] ...

[...]

a460 Lis Look at this support figure, what does it say?

a461 Ole That there is an extra angle in relation to the other figure.

a462 Lis Yes, that's right. It is angle b. We know that angle b plus d gives 180 degrees. We know that the sum of the angles in a triangle is 180. Damn. So a, b, c also gives 180. Okay, I should not have written this [*Lis erases notes that she made in the proof sheet*] Now we'll just write it straight up. b plus d is equal to 180 degrees. I am back in the flow. a + b + c is equal to b + d is equal to 180, like this ...

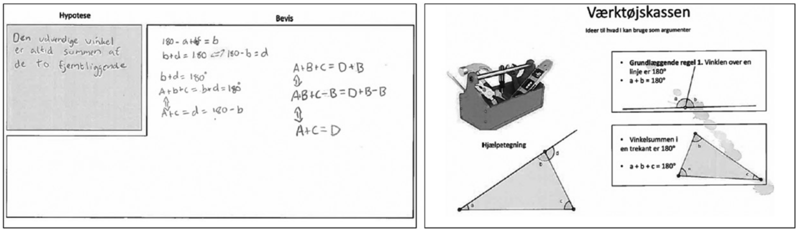


Figure 3. *The proof sheet and toolbox. Solved by Lis (mainly) and Ole*

a468 Lis	It was fun too! (Sings some words from a song with a bit of rhythm.) And then it's like this. Then we just need to have minus b on both sides. Then it becomes a + c is equal to d. Is equal to 180 — b.
a469 Lis	I think I have the proof right here... I am actually proud of myself.

Ole and Lis analysis

Ole did not follow the task steps rigidly and measured the angles before predicting as required in task 9b. However, before they add the measurements using a calculator, Lis provides the correct prediction (a258). It is not clear how she arrives at this prediction as she does not justify her prediction as required – perhaps she could quickly add the angle measurements that were already visible in GeoGebra in her head and notice the relationship. In any case, the event shows that even if *choice 3*, is implemented in the task design (requesting prediction and justification), it is not certain that the students will apply it.

We see that it is sufficient for the students to add the angles in only one example of the triangle formation in order to be convinced that their initial conjecture holds true (line a268–a270).

Ole's experimentation leads to triangle formation in which the measurement outputs, which were set to two decimal places, led to the conjecture not holding true. However, Lis shrugs it off as being a matter of rounding, which Ole accepts (line a290–a295). The example shows one of the known issues of DGE measurement experimentation previously reported by Olivero and Robutti (2007) concerning the degree of approximation of measurements and the technological limit for that precision in DGE measuring tools. The teacher's awareness of this issue is relevant in relation to *choices (1 and 2)* in the case of measurement invariance. Ole´s experimentation shows the effectiveness of *choice (4)*, requiring the students to drag and explain.

It is evident that during the time at which Lis mistakenly uses the toolbox from the previous task, she becomes unable to make progress on the proof (line a302–a306). The fact that Lis is stuck and unable to make progress shows the value of the correct toolbox. Without it, she could not go on. The finding is consistent with *hypothesis (3)* concerning the functioning of the toolbox activity to support the students in their argumentation and it indicates that design *choice 5* is suitable. When the teacher helps Lis to find the correct toolbox, she immediately recognizes its value (line a315).

In addition, we can see that the toolbox, and in particular the support figure play an important role in supporting the students (mainly Lis since Ole is disengaged) to provide theoretical validation of their conjecture (line a460–a462).

Both of these chains of signs highlight again the suitability of *choice 5* and *hypothesis (3)*.

While Ole has gradually become disengaged with the task activity, Lis, on the other hand, clearly indicates that proving the conjecture using the toolbox puzzle approach was enjoyable for her (a468–a469), which supports *hypothesis (2)* that when theoretical validation is presented as a challenge to explain why the DGE-based conjecture is true, it can become a meaningful activity to students.

The case of Ida and Sif

Ida and Sif were described by their teacher as medium/high achieving students in mathematics. In the previous task, they found the proving activity and the toolbox to be confusing. The following excerpt ensues after Ida and Sif have constructed the figure from task 9, they have guessed, investigated and put forward the correct conjecture (9a–9f) and are about to try to explain/prove why it is true (9g).

b516 Ida	The sum of the two (remote) interior angles ... [*Writes the conjecture in the proof sheet (figure 5) translated: "The external angle is as large as the sum of the two internal angles"*]
b517 Sif	Beautiful! Okay, now we have to prove it. Oh no ...
b519 Sif	Now that again ...
b520 Ida	a plus c equals b, and see. Basic Rule 1: The angle over a line is. The angle sum of a triangle is. [*reading from the tool box*]
b521 Sif	Yes! I understand. Look ... [*points to the support figure in the tool box*]
b522 Ida	Ohh.
b523 Sif	Super! In here, that's what's missing. [*points to angle b in the support figure from the toolbox (see figure 4)*]
[...]	
b533 Sif	And add this one here, to here. [*pointing to angle b being added to a+c and to d respectively*]
b534 Ida	That's right, so it makes 180. AND it makes sense. Is there more to say?

Figure 4. Sif using the toolbox to explain

b535 Sif That is ... just how it is.

b536 Ida We know that the sum in a triangle is 180 degrees and that the sum ... the angle sum of a line is 180 degrees. Therefore, when we are missing an angle here ...

In the events that follow, they write their answer (figure 5), but express difficulty in doing so, because they expect that they must use algebra in their answer.

