A set of design principles for exercising mathematical communication competency when using a DGE

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In design research, design principles involve the development of theory and practice. This paper refines a set of humble design heuristics into a set of design principles in the third iteration of a design research project. The set of design principles aims to exercise (meaning "to put into practice") students' mathematical communication competency when using a dynamic geometry environment (DGE). Based on an analysis, which includes perspectives on the instrumental approach, semiotic registers and mathematical language, the set of design principles is refined by transforming an analysis of two 9th grade (15–16 years old) students' interactions with the task design into prescriptive principles. The overall principle of *separate – join – new separate* indicates that it is crucial to relate mathematical representations across registers in the different steps, individually and in collaboration.

In recent decades, digital tools (DT) have been implemented in mathematics classrooms (Trouche et al., 2013; Rojano & Sutherland, 2020). These include *Computer algebra system(s)* (CAS) and *Dynamic geometry environment(s)* (DGE) (Freimann, 2020; Sutherland & Rojano, 2014). In a DGE, mathematical representations are dynamically linked, making it possible to explore, test and develop conjectures (e.g. Drijvers et al., 2009). In general, digital tools may quickly produce various representations acting as the basis for students' discussions (Drijvers et al., 2016).

Empirical results show that mathematical communication may change when using DT (Schacht, 2018). For example, mathematical communication may also involve "pragmatic descriptions" of how a tool is used (Jones, 2000; Schacht, 2015) or when using a DGE, mathematical communication may become more dynamic, meaning they may use words that indicate temporality or movement (e.g. Kaur, 2015; Ng, 2019). In

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Bach, C. C. (2023). A set of design principles for exercising mathematical communication competency when using a DGE. *Nordic Studies in Mathematics Education*, 28 (3-4), 125–155.

addition, students may develop their mathematical communication by talking about mathematical objects in a more general manner (e.g. Ng, 2016; Oner, 2016).

This paper focuses on students' "mathematical communication competency" when using a DGE. Mathematical communication competency is one of eight mathematical competencies defined in the *Danish competency framework* (KOM) (Niss & Højgaard, 2011, 2019). KOM addresses what it means to master mathematics across different levels of education from a competency perspective. Mathematical communication competency focuses on students' ability to engage in mathematical communication situations.

This paper is part of a design research project, which iteratively tests a design. This paper aims to refine design principles to *exercise* students' mathematical communication competency when using a DGE. Exercising a competency does not imply practising or training. Exercising a competency instead means "udøve" (in Danish) and implies doing something active or putting it into practice. Thus, the following research question is addressed: *How does one formulate a set of design principles to exercise students' mathematical communication competency when using a DGE?* First, "humble design heuristics" are presented. Second, two students' mathematical communication and their use of a DGE are analysed, acting as a critical case (Flyvbjerg, 2006). Finally, design principles are discussed and refined.

From humble design heuristics to refined design principles

Design research has a dual aim of developing theory and practice (Cobb et al., 2003) and an iterative structure of testing designs. Across iterations, theoretically grounded design principles are tested and developed using empirical results (Prediger, 2019). Design principles are heuristic in nature and may be defined as:

> If you want to design intervention X [for the purpose/function Y in context Z], then you are best advised to give that intervention the characteristics A, B, and C [substantive emphasis], and to do that via procedures K, L, and M [procedural emphasis], because of arguments P, Q, and R. (van der Akker, 1999, p. 9)

Prediger (2019) emphasises the logical structure of design principles by building on Toulmin's (1969) argumentation structures. Thus, design elements are developed using explanatory arguments, qualifiers of conditions for realising the design elements, and purposes of the intended effects of the design elements. At the beginning of a research project and before the design principles have been tested empirically and refined, the principles are labelled ”humble design heuristics”.

This project includes three iterations of testing the same design within different classrooms and schools. The paper has two parts: it aims to develop humble design heuristics into refined design principles. Part 1 describes the humble design heuristics prepared for the first iteration. The humble design heuristics include literature on mathematical communication competency (Niss & Højgaard, 2011, 2019) and the instrumental approach to mathematics education (Artigue & Trouche, 2021). Part 2 involves additional theoretical perspectives on semiotic registers and mathematical languages. A case of two students is analysed, and finally, the humble design heuristics are refined into design principles in a discussion.

Humble design heuristics

This section introduces the theoretical background used to define the humble design heuristics, which involves the KOM framework and the instrumental approach to mathematics education.

Mathematical communication competency

Mathematical competency is generally defined as ”someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations” (Niss & Højgaard, 2019, p. 14). ”Insightful readiness” refers to the conceptual understanding of mathematics related to a given challenge and being ready to act on such understanding.

Mathematical communication competency concerns communication in, with and about mathematics, and it involves being able to express oneself mathematically and interpret and understand others' mathematical expressions (Niss & Højgaard, 2011, 2019). Niss and Højgaard (2019, p. 18) clarify what characterises *mathematical* communication: ”[...] mathematical communication oftentimes invokes mathematical notions and concepts, terms, results and theories, or other features of mathematics as a discipline and a subject, and often involves the use of one or more mathematical representations”.

Hence, there is a substantial relationship between students' mathematical communication and mathematical representations (Niss & Højgaard, 2011, 2019).

A competency may be exercised or developed in a particular situation and cannot be completed or ”possessed”, only exercised in a given situation. The term ”exercise” is defined within the introduction. A competency may be assessed using KOM's "three dimensions", which concern a student's possession of competency in various situations: "degree of coverage", "radius of action", and "technical level". The degree of coverage concerns "the extent to which all the aspects that define and characterise the competency form part of that individual's possession of the competency" (Niss & Højgaard, 2019, p. 21). The three subdimensions may be defined as 1) expressing oneself mathematically or interpreting other's mathematical expression; 2) medium of communication (i.e. visual, gesture, speech and writing); 3) how mathematical communication is expressed within and across various kinds of discourses/registers/languages (as in Bach & Bergqvist, submitted). The radius of action concerns the various mathematical contexts and situations in which a student displays competency (Niss & Højgaard, 2019). In this paper, the radius of action is restricted to mathematical task contexts, such as algebra, geometry, statistics and probability (Niss & Højgaard, 2019). The technical level focuses explicitly on the mathematical content and objects in question. Thus, the technical level "denotes the level and degree of sophistication of the mathematical concepts, results, theories and methods that the individual can bear when exercising the competency" (Niss & Højgaard, 2019, p. 21).

