The problematic equal sign

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In this study, we investigated how 2,544 Norwegian students in the fifth, sixth, eighth and ninth grades of different schools answered a missing addend task that required a relational understanding of the equal sign. Only 50 % of the students correctly solved the task (30 % in grade 5 and 80 % in grade 9). We then selected the students who managed to correctly solve two tasks that did not have an explicit equal sign but required an assessment of equality. The percentage of correct answers in the missing addend task increased to 71 %, but even among those students who successfully handled the concept of equality, a substantial portion did not solve the equal sign task correctly. Our results indicate that equality and the equal sign cannot be treated as equivalent concepts.

Many years ago, we observed a teacher and a group of students in grade 4. The teacher wrote the sign ”=” on the blackboard and asked, ”What does this sign mean?” One student raised his hand and answered, ”It means equal”. ”Yes”, the teacher replied, ”but when do we use it?” ”When we have to find the answer”, the student said. ”Yes, when we have to find the answer”, said the teacher.

Stating that ”the equal sign is the soul of mathematical operations”, Ma (1999, p. 111) acknowledged the sign’s special status. The equal sign is more than just a symbol used in mathematical operations. If Ma’s statement is correct, then people’s interpretations of the equal sign are crucial determinants of their performance in mathematics. Based on studies that found a strong relationship between one’s understanding of the equal sign and success in solving algebraic equations, McNeil et al. (2006) wrote, in line with Ma’s statement, that students must develop a relational understanding of the equal sign to prepare for success in algebra. To have a relational understanding in this context means to

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view the equal sign as ”a mathematical symbol representing a relationship between quantities rather than a signal to perform arithmetic operations” (Alibali et al., 2007, p. 223).

The recognition of the equal sign’s special position has led to extensive international research on how students understand this sign. To the best of our knowledge, few studies in Norway have been conducted on how students view or interpret the equal sign. Nevertheless, both Tøgersen (2015) and Opsal (2019) have reported that a substantial portion of students in grades five and eight had an operational understanding. Opsal (2019) used both quantitative data from a survey of students in grade 5 (n = 584) and grade 8 (n = 646) from the same project that’s presented in this article – and qualitative data from interviews with four students each in grades 5 and 8. She concluded that despite many years of emphasizing that students should develop a relational understanding of the equal sign at an early age, many students only had an operational understanding of this sign. In Sweden, Madej (2022) examined third- and sixth-grade students’ understanding of the equal sign using an assessment based on Matthews et al.’s (2012) work. Madej (2022) claimed that there is more focus on the importance of the equal sign in primary school in Sweden than in other Nordic countries. However, the study also showed that being able to describe the definition of the equal sign does not translate to being able to use the relational property of the equal sign, and vice versa (Madej, 2022).

Since 2006, the curricula of all schools in Norway have been determined by legally binding national regulations. One of the competence goals for students in mathematics after grade 4 reads as follows: ”The aims of the training are to enable the apprentice to use mathematical symbols and mathematical modes of expression to express mathematical relationships to solve equations” (Utdanningsdirektoratet, 2013, p. 6). The equal sign is not explicitly mentioned the way it is in the 2011 Swedish curriculum for students from the first to third grades under the heading ”Algebra” (Swedish national agency of education, 2018).

The 2015 version of TIMSS (Trends in international mathematics science study) showed that Norwegian students in grade 9 performed the worst on algebraic tasks compared to other content areas, below the international mean (Bergem, 2016). The same trend was previously found in the 2011 TIMSS (Grønmo et al., 2012). Knuth et al. (2006) established that success in algebra depends on a *sophisticated understanding* (i.e., a relational understanding) of the equal sign. This connection, along with the poor results in algebra and the scarce research on the equal sign in Norway, highlights the need for research on how Norwegian students understand the equal sign. We wanted to address this lack of research.