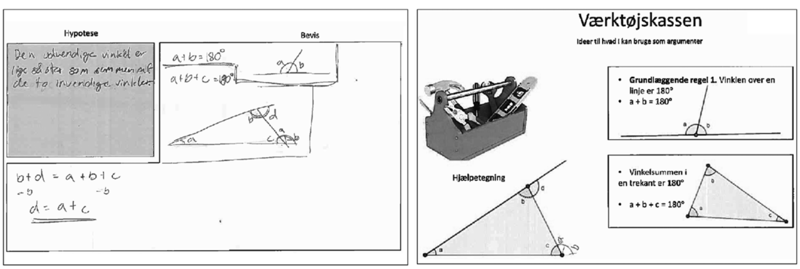


Figure 5. *Ida and Sif's proof sheet and toolbox*

b574 Ida How do we write that in mathematical language?

[...]

b595 Ida Ah okay! And a plus b and c yes. And b plus d it also gives 180

b597 Sif This one plus this one, is the same as these three. [*pointing to b+d and a+b+c*]

b598 Ida That's right. It's actually right. Oh, b plus d equals a plus b plus c because this makes 180, and this makes 180.

Ida and Sif analysis

The signs emerging from the students' discourse described in lines b517–b519 may be interpreted to mean that Sif is not excited about the prospect of having to prove the conjecture. However, the mood towards the proving activity changed when Ida and Sif had worked on 2–3 tasks of this type, which indicates that they had to get accustomed to the task design. Some of the difficulty may be attributed to the openness and unfamiliarity of the answer format, since several students could put forward their reasoning verbally, but struggled to write down their argumentation. Ida and Sif also struggle with this issue (line b574–b590). However, they find it easier to write the answer in subsequent tasks, after the teacher explained that they could write their arguments using natural language narratives. Thus, the fact that theoretical validation is not immediately met with students' enthusiasm can be attributed to several factors and does not mean that the activity is meaningless to them. In fact, the opposite becomes evident (lines b534).

From the video recording, it is evident (line b520) that Ida immediately turns to the toolbox information, reading aloud the two pieces of information provided, which indicates that she has realised the usefulness of the toolbox. It supports the suitableness of *choice 5* and sustains our *hypothesis (3)* concerning the functioning of the toolbox activity to support the students in their argumentation.

Sif listens and seems to recognize that adding angle b to $a+c$ and d respectively in both cases gives $180°$ (line b521–b533), which she manages to support Ida to grasp and elaborate as well (line b534–b536). They manage to reason deductively that their conjecture is valid, and after some struggle, write their answer algebraically (figure 5). The sequence of utterances from the students indicate that it is a sense making activity for them, and that there seems to be intellectual satisfaction attached to their experience (line b534–b536, and line b595–b598). These chains of signs emerging from Ida and Sif's work support our *hypothesis (2)*, that theoretical validation presented as an explanatory activity of the conjectures that are empirically evident for the students can be meaningful.

Discussion

On the basis of the analysis of the unfolding of the semiotic potential, the intended goals, which are described in the task design principles, can be evaluated and discussed. It seems reasonable to claim that the task design functioned more or less as intended in the two cases. Especially *choice 5* and *hypothesis (3)* concerning the functioning of the toolbox was substantiated in several episodes, for example as Lis was unable to progress without the correct toolbox. *Hypothesis (2)* is also substantiated in both groups, seeing as both groups expressed that the activity of theoretically validating their conjecture was meaningful, however, it is difficult to interpret if Ole found it meaningful, since he was mainly preoccupied with non-task related issues.

It was apparent that Ida and Sif had to become acquainted with the structure of the toolbox puzzle approach, before it became a sense making activity for them. This point was also evident in other groups. Additionally, several groups found it difficult to write down their arguments (line b547) even though they could convince each other verbally and with the help of gestures (e.g. figure 4).

Most groups of students succeeded and seemed to enjoy the exploration and conjecturing part of the tasks in the sequence. However, medium-low achieving students struggled to string together coherent deductive reasoning, and some never managed to overcome the toolbox puzzle part of the task on their own.

Returning to the unresolved issue in mathematics education research, which was raised in the introduction concerning conjecturing and proof in a DGE, the results from this study suggest that DGE explorations do not in and of themself impede deductive reasoning. This result is coherent with the view put forward by some researchers (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). In fact, the study indicates that DGE can be exploited to foster initial development of theoretical validation, in tasks that are designed to exploit the semiotic potential of such an artefact.

In the proposed task design principles, the role of DGS is most significant in relation to choices 1–4, which concern exploration and conjecturing. Other aspects of interest in this study are to what degree the students use DGE as they are trying to make a deductive argument; and what role the DGE plays in this regard. There are some indications that the students go back to the DGE in order to exemplify arguments to each other. Notably, some students return to DGE in order to verify what they have proven.

This interplay between the theoretical validation and ensuing DGE actions needs to be investigated further. In addition, the important mediating role of the teacher, which is described in *hypothesis 4*, was not discussed in this paper, but will be the focus of attention in future publications.

Conclusion

The study indicates that the "toolbox puzzle" approach can bridge a connection between conjecturing activities in DGE and deductive reasoning. The students explained theoretically, what they initially guessed visually and secondly investigated empirically in DGE. Importantly, the activity of conducting the theoretical validation seemed to make sense for the students, and to some students the activity was entertaining and intellectually satisfying. The study contributes to shed further light on the interplay between DGE conjectures and proof, and the results are aligned with the claim that DGE does not have to risk development of deductive reasoning (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). The results from the implementation of the toolbox puzzle approach in this study indicate that de Villiers (2007) proof as an explanation method, can be implemented into the task design itself, in order to support students' initial development of theoretical validation of DGE-based conjectures (Trocki, 2014). The systematic account of the task design principles in the form of objectives, hypothesis and choices, was found to be a useful frame to allow for a systematic evaluation based on the empirical data.

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