If looking at *one* or *two* situations, a student's exercise of communication competency may be analysed in these situations. If looking at situations over time, or changes in, for instance, how a student communicates mathematically, a student may develop communication competency (Niss & Højgaard, 2011, 2019). Niss (personal communication, 11 November 2021) adds that developing a competency requires "exercising and reflecting".

The use of digital tools

The instrumental approach to mathematics education concerns students' use of digital tools. A fundamental process of this approach is "instrumental genesis", which includes four concepts: artefact, instrument, instrumentalisation and instrumentation (Artigue & Trouche, 2021). An artefact is a material thing that an individual cannot yet use for a given purpose. An instrument is an artefact that the student uses for a particular purpose in a specific situation (Verillon & Rabardel, 1995). Hence, instrumental genesis refers to the process of an artefact becoming an instrument for a student (Trouche, 2005a).

Instrumentalisation and instrumentation are two opposite processes. Instrumentation is directed from the artefact to the student (Trouche, 2020b) and is closely related to conceptualisation (Trouche & Drijvers, 2010). Instrumentation depends on the various information tools involved (e.g. the tool, communication with peers and mathematical theory), the mathematical knowledge in question and the strategies applied to completing a task (Guin & Trouche, 1998). Instrumentalisation is directed from the student toward the artefact and concerns the student's manipulation of the artefact (2020a). Thus, it is related to the student's use of the digital tool and what the student believes is the intention of using it (Trouche, 2005a). Two such processes are time-consuming (Trouche, 2020a; 2020b).

Students may use various information tools during instrumental genesis, such as multiple artefacts/calculators, other students or teachers, and mathematical knowledge (Guin & Trouche, 1998). A DGE offer several artefacts within one artefact (Drijvers et al., 2013). The multiple artefacts in a DGE have different potentials. In a review on DGEs, Højsted (2020) identifies dragging, measuring, feedback and tracing. Thus, it is crucial to identify a tool's various affordances for instrumental genesis (Leung, 2017). Nevertheless, making sense of dragging is a rather complex task involving interpreting different representations and changes within them (Arzarello et al., 2002).

When a digital tool is used, it represents many representations of the same object (Lagrange, 1999), and mathematics learning is closely related to interpreting and translating between representations. Artigue (2005, p. 239) adds that: "An efficient instrumentation requires some sensitivity to these problems which are linked to the representation of formal objects, and to the development of instrumented schemes allowing the user to take these phenomena into account in computations."

For instrumental genesis, students must explore and test conjectures (Guin & Trouche, 1998; Lagrange, 1999; Leung, 2011). Tasks must promote interaction between the computer, technical results, and paper and pencil calculations. By doing this, students conjecture, test, complete a task and check results. The focus is then on the role of mathematics, not on the computer (Guin & Trouche, 1998).

Presentation of humble design heuristics

Four humble design heuristics were defined based on the theoretical background of mathematical communication competency and the instrumental approach. The criteria for each heuristic was that it involved aspects related to both mathematical communication competency and the instrumental approach. For both students' mathematical communication competency and the use of a DGE, it is necessary to focus on different task contexts, mathematical representations, and procedures for answering a task. In addition, the dialectic individuality and collaboration also seemed to be relevant. Using van der Akker's concepts (1999), humble design heuristics involved the design characteristics of a possible task (Prediger, 2019). Tasks must be designed so that:

DH1: There is a progression across situations and content, having various task contexts, increased complexity of the content, and collaboration in pairs.

DH2: There are various representations offered by tasks for exploration, interpretation, linking, and choosing.

DH3: Various artefacts with different affordances that support tracing, dragging, measuring, and feedback in a DGE are included in the tasks and artefacts must be introduced gradually.

DH4: There is room for individual and collective processes, achieved by tasks involving individual problem solving and tasks focusing on communicating.

The conditions for success are that students use a DGE and develop or express an understanding of the mathematical object in question.

Semiotic registers and mathematical language

As mentioned, mathematical representations are vital in mathematical communication (Niss & Højgaard, 2011) and instrumental genesis. Guin and Trouche (1998, p. 207) note that "the two key points for learning are the functional differentiation and coordination of semiotic registers", and Duval (2006; 2017) describes an essential characteristic of mathematics: our only access to the objects is through their representations.

Duval (2006) describes four semiotic registers that depend on whether the representations are discursive or not and whether they are algorithmic or not. A crossing table presents the semiotic registers (see table 1).

The distinction between the monofunctional and multifunctional register is closely related to what is mathematics and what is not. Representations, which are characteristic of mathematics, are monofunctional and controlled by algorithms, such as equations and graphs. Yet, multifunctional representations are not less important, particularly for learning, communicating and processing. The discursive representations include signs and words, such as letters and numbers, whereas non-discursive representations do not.

Transformations between different representations may be either treatments or conversions. Treatments are transformations within a

Table 1. *Four registers of semiotic representation (Duval, 2006, p. 110)*

	Discursive	Non-discursive
Multifunctional	Linguistic representations: written and oral language	Figurative representations, e.g. images, figures
Monofunctional	Symbolic representations: arithmetic calculation, equations	Graphical representations, e.g. graphs, diagrams

register, for instance, when reducing symbolic expressions. Conversions are transformations across registers, for example, from an equation to a graph, and these are important to mathematical understanding, which "is a cognitive process, which involves 'coordination of at least two registers' " (Duval, 2017, p. 110). A computer performs constant treatments, which means that students can manipulate semiotic representations as real objects (e.g. rotating and zooming in/out). Furthermore, the linguistic register is often neglected when using a computer, as verbal expressions may be reduced to buttons or commands (Duval, 2017).

Duval's (2006, 2017) semiotic registers are related to the mathematical representations, but they do not offer tools for analysis within the linguistic register. Yet, the linguistic register is relevant to mathematical communication competency (Niss & Højgaard, 2019).

An existing framework relates Duval's (2006) semiotic registers to three languages: "everyday language", "school language" and "technical language" (initially presented by Prediger & Wessel, 2011, 2013)[1] (Prediger & Neugebauer, 2021). The three languages exist on a continuum from contextualised to generalised, decontextualised language (Gibbons, 2015; Schleppelgrell, 2004).