Our aim was to find out if understanding equality implies understanding the equal sign. If this is not the case, these two concepts need to be considered separately. In this article, we present the results of a study in which students who managed to solve mathematical tasks requiring the assessment of equality responded to a task requiring a relational understanding of the equal sign. Our research question was as follows: How do Norwegian students in grades 5, 6, 8 and 9 who have correctly solved two mathematical tasks requiring the assessment of numerical equality solve a *missing addend* task with operations on both side of the equal sign?

Numerical equality

To address the research question, we first distinguish between the concept of numerical equality and the equal sign. This distinction is in line with Schwartzkopf et al.'s (2018) argument that a concept is more than its associated sign and one should first work with the concept of equality before introducing the equal sign in primary school. We consider numerical equality to be a relation on the real numbers with the reflexive, symmetric and transitive properties. Two mathematical expressions are numerically equal if they have the same numerical value. As mentioned earlier, the equal sign plays a vital role in mathematics as a semiotic symbol of equality. Nevertheless, it is merely a symbol. The equality relation is the fundamental concept and is itself independent of the symbol used. Before children can fully grasp the meaning of (and make reasonable use of) the equal sign, they should be able to understand the concept of equality, and this understanding should develop alongside their mathematical proficiency.

Young children may develop insights based on an understanding of certain aspects of numerical equality even before they are introduced to the equal sign. Through a set of experiments, Izard et al. (2014) showed that children as young as 36 months were able to use one-to-one correspondence to reconstruct a set of five or six objects. One-to-one correspondence is central to the concept of equal cardinality of two sets, but these were children who had not mastered the exact numerical meanings of *five* and *six*. At the age of five, many children know that a specific number word describes a specific number of elements in a set, and if the number of elements changes, then the number word must change too, even for numbers well beyond their counting range (Izard et al., 2008). This understanding is necessary to grasp the concept of numerical equality. Thus, the results of the above study indicate that children's understanding of equality develops independently of their understanding of the equal sign. Some decades earlier, Kieran (1981) found that students may have an adequate understanding of word problems involving equality while simultaneously having difficulties with the equal sign at the symbolic level. Therefore, treating *an understanding of the equal sign* and *an understanding of equality* as equivalent concepts is an insufficiently nuanced approach.

The equal sign

We started this article by quoting Ma's (1999) statement about the equal sign's vital role in mathematical operations. According to Falkner et al. (1999), *understanding* the equal sign is a core element of understanding algebra. Success in algebra depends on a *sophisticated understanding* of the equal sign (Alibali et al., 2007; Knuth et al., 2005; Knuth et al., 2006). Falkner et al.'s (1999) mention of *understanding* can be interpreted as relational understanding, and this concept corresponds to what Alibali et al. (2007) called *sophisticated understanding*.

In mathematical texts, the equal sign is the symbolic representation of the reflexive, symmetric and transitive relation called equality. In this article, we restrict ourselves to numerical equality, a relation between real numbers. The equal sign symbolizes sameness. The quantities or expressions on each side of an equal sign must have the same numerical value. Therefore, the expression $A = B$ is either true or false, depending on the numerical values of A and B. This relational understanding represents a rather stringent and unambiguous definition, but it is not, as discussed below, the only interpretation of this sign among students.

O*perational* and *relational* understanding are often distinguished in research on students' understanding of the equal sign (e.g. Kieran, 1981; Knuth et al., 2006). Some researchers, arguing for a more elaborate categorization of these concepts, split relational understanding into two or more categories (Prediger, 2010; Stephens et al., 2013). However, the operational–relational dichotomy was deemed adequate for the present study. *Relational* means that the equal sign is understood and used as the mathematical symbol for the equality relation, as explained above. According to Matthews et al. (2012), a well-developed conception of the equal sign is characterized by relational understanding.