Everyday language is contextualised and personal (Schleppelgrell, 2004). It involves everyday terminology or one's own invented words (Prediger & Wessel, 2013). The representations accompanied by everyday language are often concrete and verbal (Prediger et al., 2016). School language is a medium for learning and is often used by teachers (Prediger & Wessel, 2013). It is less contextualised and personal (Schleppelgrell, 2004) and may include graphical or numerical representations (Prediger et al., 2016). Finally, technical language is even more decontextualised and impersonal and may involve more complex sentence structures (Schleppelgrell, 2004). In contrast to everyday language and school language, symbolic and algebraic expressions may appear in the technical language (Prediger et al., 2016). Within this perspective, a sign of learning is if students express themselves in a less contextual language and describe the representations involved. Through language, the students must reason out the representations' relations (Prediger & Wessel, 2013).

Method

This design research project includes three iterations, which test a similar task design. As this paper presents the results of a third iteration, two previous iterations involved continuous design adjustments based on analyses of students' mathematical communication and their use of DGE. The task design was tested in 8th- and 9th-year classes in Denmark. For iteration 3, the testing involved 9th-grade students (15–16 years old). The adjustments of the tasks are explained when presenting the task design in the next section. The project uses previously described and tested tasks by Johnson and McClintock (2018). The tasks were adjusted to the humble design heuristics and for the classrooms in which they were tested.

As part of the study design, the students collaborated in pairs. The students had a computer each. The teacher paired the students to establish the best-functioning teams based on who they usually collaborate with and their proficiency in mathematics. GeoGebra is widespread in Danish schools, and the students were all familiar with it. They use GeoGebra for every written assignment (at least once a month) and sometimes in class. Mathematical communication competency is included as a goal in the Danish curriculum; thus, the teachers must aim to exercise the competency when teaching (Ministry of Education, 2019).

Data collection included written answers on worksheets and screencasts, including videos of students' screens (to observe their use of the GeoGebra), their voices and their webcams (to identify who said what). Finally, a camera was placed behind selected pairs to observe the gestures, which were not captured by the webcams. In addition, the voice recording quality was often better at the external camera than the computers' voice recording. All communication (oral and written) was originally in Danish but was translated into English by the author when presented in this paper – for words, which was challenging to translate, the author consulted with colleagues. The data presented in this paper cover two students' work and communication (i.e. written and oral communication, representations, and gestures, when applicable). The student pair was chosen as they worked together throughout the task, and data included both screencasts and a video camera behind them. Thus, it was possible to triangulate data sources and investigate the whole task design and the three dimensions of communication competency of the two students.

Nevertheless, different phases of students' work are presented to identify the students' mathematical communication competency across situations. The selection of the two students provides a potential "critical case", which "can be defined as having strategic importance in relation to the general problem" (Flyvbjerg, 2006, p. 229). Such case makes it possible to generalise in the following manner: "To achieve information that permits logical deductions of the type, 'If this is (not) valid for this case, then it applies to all (no) cases.' " (Flyvbjerg, 2006, p. 230). As the students in this paper are carefully chosen, and the case aims to show students exercising mathematical communication when using the DGE – using different languages and uses of the DGE. Hence, if results do not apply to this case, it does not exist for others. On the other hand, if the results appear for these two students, they may also appear in others.

Task design

The tasks were built on Johnson and McClintock's (2018) dynamic interpreting of "the bottle problem". The problem presents a "bottle" as a polygon, which may be filled by dragging in a point. The particular DGE used in this project is GeoGebra, and the students were provided with GeoGebra activity templates, which they access using links, designed to address such problems. The activity template presents GeoGebra but with pre-constructed polygons on the one hand and a point in a coordinate system on the other. The two elements are related, and the task is for the students to investigate such a relationship using GeoGebra.

The mathematical content of the tasks concerned functions such as covariation, meaning that a "student using a covariation perspective could conceive of functions as specialised relationships between quantities" (Johnson & McClintock, 2018, p. 303). The tasks lasted 45-minute lessons. The tasks were split into two parts: the first part concerned a linear function, and the second part concerned quadratic functions.

The first part, "Filling the rectangle", concerned a linear functional relationship. Students were asked to explore, conjecture and test the

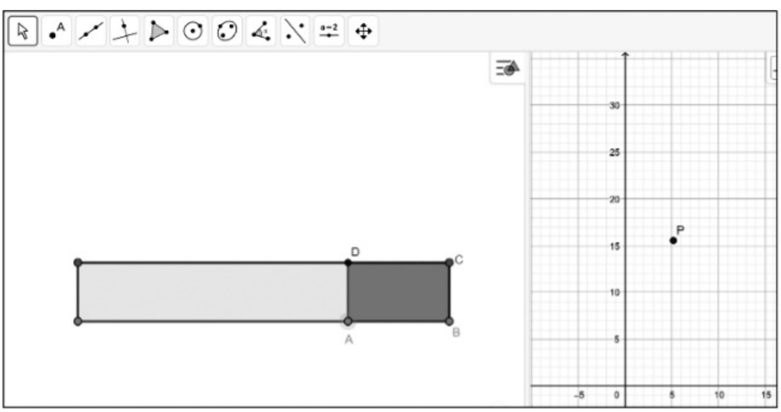

Figure 1. *The template when the students access it at the beginning of the task sequence (inspiration from Johnson & McClintock, 2018, p. 305). Link to the template:* https://www.geogebra.org/classic/jqjt7ecs

relationship between the representations in the template. The representations were a rectangle *DABC* and a point *P*. See figure 1. At first: a rectangle *DABC* was placed within a longer rectangle. These belong to the figurative register and are essential to the "bottle problem" (Johnson & McClinktock, 2018; Swan, 1985). The point *P* belonged to the graphical register, and the properties of rectangle *DABC* defined it: the x-coordinate was the length of *AB*, and the y-coordinate was the area of rectangle *DABC*.

It was possible to drag point *A*, measure properties in the rectangle and trace point *P*. Hence, the task design included affordances for DGE use. The traces of point *P* showed a linear graph, which is defined as $y = 3x$ (the height of *BC* is 3). See figure 2.

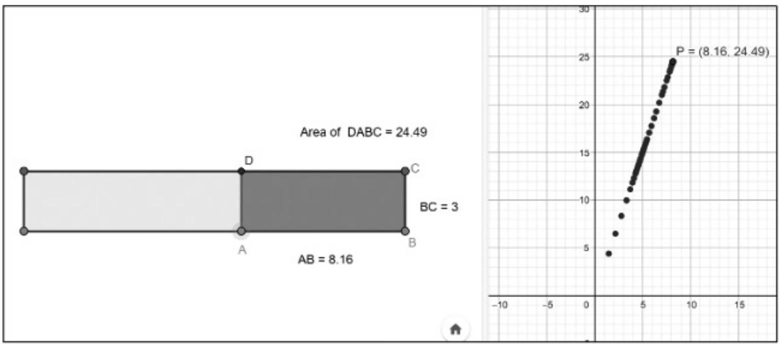

Figure 2. *Template with measures on the rectangle DABC and coordinates and traces for point P (inspiration from Johnson & McClintock, 2018, p. 305)*

The students are asked to answer the following questions:

1. Together. Which figures can you see? Explain why you think it is the figure in question.
2. Together. Identify the height, width and area of the figure
3. (From now on, you may drag point *A*) Individually. What happens when you drag in point *A*? Describe your conjecture about the relationship between the figure and point *P*'s coordinates. Give as many arguments as possible.
4. You can turn on "show track" or have GeoGebra show point *P*'s coordinates.
5. Together. Present your assumptions from task 3 to each other. Write below what you agree on.

6. Individually. The table (see table 2) shows the lengths of the length of *AB*. Try to fill the table by finding the area of the *DABC* for a given length.

Table 2. *The table for task 6*

Area of *ABCD*	1	3	8	15	20
Length of *AB*					

7. Together. What is the equation of the function?

8. Together. View the graph of the function in GeoGebra. Try to explain what the function's graph looks like and why it looks the way it does. You may draw the graph on paper.

9. Together. Which variable is dependent and which is independent? And why?

10. Together. How can you test that you have found the correct equation of the function and the graph you have found?

11. Together. What do you expect to happen to your function if you change the length of *BC* increases?

Individual and collective questions were emphasised throughout the task design, as our results indicated that students might focus either on using GeoGebra and interpreting representations or on having a conversation with their peers (Bach & Bikner-Ahsbahs, 2022).

In iterations 1 and 2, the height and area of the rectangle were infinite, and the rectangle was rotated 90°. For iteration 3, the rectangle was rotated to appear as in figures 1 and 2, as some students were confused with the height of the rectangle increasing on the *y*-axis, which in point *P* were represented on the *x*-axis (Bach & Bikner-Ahsbahs, 2020, 2022). Previous editions of the template also had a visible algebra view that showed all points and figures. However, words in this algebra view also appeared to confuse some students and were a constraint in the instrumentation process (Bach et al., 2022b).

In Iteration 3, an additional task was included to investigate a particular way of mathematical communication while using DGE. This task concerned 2nd order covariation. Yet, this part was not included in this paper as it is not part of the original task design.

The second part, "Filling the triangle", concerned a quadratic function and built on the same structure as in part 1. The students were provided with a new template showing a triangle, a right trapezoid and a point, *P*. See figure 3.

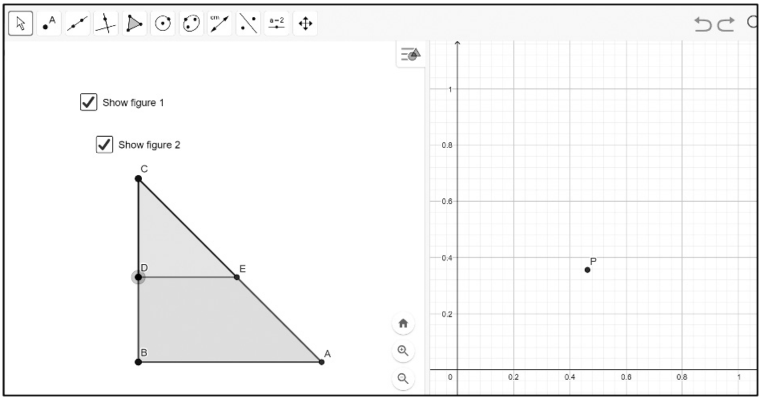

Figure 3. *Screen dump of the template for phase 3, filling the triangle (inspired by Johnson & McClintock, 2018, p. 306)*
Link to the template: https://www.geogebra.org/classic/zatbex7u

The second part used the same DGE affordances as part1, namely measuring, dragging and tracing. The traces of point *P* showed a parabola, which could be converted into the equation $f(x) = -½x^2 + x$. The students were given a drawing (figurative register) to ease the conversion from the non-discursive register to the symbolic register. As in "filling the rectangle", the students may also turn on tracing for point *P*. See figure 4.

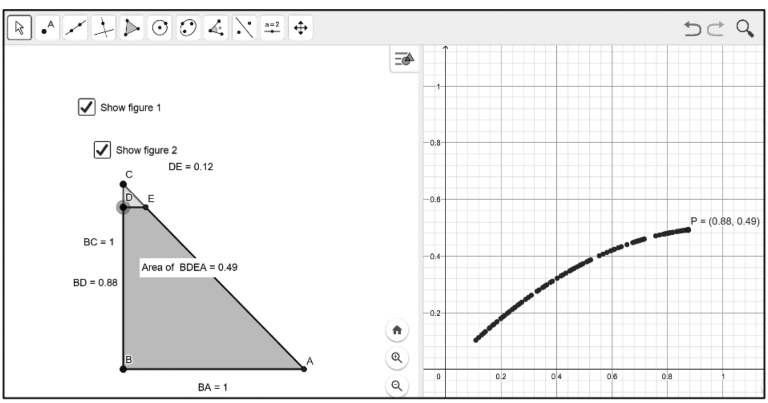

Figure 4. *The template for "Filling the triangle" with measures and traces for point P (inspired by Johnson & McClintock, 2018, p. 306)*

The students are asked to answer the following questions:

1. Together. What figures can you see? Please explain why you think it is the figures in question.

2. Together. Find the side lengths and area of the figures. You can use GeoGebra to find it.

3. Individually. What happens when you pull in point *D*? Describe your conjecture about the relationship between the figures and the coordinates of point *P*. Give as many arguments as possible.

   You may turn on ”show track” or have GeoGebra show point *P*'s coordinates.