The relational nature of the equal sign is crucial for equations. Specifically, the equal sign represents a relation between the two sides of an equation; it is a statement of numerical equality between the two sides. To solve an equation is to find the numerical values for the unknowns that make the statement of equality true. However, merely understanding that the expressions on each side of the equal sign have the same value is insufficient for understanding the entire solution process; the equation solver must also recognize that each equation can be replaced by an equivalent equation (Kieran, 1981).

Students who interpret the equal sign as an order to do something – that is, an imperative – are said to have an *operational* understanding of the sign (Kieran, 1981; Knuth et al., 2006). For them, the equal sign is a signal to do something, to operate. The operation may be to find an answer to a calculation, often a total. This understanding is adequate for tasks such as $345+215=$ __ and $25 \times 84=$ __. In such cases, students will solve the given tasks correctly if they obey the equal sign's signal to calculate the answers. An operational understanding is well known for leading to errors, such as $34+5=39-7=32 \times 2=64$, and for being inadequate when solving equations such as $3x+5=2-x$. Thus, having only an operational understanding of the equal sign is also associated with difficulties in solving equations (Alibali et al., 2007; Knuth et al., 2005; Knuth et al., 2006).

Studies have shown that different students view the equal sign in different ways. Knuth et al. (2008) asked middle school students (grades 6–8) in the USA to first name the symbol "=" and then explain what the symbol meant. They also asked the students whether the symbol could have another meaning. They categorized a student response as relational if the general idea was that the equal sign represented an equivalence relation between two quantities. They categorized responses such as "add the numbers" or "the answer" (Knuth et al., 2008, p 515) as operational. The answers of more than 50 % of the students in grades 6 and 7 and 45 % of students in grade 8 were categorized as operational. Only 29 % of sixth-grade students provided answers categorized as relational (Knuth et al., 2008). In a previous study, Knuth et al. (2006) gave the participants number sentences with missing numbers and asked them to find the value that made the sentences true (i.e. to solve the equation), such as "$4m+10=70$" (p. 301). Knuth et al. (2006) concluded that "a greater proportion of students who exhibited a relational understanding of the equal sign solved the equations correctly" (p. 305). According to McNeil et al. (2006), teachers who want to prepare students for success in algebra should facilitate students' relational understanding of the equal sign at an early age.

Equality versus the equal sign

Our literature review revealed that many studies on how students understand equality have focused on how students interpret the equal sign or their understanding of equality in problems where the equal sign is explicitly present. Even some articles that seemed to discuss studies of how students understand the concepts of equality or equivalence were mostly about the equal sign (e.g. Falkner et al., 1999; Knuth et al., 2005; Li et al., 2008; McNeil, 2014; Barlow & Harmon, 2012; Leavy et al., 2013). The titles of these studies included the words ”equality” or ”equivalence” but did not mention the equal sign. However, their focus was on children's understanding of the equal sign.

The equal sign is a symbolic representation of equality and is not as fundamental as the concept of equality itself. Therefore, one cannot automatically use students' understanding of the equal sign as a measure of their understanding of equality. It is also possible for students to manage tasks involving the concept of equality and, a few moments later, demonstrate an operational understanding of the equal sign by giving the answer 9 to the missing addend task $14 - 5 = \square + 7$. In the present study, we investigated how students who correctly answered two tasks that involved assessing equality answered the missing addend task given above.

Methods

Our analysis was based on data collected during the Norwegian project SPEED (The function of special education) (Haug, 2017). We collected data from 29 schools in two medium-sized municipalities in different parts of Norway, with students representing a variety of cultural, social, and other backgrounds. We invited all students in grades 5, 6, 8 and 9 (ages 10–14) to participate in the study in spring 2013. The present study's findings are based on mathematics tests completed by 2,544 students (75 % of all students; 92 % of those who consented to participate).

An information letter and a parent consent form were sent to each student invited. The project was approved by the Norwegian centre for research data (NSD).