4. Together. Read your thoughts from task 3 aloud to each other. Write below what you agree on.

5. Individually. The table (table 3) shows the size of *AD*. See if you can fill in the table by finding the area of *ABDE* to a given size of *AD*.

Table 3. *The table for task 5*

Area of *ABCD*	0,2	0,4	0,6	0,8	1
Size of *AB*					

6. Together. What is the equation of the function? And why can it be written like that?

7. Together. Try to explain what the graph of the function looks like and why it looks the way it does. You are welcome to draw.

8. Together. Which variable is dependent and which is independent? And why?

9. Together. What type of function is it? And why?

10. Together. Describe how you can use GeoGebra to test if your results are correct.

11. Individually. What will happen to your function if you change the lengths of *BC* and *AB* to become longer or shorter? They must still be the same length.

12. Together. Present your thoughts from task 11 to each other. Write below what you agree on.

13. Together. Use this link https://www.geogebra.org/classic/k2qff6wb by dragging in point *F* to change lengths *AB* and *BC*. See if you can denote *P*'s *x*- and *y*-coordinates, the graph, and the equation of the function when the sizes vary.

To summarise, there was a progression in the content provided by various task contexts, and the complexity of the content was achieved by including polygons and functions and by providing two tasks (one on linear functions and one on quadratic functions). In the templates, polygons and points in a coordinate system were present, and the students were asked to fill tables and find equations of the functions. The tasks included measuring polygons, tracing and dragging.

Method of analysis

The analysis is twofold. It includes analysing the use of a DGE and mathematical communication. Thus, the processes of instrumental genesis and mathematical communication by distinguishing between everyday language, school language and technical language, as presented by Prediger and Wessel (2011, 2013).

Beginning with instrumental genesis, the students' "information tools" are described to identify their resources for completing the task and determining how they investigated the template. These may include their use of theoretical knowledge, the DGE, their peers and the internet (Guin & Trouche, 1998).

Second, the students' mathematical knowledge (i.e. their conceptualisation) was analysed from the perspective of semiotic registers, focusing on treatments and conversions within and across representations (i.e. the graphical register, the figurative register, the symbolic register and the linguistic register) (Guin & Trouche, 1998; Duval, 2006; 2017). The students' languages were also analysed, relating the different representations in play and the levels of contextuality within students' language. Such categorisation makes it possible to determine if students use everyday, school or technical language. See table 4.

Finally, the students' instrumentation, instrumentalisation and mathematical communication are summarised. Students' instrumentation base

Table 4. *An overview of language and representations used to determine the students' language, both focusing on the contextuality of the language and the representations used (Prediger & Wessel, 2013; Prediger et al., 2016; Schleppelgrell, 2004)*

	Everyday language	School language	Technical language
Language characteristics	Contextualised and personal	Less contextualised and personal	Decontextualised, impersonal and complex
Representations	Concrete and verbal	Graphical and numerical	Symbolic and algebraic

on their information tools, mathematical knowledge and strategies. As instrumentation is related to conceptualisation (Trouche & Drijvers, 2010), the mathematical knowledge is analysed by applying Duval's (2017) concepts of transformations of representations to identify if the students coordinate the registers of the template, which indicate learning. Students' instrumentalisation is analysed by identifying the various features used in GeoGebra and the indications of what the students may believe to be the intended uses of this tool (Trouche, 2005a). Finally, the students' mathematical language is related to their work.

Data and analysis

Two students, Amelia and Charlie (fictitious names), collaborated throughout the task sequence. This section is split into two parts: a presentation of students' dialogue and related written answers and the subsequent analysis of these excerpts. The students' written answers are presented as close to how they wrote – for instance, using a lower-case *a* (for point *A*) instead of a capital letter.

Individual investigations of "Filling the rectangle"[2]

The students began by defining the rectangles (see figures 1–2), and then they used commands in GeoGebra's graphical view to identify the rectangle's length and area. Next, the students worked on the individual task of investigating the template by dragging point *A* and thus conjecturing about the relationship between point *P* and rectangle *DABC* (question no. 3).

1	Amelia	Can I drag point *A*?
2	Charlie	Yes. [he drags point *A* back and forth, both quickly and slowly]
3	Amelia	[begins to drag point A] Oh see [she drags very quickly back and forth], I don't understand this

The two students briefly discussed whether this part of the task should be done individually or together. After being told that it was to be done separately, they worked on their own. Amelia dragged a bit more before writing down her answer.

Charlie's friend came over, and they discussed "what is the equation of the function". Together, the two friends agreed that it is $y = 3x$. The friend left. However, Charlie was uncertain how to present his results and asked the teacher, who told him to just try to do the task on his own. Finally, Charlie drew a line ($y = 3x$) and tested the results by dragging.

Amelia's written answer to the individual task: If one drags A to the left, point P moves upwards, if one drags A to the right, point P moves downwards.

Charlie's written answer to the individual task: P goes between 0 and 57 because $A = 57$. P moves by $3x$ because the rectangle is 3 high.

Analysis of individual investigations of "Filling the rectangle"

Beginning with Amelia, her primary information tool is dragging in GeoGebra utilised to investigate the template. When considering the mathematical knowledge in question, Amelia referred to both point A and point P. Yet, she did not refer to the properties of the rectangle. Amelia's answer did not include deep descriptions of the representations used, and it included everyday words such as "moves', "upwards", "downwards", "left" and "right". Her answer expressed particular actions when using GeoGebra ("If one drags"). The use of "one" instead of "I" depersonalised her answer, but the reference to dragging made it more contextualised. Amelia's strategy for investigating the template was dragging and relating the representations to each other. However, the representations were not deeply interpreted, nor were the treatments and conversions caused by dragging. During the process of instrumentation, Amelia struggled to understand the representations and their relations. Relates to instrumentalisation, Amelia seemed to be unfamiliar with some parts of GeoGebra, and her understanding of its purpose in the given situation was limited. Thus the GeoGebra template appeared not to be an instrument. Amelia used everyday language due to her contextualised descriptions, use of everyday terms and superficially described relations between the representations.

Charlie's information tools included dragging, measuring when identifying the properties of the rectangle, calculation, when writing the equation $y = 3x$, and his friend (which was not consistent with the guidelines of this task). In terms of mathematical knowledge, Charlie's written answer included symbolic representations and verbal language. Charlie referred to the area and the height of rectangle $DABC$ and its relationship to point P. Hence, Charlie made conversions from the graphical to the figural register and the symbolic register (i.e. $3x$). The language was impersonal, without directly referring to dragging, and it included mathematical terms, symbols and argumentation ("because" is a conjunction indicating reason). Charlie used several strategies: dragging for investigating the relationship and validating symbolic and verbal explanations. Charlie's process of instrumentation was less complex. The rectangle and point P were related by interpreting and describing treatments and conversions, revealing a conceptual understanding of linear functions. Charlie used technical language, which uses algebraic expressions; it is decontextualised and impersonal.

Collective work on "Filling the rectangle"

After investigating the template separately, Amelia and Charlie began to discuss question no. 4. They were a little confused about what to do and asked for help to clarify the goal of the task (i.e. presenting their conjectures to each other).

4 Amelia But I don't really know what the relationship is ... I have written that "if you drag *A* to the left, point *P* goes upwards, and if you drag *A* to the right, point *P* goes down."

5 Charlie Okay, Amelia, what I have written ... [he drags point *A* quickly back and forth] "*P* goes by 3*x*. It is like 3*x* when it goes one to the right. It goes 3 up." Try to see [while pointing with a pen on his graph]. Try to see, can you see it? [he drags slowly for six seconds] And then if you drag it a little further ...

6 Amelia So, we agree that if it goes three up. It goes 3*x* up [as she points at her screen with her pen without a graph] or?

7 Charlie If you zoom out, then you can see that.

8 Amelia The equation is ... At least ...

9 Charlie So it goes up to 57, and here it goes down to 0 [he points at the graph and drags point *A* until the area of *DABC* is 0, then 57, and finally he returns to the graph to illustrate this]

10 Amelia Okay. So, it is actually ...

11 Charlie It is the area of the rectangle.

12 Amelia It is the area of it [she drags very quickly back and forth for two seconds] ...

13 Charlie Yes.

14 Amelia Mmm. So, when you drag it, it can go up to 57 and down to 0. The area. Okay. When you drag point *A*, when you drag point *A* to the left [she is talking slowly, as she is writing at the same time], left, *P* can move up to 57 ... [mumbling]

15 Amelia Mmm. What do you say? That *P* follows 3*x*?

16 Charlie Yes

The dialogue continued for a few minutes.

Amelia's written answer for the collective task: When one drags a to the left, *p* goes up to 57, which is also the area of the figure. *P* follows 3*x*. The rectangle *ABCD* has a direct relationship to *P*'s position.

Charlie's written answer for the collective task: P follows $3x$. The rectangle $ABCD$'s area has a direct relationship to P's y position.

Analysis of the collective work on "Filling the rectangle"

Amelia's and Charlie's information tools were dragging, measuring, and communicating with each other. For the mathematical knowledge, they referred to representations in all four registers. They made the conversions from rectangle to point P when dragging (lines 5 and 12) and pointing at the screen (lines 5, 6 and 9). In the discussion, the students' communication was contextualised and addressed what happens when dragging point D. For instance, Charlie utilised deictic terms such as "it" in statements such as "can you see *it*", and "if you drag *it* a little further" (line 5). It was used often, sometimes referring to the rectangle and sometimes to point P (e.g. lines 5, 6, 9, 10, 11, 12 & 14). The students also used technical terms such as "rectangle", "area", and "point". Also, Amelia repeated some of Charlie's sentences (e.g. 11–12). Thus, the language lacked mathematical specificity and was contextual and more personal (the use of "you"). In contrast, if we consider the students' written answers based on their discussion, their answers were less contextualised, as there were no deictic words and fewer personal references. Charlie's answer may be defined as a technical language, with symbolic expressions (e.g. "$3x$"), which reduced the context references in the texts, and compact language. Amelia's written answer also repeated some of Charlie's verbal statements. Yet, it also presented transformations between and within the registers. However, this was decontextualised as Charlie's technical usage. Thus, it was school language with a few symbolic expressions.

All in all, the two students had different experiences to bring into their discussion after their individual investigations. Yet, both Charlie and Amelia could both use GeoGebra as an instrument. They interpreted representations and made transformations within and across registers, indicating instrumentation. They also utilised various features of GeoGebra, and they could use it for their intended purposes, indicating instrumentalisation. When comparing the dialogue to the written answers, there appears to be a shift from contextualised to decontextualised language.

Individual investigations of "Filling the triangle"

Charlie and Amelia began the "Filling the triangle" task by looking at the polygons shown in graphical view in GeoGebra and finding both heights, lengths and area (questions no. 1 and 2; see figure 4). They wrote that the figures were a right-angled triangle and a trapezoid. They also noted that the trapezoid's height varied from 0 to 1. Hence, the students had to individually investigate the relationship between the trapezoid and point *P* by dragging point *D* in the trapezoid (question no. 3). When they began to do so, they turned on tracing for point P.

Charlie dragged point *D* slowly up and down for some time (about 15 seconds) for *BD* being 0–0.2. Charlie dragged for approximately 1 minute. While dragging, he pointed at *AB* and said, "it is because down here, it is wider".

Amelia primarily looked at Charlie's template as he dragged and tracing was on. Then she asked Charlie to drag it again. She talked to herself, "so it goes along like that. It goes one ... [she made an arc with her hands]. It rises and goes like that ... It rises and then".

Amelia's written answer for the individual task: When you pull in *D*, *p* goes in an arch.

Charlie's written answer for the individual task: *P* comes further away from 0.0 as figure 2's [his labelling of the circumscribed triangle for the trapeziod] area is bigger.

Analysis of individual investigations of "Filling the triangle"

Amelia again used dragging, including tracing point *P*, as an information tool. For her mathematical knowledge, Amelia was focused on the treatments of point *P* and the slope/curve that the tracing made. She made gestures with similar forms to the traces for point *P*. The answer indicates that she did not make conversions between the trapezoid and point *P*. Amelia used everyday language with everyday terms, such as "arch". It is difficult to understand without the template. Amelia's strategy was to describe the graph that point *P*'s traces make. Amelia did not use GeoGebra. Her answer did not indicate instrumentation (i.e. conceptual understanding of the concepts in play), and a process of instrumentalisation was not evident.