The mathematics tests

All students were asked to complete 40 mathematics tasks; students in the eighth and ninth grades were given 12 additional tasks. Every task was in a multiple-choice format with seven responses, including ”Don't know”. One of the alternatives was the correct answer, and the rest were distractors. The assignments were paper-based with checkboxes. The tests were developed during the SPEED project and discussed in detail in the work by Opsvik and Skorpen (2017).

We piloted the tasks in three stages. First, the tasks were tested with a small group of students from grades 5–8 to estimate time consumption and to obtain feedback. The tasks were then adjusted and tested with 84 students from grades 8 and 9. The last step in the pilot study was to select the 12 tasks to use only for grades 8 and 9 and to test the rest with 40 students from grades 5 and 6. Small adjustments were made as a result of the last two steps. The results of the final sample of 2,544 students showed high internal consistency, with Cronbach's α scores between 0.88 and 0.91 calculated separately for all four grades.

In the present study, we focused on the students' responses to three tasks. Two were used solely to select students who were able to solve tasks requiring assessments of equality. Neither of the two tasks had an explicit equal sign. The first, task 32 (figure 1), tested whether students could assess equality between two different representations of a fraction – that is, between the symbolic representation $\frac{2}{3}$ and a partly coloured rectangle.

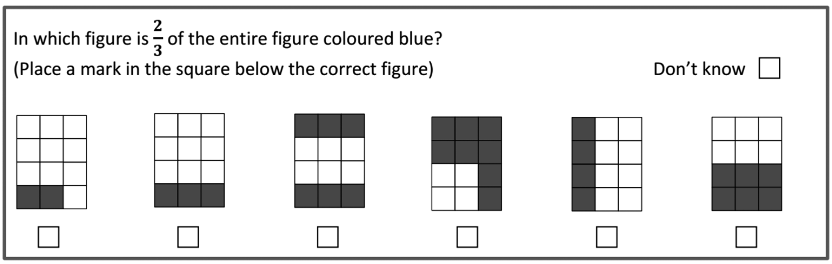

Figure 1. *Equality between two different representations of a fraction; the students received a Norwegian version of the task*

There is a general possibility that students may memorize specific procedures to solve such tasks. The most obvious procedure for solving tasks with fractions and figures is to count the squares and compare them with the nominator and denominator of the fraction. In this case, counting the correct figure would result in the fraction $\frac{8}{12}$, which would need further calculation to transform it into the equivalent $\frac{2}{3}$. Another possible issue is that some students might choose the wrong answer because they have problems with the concept of fractions. Since our main use of this task was to select students who provided correct answers, and could thus handle equality in this context, the issues were irrelevant.

The second task, task 33 (figure 2), was an arithmetic word problem with more than one step.

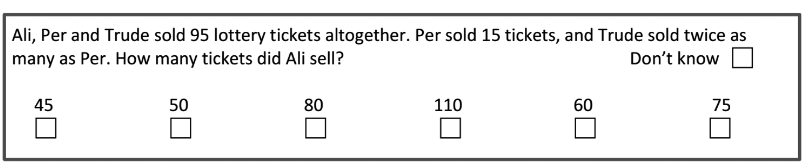

Figure 2. *Equality in an arithmetic word problem; the students received a Norwegian version of the task*

In this task, at least two steps in the solving process rested on the concept of equality. The first sentence in the task implied equality between the numerical sum of three unknown numbers and 95. To solve this task, students had to find the answer to a missing addend problem in which the sum was 95 and the known addend was the combined number of tickets sold by Per and Trude. Students were less likely to be able to solve this task using only memorized procedures.

The students had to make implicit use of equality in both tasks 32 and 33, and those who solved both tasks correctly demonstrated that they could handle the concept of equality in these two contexts.

The actual object of analysis was a task that presupposed a relational understanding of the equal sign, task 21 (figure 3).