Charlie used dragging and tracing as information tools, but he also measured the lengths of the polygons. For instance, he referred to the area of the trapezoid when investigating the relationships. Concerning Charlie's mathematical knowledge, he made conversions between point *P* and the area of the trapezoid. The description used everyday language, such as "*P* moves ...". It was closely related to the context, and changes in the representations were not profoundly done. Charlie used dragging and relating the representations as strategies for completing the task. Charlie used various information tools, yet he appeared to lack a mathematical understanding of the object in question (i.e. quadratic functions). His work showed a process of instrumentalisation that used several features of GeoGebra (i.e. information tools). Thus, his written answer revealed difficulties in understanding and communicating the relationship between the trapezoid and point P.

Collective work on "Filling the triangle"

After the two students wrote their answers individually, they had the following discussion (question no. 4).

18	Charlie	Okay. I have written that it comes further away from 0.0 [pointing at the screen].
19	Amelia	I have written that when you drag in *D*, *p* goes in an arch.
20	Charlie	That's technically right, but there's not that much about it.
21	Amelia	But what did we find out then?
22	Charlie	It is not exponential, but it is *decreasing* ["decreasing" is not translated]
23	Amelia	Exponential.
24	Charlie	That's how it is. You know where ... it's like a power.
25	Amelia	Power?
26	Charlie	You know, like 2 + 2 it is 4, 4 + 4 it is 8, 8 + 8 it is ... it does not get larger and larger.
27	Amelia	When it's like that, where it's like 1 to 1, uh no.
28	Charlie	Uhm, but it's like that ... the slope number, it drops [the pencil moves in jerks upwards].
29	Amelia	Yes, okay.

Amelia's written answer for the collective task: *p* becomes larger the further you pull in *d*, but the coefficient decreases. It is a parabola.

Charlie's written answer for the collective task: *P* becomes larger together with the area of figure 2 but the coefficient decreases.

Analysis of the collective work on "Filling the triangle"

The students' information tools included dragging, measuring, tracing (from previous investigations), calculations (line 26) and each other. Concerning the mathematical knowledge, the students tried to communicate the equation of the function based on the traces of point *P*. The students made different conversions using the traces for point *P*. For instance, in lines 26 to 27, the students described it as a quadratic function without mentioning parabola or power. Also, in line 28, Charlie stated that ”the slope drops”. He probably meant that it did not increase as for a linear function because a parabola does not have a slope. They displayed everyday language as it was contextualised, and they used everyday words and improvised usage (e.g. decreasingly ”drops”). When we consider the written answers, these neither included symbolic or algebraic representations. Both answers include ”slope decreases”, which was incorrect. Both written answers were contextualised (e.g. referring to ”figure 2” instead of writing ”trapezoid”; referring to dragging), yet impersonal. It is unclear how ”parabola” ended up in Amelia’s answer. Their strategies may be interpreted as applying semantic interpretation based on their dragging, but they do not validate their results.

The students’ instrumentalisation was lacking in this task, and they did not explore additional artefacts in GeoGebra to help them to complete the task. The instrumentation process revealed difficulties in making transformations within and between registers and a lack of a specific vocabulary. Hence, the students used everyday language, including mistakes in the mathematical representations and contextualisation.

Summarising the results

There appeared to be a relationship between Charlie’s and Amelia’s ways of communicating mathematically and their instrumental genesis, particularly when giving their explanations in writing. For instance, Amelia used everyday language in both tasks. Also, the treatments and conversions in the representation were not done, and her process indicated that the instrumentation of the DGE template was absent. Their school language indicated that they made conversions and treatments, and it did not include as many everyday terms. Amelia used school language in the written answers following a discussion with Charlie. Charlie used technical language in the first tasks. The relationship between their mathematical language, the mathematical representations and the processes of instrumental genesis are summarised in table 5.

For table 5, the use of everyday language, school language, and technical language appear to be closely related to students’ processes of instrumental genesis.

If we consider the students’ oral communication during the first task concerning the rectangle, they appeared to use school language, but it was contextualised and personal when communicating. However, when the students wrote after communicating, their communication was less contextualised and without deictic terms. On the other hand, when they discussed the second task concerning the triangle, they appeared to use

Table 5. *An overview of relations between written language, representations and instrumental genesis*

Language	Language characteristics	Representations	Instrumental genesis
Everyday language	Contextualised, everyday terms and improvised usage.	Transformations within and across registers were done superficially.	The initial phase of the instrumental genesis. The student were at the beginning of both instrumentalisation and instrumentation.
School language	Mathematical terms, less contextualised and personal.	Transformations within and across registers were done, and the representations were described dynamically.	The tool was developing into an instrument, including instrumentation and instrumentalisation. Thus, students understood the representations, used different features in the GeoGebra.
Technical language	Decontextualised, impersonal, complex structure.	Transformations within and across registers were done, and the representations were described statically.	The tool was an instrument for the student, including instrumentation and instrumentalisation. Thus, students understood the representations, used different features in the GeoGebra and had the goal of the task in mind.

everyday language that lacked mathematical terms and improvised their own. Hence, the students' oral language appears to be more contextual and personal than their written language.

Adjusting design principles for mathematical communication competency and the use of the DGE

Charlie and Amelia can be considered as a critical case showing how students' mathematical communication may develop – and differ – depending on the situation. Such situations regard both individual or collective tasks, the process of instrumental genesis, and the mathematical content in question. Hence, their mathematical communication competency is exercised in the collective work when presenting the results of individual investigations, and their written answers develop based on these dialogues. Results also indicate that their mathematical communication competency is closely related to the processes of instrumental genesis (see table 5).

This section discusses the results and refines the humble design heuristics (DHs) into a set of design principle(s) (DPs). Thus, the design heuristics are transformed based on the analysis of Amelia and Charlie (Prediger, 2019).

Progression across situations and content (DH1)

When we begin by discussing DH1, KOM's three dimensions of mathematical communication competency are interesting. Related to the degree of coverage, Amelia and Charlie were asked to express themselves mathematically and to interpret each other's mathematical communication across two media (oral and written), and they communicated mathematically across different languages – everyday, school and technical. Related to the radius of action, the tasks included geometry and functions and related to the technical level; the students were faced with tasks on linear function and quadratic functions.

The results indicate that how they exercised mathematical communication competency depended on the situation. Charlie communicated using more registers than Amelia when focused on linear functions. Thus, Charlie's degree of coverage was greater than Amelia's when communicating about linear functions. Looking at the degree of coverage, Amelia showed progress in both tasks: from using everyday language to using school language. The data presented is not enough to display the radius of action for Charlie and Amelia, but the results indicate that both students could use GeoGebra to measure the figures in both tasks but not find the functional relationships. At the technical level, they communicated less contextualised when working with linear functions than with quadratic functions. Thus, designing tasks with a progression of content makes it possible to exercise mathematical communication competency in various situations.

DP1. Tasks must be designed so that there is a progression across situations and content, having various task contexts (e.g. geometry and functions) and increased content complexity (e.g. from linear functions to quadratic functions). Hence, the three dimensions of competency can be exercised when using a DGE.

Using of various representations (DH2)

If we focus on using different representations and registers (DH2), transformations within and across registers are closely related to mathematical understanding (Duval, 2017), which entails the process of instrumentation when using a DGE. The use of a DGE automatically offers treatments of representations (Duval, 2017), which may result in dynamic mathematical communication (Bach et al., 2022a), also found in Amelia's mathematical expressions. The dynamic features are embedded in everyday and school language, which may be due to the contextualisation of these two languages. However, Charlie did not embed dynamic features in his mathematical communication when using technical language, which may be due to the decontextualised language, and his instrumentation of the DGE, including the understanding of linear functions.

When Amelia and Charlie investigated the relationship between point *P* and the rectangle, they needed to address a variety of properties of the representations. Hence, if they were to identify the relationships in small steps (e.g. introducing and focusing on one representation at a time), more students might be able to follow this (e.g. also in Prediger & Wessel, 2013). The covariation perspective, involving coordination of more than one register, supports the potential to develop a mathematical understanding (Duval, 2017).

Using the Prediger and Wessel's model (2011; 2013), the goal of mathematical communication is to communicate in technical language, which may be reasonable, given that results indicate that instrumentation is necessary to use such language. However, when using school language, students may also show mathematical communication competency, as they use a vocabulary that is accepted and used in the classroom. Hence, both school language and technical language may include more sophisticated mathematical language. To design a task aiming for these languages, students need to relate the various registers and languages, for instance, when formulating the task description in everyday language (Prediger & Wessel, 2013) or breaking it down for the students to not describe the whole situation at once, since dragging is highly demanding (Arzarello et al., 2002). By asking to relate all representations and tasks in small steps, the subdimensions of communication may also be more effectively taken into account (e.g. evoking everyday language, school language and technical language).

DP2. Tasks must be designed to address covariation, which involves at least two registers that support various languages by offering tasks for exploring, understanding and linking each representation and language in small steps, as mathematical representations are key to mathematical communication and instrumentation that involves conceptualisation.

Various artefacts in a DGE with a variety of affordances (DH3)

Concerning DH3, the tasks include various kinds of potential for using a DGE, such as dragging, tracing and measuring (Højsted, 2020). However, when dragging constant treatments and conversions within and across register in the DGE appear (Duval, 2017). For instance, Amelia also noted her dragging in her written responses (i.e. indicating both dynamic and pragmatic mathematical communication). Dragging is related to instrumentation, which means making sense of the representations and the transformations. Dragging also relates to the instrumentalisation of the DGE, meaning whether a student can manipulate the DGE and how. However, the many potentials of a DGE should not be deployed all at once, given the complexity of the representations involved, which is closely related to DP2.

DP3. Tasks must be designed so that various artefacts in a DGE with a variety of affordances, which gradually support tracing, dragging, measuring and feedback, are used in the task design. Students' varying use of such artefacts are relevant to instrumentation and instrumentalisation and offer various opportunities for mathematical expression and representations when communicating mathematically. For instance, dragging may be challenging, and consequently, its affordances should not be introduced simultaneously to ensure that its potential is fully exploited.

Making room for both individual and collective processes (DH4)

In particular, Amelia developed her language when communicating with Charlie, which is related to DH4. When working in pairs, students may test their results and exercise different subdimensions of the communication competency throughout the task. Thus, students must work in pairs, as they may help each other, which is essential to both mathematical communication (Prediger & Wessel, 2013) and instrumental genesis (Guin & Trouche, 1998).

The process of instrumental genesis is time-consuming (Trouche, 2020a; 2020b), and students need time to investigate the template. When they have time to investigate the template, it limits the risk of having a dialogue that is too contextualised and reduce listening. Thus, when students communicate after investigating the template, they focus better on communicating instead of using the DGE (Bach & Bikner-Ahsbahs, in press, 2022).

DP4. Tasks must be designed to allow individual and collective processes by including individual tasks and tasks that focus on communication. Thus, students have time for the process of instrumental genesis, as well as time to focus on mathematical communication.

Using a DGE and exercising mathematical communication competency are demanding tasks. They are dependent on instrumental genesis, particularly the process of instrumentation, which is closely related to mathematical understanding (Trouche & Drijvers, 2010). Hence, understanding involves coordinating at least two registers (Duval, 2017).

Conclusion

When designing tasks intended to exercise, or even develop, students' mathematical communication competency when using a DGE, it is important to *separate – join – new separate*[3] in steps relating mathematical representations (symbolic, graphical, figurative and verbal) between the registers in the DGE and making room for both collective and individual processes.

Applying the set of design principles presented in this paper offers the opportunity to exercise – and possibly develop – mathematical communication competency when using a DGE. Moreover, as this paper only includes one case, the design principles could be tested on other additional cases and possibly other tasks and mathematical content than what has been presented in this paper.

Acknowledgement

I would like to express my gratitude to Angelika Bikner-Ahsbahs, who has helped when designing and discussing the tasks and the design principles.

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Notes

1 The original terminology referred to these three kinds of language as ”registers”, but when using the concepts with Duval’s terminology, two different understandings of registers are rather difficult (Prediger & Neugebauer, 2021). In addition,the framework was initially developed for language learners (Prediger & Wessel, 2011, 2013).

2 This first part of the students’ work on ”Filling the rectangle” have been presented and analysed in Bach (submitted), but not focusing on mathematical communication, describing the development of instrumental genesis and mathematical communication competency.

3 *Separate – join – new separate* is a reference to Shvarts et al.’s (2022) design principle, concrete-abstract-new-concrete, based on Davydov’s (1990) work. Such a labelling refers to that the different phases of separating and joining brings new perspectives to the new phase of separating (e.g. when writing).

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