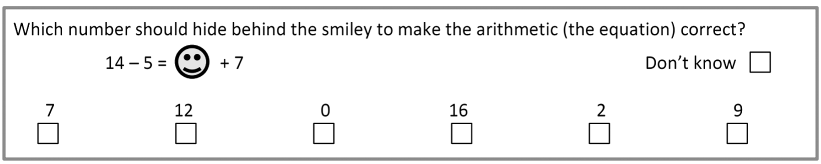

Figure 3. *The equal sign task; the students received a Norwegian version of the task*

The formulation of this task involved a missing addend. On the right side of the equal sign, the addition of two numbers (one of which was missing) had to be equal to 14 – 5 on the left side. The students were required to find the missing number, which they could go about in several ways: The first method involves subtracting 5 from 14 to get 9 and then subtracting 7 from that number. Another approach after subtracting 5 from 14 to get 9 is to wonder what needs to be added to 7 to get 9; this is in line with the missing addend idea of the problem. Alternatively, students could try the listed alternatives until one of them fits and provides equality.

The equal sign task was mathematically simple, requiring only the addition and subtraction of small natural numbers. In Norway, students in second grade (ages 7–8) should have learned enough arithmetic to determine that the difference between 5 and 14 is 9 and that 2 must be added to 7 to give 9. Similar tasks with larger numbers often appear in Norwegian textbooks for students in grade 4. Therefore, the actual calculations involved in solving task 21 should pose few problems for students in grade 5 and above. The only possible obstacle could be the students' bias towards misunderstanding the equal sign.

Results

Table 1 presents the results of task 21. In total, just above half of the students chose the correct answer, with the proportion increasing from 30 % in grade 5 to 80 % in grade 9. In line with the reports of Falkner et al. (1999) and other researchers, the most conspicuous distractors were 9 (subtract 5 from 14 to get 9) and 16 (subtract 5 from 14 to get 9, and then add 7 to get 16). The occurrence of these two distractors decreased as school grade increased. The occurrence of "9" as an answer fell from 43 % among fifth-grade students to 9 % among ninth-grade students, with an average of 28 % across all four grades, and the occurrence of "16" as an answer fell from 17 % among fifth-grade students to 3 % among ninth-grade students, with an average of 10 %. The percentages of students who responded otherwise ("7", "12", "0" or "Don't know") were all low.

Table 1. *Results of task 21 – number and percentage of students with different answers*

	All students		grade 5		grade 6		grade 8		grade 9	
Multiple x	2	0 %	0	0 %	1	0 %	1	0 %	0	0 %
7	56	2 %	14	2 %	21	3 %	13	2 %	8	1 %
12	43	2 %	13	2 %	10	2 %	8	1 %	12	2 %
0	5	0 %	0	0 %	2	0 %	3	0 %	0	0 %
16	238	10 %	100	17 %	78	12 %	42	6 %	18	3 %
2	**1,334**	**53 %**	**173**	**30 %**	**268**	**41 %**	**402**	**62 %**	**491**	**80 %**
9	697	28 %	250	43 %	240	37 %	149	23 %	58	9 %
Don't know	120	5 %	34	6 %	30	5 %	29	4 %	27	4 %
Total	2,495		584		650		647		614	

As described above, we used tasks 32 and 33 to select students who could handle tasks involving the assessment of equality. Table 2 presents the distribution of answers for task 21 among the students who correctly answered tasks 32 and 33. Of these students, 71 % provided correct answers to task 21, 20 % provided the distractor 9 as the answer, and 6 % provided distractor 16 as the answer. When we arranged the results by grade, the percentage of correct answers to task 21 increased from 47 % in grade 5 to 90 % in grade 9. We noted a large and unambiguous increase in percentages from grade 6 to grade 8, which may be because the eighth graders were two years older than the sixth graders and would have worked on formal algebra, with a stronger focus on a relational understanding of the equal sign.

Table 2. *Students' responses to the equal sign task (only students who answered correctly on tasks 32 and 33)*

Task 21	All students		grade 5		grade 6		grade 8		grade 9	
7/12/0	22	2 %	2	1 %	5	2 %	7	3 %	8	3 %
16	54	6 %	15	11 %	26	10 %	11	4 %	2	1 %
2	**667**	**71 %**	**65**	**47 %**	**134**	**53 %**	**208**	**78 %**	**260**	**90 %**
9	187	20 %	54	39 %	84	33 %	32	12 %	17	6 %
Don't know	15	2 %	1	1 %	4	2 %	7	3 %	3	1 %
Total	945		137		253		265		290	

Discussion

Many students did not correctly solve the task that required a relational understanding of the equal sign. Only 30 % of the students in grade 5 showed such a relational understanding, although this increased with grade to 80 % of the students in grade 9. This result is in agreement with previous research (Knuth et al., 2008; Tøgersen, 2015). To prepare students for success in algebra, they must develop a relational understanding of the equal sign (McNeil et al., 2006). Because of this our result is also consistent with the findings from the TIMSS 2015 that algebra is the topic where Norwegian students perform most poorly (Bergem, 2016). Based on our results, we suggest paying attention to the potential for development in algebra and facilitating a relational understanding of the equal sign among younger students.

The main goal of the present study was to investigate the connection between equality and the equal sign based on how students who managed to solve tasks requiring an assessment of equality (tasks 32 and 33) solved a missing addend task (task 21). The results showed that a rather large fraction of the students who provided correct answers on tasks 32 and 33 still chose an incorrect answer on the equal sign task. Of these students, as many as 53 % in grade five and 47 % in grade six chose an incorrect answer on the equal sign task. Their ability to handle equality in one specific context did not necessarily help them to handle equality in a context involving an equal sign. While solving tasks in the same test, students handled the equal sign operationally, and correctly solved tasks involving the assessment of equality. Equality and the equal sign did not seem to represent the same thing for these students. Further, our results indicate that the ability to handle equality at the level of tasks 32 and 33 does not necessarily imply that students have a relational understanding of the equal sign. This finding is in agreement with the findings of Kieran (1981).

As mentioned earlier, many published studies have titles suggesting that they focused on students' understanding of equality or equivalence, but in fact, they investigated students' understanding of the equal sign (e.g. Barlow & Harmon, 2012; Falkner et al., 1999; Knuth et al., 2005; Leavy et al., 2013; Li et al., 2008; McNeil, 2014). Our results suggest that the approach taken by those studies was oversimplified. Equality and the equal sign are not equivalent concepts. The equal sign is one of many possible semiotics that can be used to express the fundamental mathematical object of equality.

Due to this scenario, teachers may be misled by the answers that students give to questions such as, "What number would you put in the box to make this a true number sentence? $8 + 4 = \square + 5$". The most common incorrect answers to this problem are 12 and 17 (Falkner et al., 1999). These errors clearly indicate the possession of an operational understanding of the equal sign but do not necessarily say anything about a student's understanding of equality. The problem does not seem to be a lack of understanding of equality per se, but an inadequate understanding of the semiotic role the equal sign plays in the mathematical language. The operational understanding of the equal sign overrules students' understanding of equality in these situations.

Equality is a more fundamental mathematical object than its semiotic representations. Thus, not understanding equality is a problem on a deeper level than having an operational understanding of the equal sign. Both misconceptions, with equal sign and with equality, have consequences. The consequences might look similar, but they must be treated differently. It is necessary to be careful when drawing conclusions about students' understanding of equality based on tasks involving the equal sign. The conclusions could be hasty and wrong.

After data for this study were collected, Norway implemented a new primary school curriculum in 2020. In this new curriculum, like Sweden's, the equal sign is explicitly mentioned (Utdanningsdirektoratet, 2020, p. 7) in a competence goal that applies to students in grade 3. This will hopefully lead to more uplifting results in future research on students' understanding of the equal sign.

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